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Recast combination functions of coordinate and momentum operators into their ordered product forms*

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By using the parameter differential method of operators, we recast the combination function of coordinate and momentum operators into its normal and anti-normal orderings, which is more ecumenical, simpler, and neater than the existing ways. These products are very useful in obtaining some new differential relations and useful mathematical integral formulas. Further, we derive the normally ordered form of the operator $(fQ+gP)^{-n}$ with n being an arbitrary positive integer by using the parameter tracing method of operators together with the intermediate coordinate–momentum representation. In addition, general mutual transformation rules of the normal and anti-normal orderings, which have good universality, are derived and hence the anti-normal ordering of $(fQ+gP)^{-n}$ is also obtained. Finally, the application of some new identities is given.

Keywords: normal and anti-normal orderings, parameter differential method of operators, intermediate coordinate–momentum representation

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1. Introduction

The operator ordering problem,[1,2] which is one of the consequences of non-commutativity, plays an important role in describing the phase space of quantum mechanics[3–5] and constructing quantum mechanical operators.[6–8] There are some definite operator ordering rules, such as normal ordering, anti-normal ordering, and Weyl ordering. Among them, the normal ordering (denoted by the symbol : :) of operators is often used to calculate the expectation values in coherent state,[9–12] i.e., $\langle z|:F(\hat{a},\hat{a}^{\dagger}):|z\rangle=F(z,z^{*})$, while the P-representation of any density operator ρ can be obtained directly by its anti-normally ordered form (denoted by the symbol $\vdots\ \vdots$), i.e., $\rho=\int\frac{\mathrm{d}^{2}z}{\pi}P(z,z^{*})\,|z\rangle\langle z|=\vdots P(\hat{a},\hat{a}^{\dagger})\vdots$.[13,14] Therefore, it is important to obtain the normal and anti-normal ordering products of operators in quantum optics.

To our knowledge, there are three main methods of handling operator ordering problems in previously published literatures, namely, Louisell's differential operation method,[15] the Lie algebra method,[16] and the method of integration within ordered product (IWOP) of operators.[17–21] However, the above mentioned methods have some inconveniences. For example, Louisell's differential operation method must utilize the coherent state representation, the Lie algebra method has to use the decomposition of angular momentum or exponential operators, the IWOP method must first know the completeness relation of the eigenvector of the operator and then make use of some complex mathematical integral formulas. In this paper, in order to avoid these inconveniences, we directly propose the parameter differential method of operators and use it to arrange the combination functions of coordinate and momentum operators in their normal and anti-normal orderings conveniently. It is found that this way is much simpler and neater than those of Refs. [22–25]. Also, the mutual transformation rules between the normal and anti-normal orderings are derived. In addition, we obtain some mathematical integral formulas without really performing these integrations by virtue of the deduced operator ordering identities and some new operator differential relations.

2. Normally and anti-normally ordered expansions of $(fQ+gP)^{n}$

We first derive the normally and anti-normally ordered expansions of $(fQ+gP)^{n}$ (here $n=0,1,2,\ldots$, f and g are two arbitrary parameters) by using the parameter differential method of operators, and then convert a general operator function $F(fQ+gP)$ into its normal and anti-normal ordering products. To achieve these aims, we now use the Baker–Hausdorff formula[26] $\mathrm{e}^{A+B}=\mathrm{e}^{A}\mathrm{e}^{B}\mathrm{e}^{-\frac{1}{2}[A,B]}=\mathrm{e}^{B}\mathrm{e}^{A}\mathrm{e}^{\frac{1}{2}[A,B]}$ with $[A,[A,B]]=[B,[A,B]]=0$ and the parameter differential method of operators to obtain the normal ordering of

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$(fQ+gP)^n$ as follows:

$$\begin{aligned}(fQ+gP)^n &= \frac{\partial^n}{\partial t^n}\mathrm{e}^{t(fQ+gP)}\Big|_{t=0}\\ &= :\frac{\partial^n}{\partial t^n}\mathrm{e}^{\frac{1}{4}(f^2t^2+g^2t^2)}\mathrm{e}^{t(fQ+gP)}\Big|_{t=0}:\\ &= :\mathrm{e}^{\frac{1}{4}\left(\frac{\partial^2}{\partial Q^2}+\frac{\partial^2}{\partial P^2}\right)}\frac{\partial^n}{\partial t^n}\mathrm{e}^{t(fQ+gP)}\Big|_{t=0}:\\ &= :\mathrm{e}^{\frac{1}{4}\left(\frac{\partial^2}{\partial Q^2}+\frac{\partial^2}{\partial P^2}\right)}(fQ+gP)^n:. \end{aligned} \tag{1}$$

Further, by using the differential expression of single-variable Hermitian polynomial $H_n(q)$[27,28] (see Appendix A for details)

$$H_n(q) = \frac{\partial^n}{\partial t^n}\mathrm{e}^{-t^2+2qt}\Big|_{t=0} = \mathrm{e}^{-\frac{1}{4}\frac{\partial^2}{\partial q^2}}(2q)^n, \tag{2}$$

we may derive the normally ordered expansion of $(fQ+gP)^n$, i.e.,

$$\begin{aligned}&(fQ+gP)^n\\ &= :\mathrm{e}^{\frac{f^2+g^2}{4}\frac{\partial^2}{\partial(fQ+gP)^2}}(fQ+gP)^n:\\ &= \left(\frac{\sqrt{f^2+g^2}}{2\mathrm{i}}\right)^n:\mathrm{e}^{-\frac{1}{4}\frac{\partial^2}{\partial[\mathrm{i}(fQ+gP)/\sqrt{f^2+g^2}]^2}}\left[2\frac{\mathrm{i}(fQ+gP)}{\sqrt{f^2+g^2}}\right]^n:\\ &= \left(\frac{\sqrt{f^2+g^2}}{2\mathrm{i}}\right)^n:H_n\left[\frac{\mathrm{i}(fQ+gP)}{\sqrt{f^2+g^2}}\right]:, \end{aligned} \tag{3}$$

which is an n-order single-variable operator Hermitian polynomial within normal ordering.

In the similar way to deriving Eq. (1), we can obtain the differential formula for converting the operator $(fQ+gP)^n$ into its anti-normal ordering, i.e.,

$$(fQ+gP)^n = \vdots\mathrm{e}^{-\frac{1}{4}\left(\frac{\partial^2}{\partial Q^2}+\frac{\partial^2}{\partial P^2}\right)}(fQ+gP)^n\vdots. \tag{4}$$

Using Eq. (2) again, we have

$$(fQ+gP)^n = \left(\frac{\sqrt{f^2+g^2}}{2}\right)^n\vdots H_n\left(\frac{fQ+gP}{\sqrt{f^2+g^2}}\right)\vdots, \tag{5}$$

which is the anti-normally ordered expansion of $(fQ+gP)^n$, similar to the normal ordering of $(fQ+gP)^n$ in form.

Further, letting $\mu=(f-\mathrm{i}g)/\sqrt{2}$ and $\nu=(f+\mathrm{i}g)/\sqrt{2}$, we can respectively identify Eqs. (3) and (5) as

$$\left(\mu a+\nu a^\dagger\right)^n = \left(-\mathrm{i}\sqrt{\frac{\mu\nu}{2}}\right)^n:H_n\left(\mathrm{i}\sqrt{\frac{\nu}{2\mu}}a^\dagger+\mathrm{i}\sqrt{\frac{\mu}{2\nu}}a\right):, \tag{6}$$

$$\left(\mu a+\nu a^\dagger\right)^n = \left(\sqrt{\frac{\mu\nu}{2}}\right)^n\vdots H_n\left(\sqrt{\frac{\nu}{2\mu}}a^\dagger+\sqrt{\frac{\mu}{2\nu}}a\right)\vdots, \tag{7}$$

which are respectively the normal and anti-normal ordering products of a new boson operator $\left(\mu a+\nu a^\dagger\right)^n$.

So, for any operator function $F(fQ+gP)$ which can be expanded as a power series of the combination operator $(fQ+gP)$, equations (1) and (4) can be generalized as

$$F(fQ+gP) =: \mathrm{e}^{\frac{1}{4}\left(\frac{\partial^2}{\partial Q^2}+\frac{\partial^2}{\partial P^2}\right)}F(fQ+gP):, \tag{8}$$

$$F(fQ+gP) = \vdots\mathrm{e}^{-\frac{1}{4}\left(\frac{\partial^2}{\partial Q^2}+\frac{\partial^2}{\partial P^2}\right)}F(fQ+gP)\vdots, \tag{9}$$

which are respectively the formulas for converting operator function $F(fQ+gP)$ into its normal and anti-normal ordering products. And, as noted, $\partial/\partial Q$ and $\partial/\partial P$ are merely differential operations, not quantum mechanical operators, so their consequences are permutable and they can enter into or go out of the ordered symbols : : and $\vdots\ \vdots$ freely.

Comparing both sides of Eqs. (8) and (9), we can naturally have

$$:F(fQ+gP):=\vdots\mathrm{e}^{-\frac{1}{2}\left(\frac{\partial^2}{\partial Q^2}+\frac{\partial^2}{\partial P^2}\right)}F(fQ+gP)\vdots, \tag{10}$$

which is just the mutual transformation formula between normally and anti-normally ordered products of the operator $F(fQ+gP)$ (see Appendix B for the detailed proof), and is the same as Eq. (23) in Ref. [29].

3. Normally and anti-normally ordered expansions of $(fQ+gP)^{-n}$

In the present section, we want to transform the negative exponential operator function $(fQ+gP)^{-n}$ to its normally and anti-normally ordered expansions by virtue of the intermediate coordinate–momentum representation $|\eta\rangle_{f,g}$.[30–32] It needs to point out that the parameters f and g are both real numbers, different from the case in Section 2, otherwise $|\eta\rangle_{f,g}$ might not satisfy the completeness relation and $(fQ+gP)^n$ might be irreversible (or say $(fQ+gP)^{-n}$ might not exist).

3.1. Normal ordering and anti-normal ordering of $(fQ+gP)^{-1}$

To begin with, we derive the normal and anti-normal orderings of $(fQ+gP)^{-1}$ with the help of the intermediate coordinate–momentum representation and the mutual transformation formula in Eq. (10).

As described in Ref. [30], there exists the intermediate coordinate–momentum state $|\eta\rangle_{f,g}$, defined as

$$\begin{aligned}|\eta\rangle_{f,g} &= \left[\pi\left(f^2+g^2\right)\right]^{-1/4}\\ &\times\exp\left[-\frac{1}{2\left(f^2+g^2\right)}\eta^2+\frac{\sqrt{2}}{f-\mathrm{i}g}\eta a^\dagger-\frac{f+\mathrm{i}g}{2(f-\mathrm{i}g)}a^{\dagger2}\right]|0\rangle \end{aligned}$$

in the single-mode Fock space. The state $|\eta\rangle_{f,g}$ is the eigenvector of operator $(fQ+gP)$ and satisfies the orthogonal completeness relation, that is,

$$(fQ+gP)|\eta\rangle_{f,g} = \eta|\eta\rangle_{f,g},$$

$$\int_{-\infty}^{\infty} \mathrm{d}\eta |\,\eta\rangle_{\mathrm{f,gf,g}}\langle\eta| = 1,$$
$$_{f,g}\langle\eta|\eta'\rangle_{f,g} = \delta(\eta-\eta'). \tag{11}$$

For $(fQ+gP)^{-1} \equiv \frac{1}{(fQ+gP)}$, we thus have

$$\frac{1}{(fQ+gP)} = \int_{-\infty}^{\infty} \mathrm{d}\eta \frac{1}{\eta} |\,\eta\rangle_{f,gf,g}\langle\eta|$$
$$= \left[\pi\left(f^2+g^2\right)\right]^{-1/2} : \int_{-\infty}^{\infty} \mathrm{d}\eta \frac{1}{\eta} \times \exp\left\{-\frac{1}{f^2+g^2}\left[\eta-(fQ+gP)\right]^2\right\} : . \tag{12}$$

The integral in Eq. (12) has a singular point at $\eta = 0$ on the contour, so we must use the Cauchy principal value integral

$$\begin{aligned}
\frac{1}{(fQ+gP)} &= \frac{1}{\sqrt{\pi(f^2+g^2)}} \lim_{\varepsilon\to 0^+} : \int_{\varepsilon}^{\infty} \mathrm{d}\eta \frac{1}{\eta} \exp\left\{-\frac{1}{f^2+g^2}\left[\eta-(fQ+gP)\right]^2\right\} + \int_{-\infty}^{-\varepsilon} \mathrm{d}\eta \frac{1}{\eta} \exp\left\{-\frac{1}{f^2+g^2}\left[\eta-(fQ+gP)\right]^2\right\} : \\
&= \frac{1}{\sqrt{\pi(f^2+g^2)}} \lim_{\varepsilon\to 0^+} : \int_{\varepsilon}^{\infty} \mathrm{d}\eta \frac{1}{\eta} \exp\left\{-\frac{1}{f^2+g^2}\left[\eta-(fQ+gP)\right]^2\right\} - \int_{\varepsilon}^{\infty} \mathrm{d}\eta \frac{1}{\eta} \exp\left\{-\frac{1}{f^2+g^2}\left[-\eta-(fQ+gP)\right]^2\right\} : \\
&= \frac{1}{\sqrt{\pi(f^2+g^2)}} \lim_{\varepsilon\to 0^+} : \mathrm{e}^{-\frac{1}{f^2+g^2}(fQ+gP)^2} \int_{\varepsilon}^{\infty} \mathrm{d}\eta\, \mathrm{e}^{-\frac{1}{f^2+g^2}\eta^2} \frac{\mathrm{e}^{\frac{2}{f^2+g^2}(fQ+gP)\eta} - \mathrm{e}^{-\frac{2}{f^2+g^2}(fQ+gP)\eta}}{\eta} : \\
&= \frac{1}{\sqrt{\pi(f^2+g^2)}} : \mathrm{e}^{\frac{-1}{f^2+g^2}(fQ+gP)^2} \sum_{k=0}^{\infty} \frac{2^{2k+1}}{(2k+1)!} \frac{(fQ+gP)^{2k+1}}{(f^2+g^2)^{2k+1}} \left(2\int_0^{\infty} \mathrm{d}\eta\, \mathrm{e}^{-\frac{1}{f^2+g^2}\eta^2} \eta^{2k}\right) : \\
&= \frac{1}{\sqrt{\pi(f^2+g^2)}} : \mathrm{e}^{\frac{-1}{f^2+g^2}(fQ+gP)^2} \sum_{k=0}^{\infty} \frac{2^{2k+1}}{(2k+1)!} \frac{(fQ+gP)^{2k+1}}{\left(\sqrt{f^2+g^2}\right)^{2k+1}} \left(2\int_0^{\infty} \mathrm{d}\xi\, \mathrm{e}^{-\xi^2} \xi^{2k}\right) : \\
&= \frac{1}{\sqrt{\pi(f^2+g^2)}} : \mathrm{e}^{\frac{-1}{f^2+g^2}(fQ+gP)^2} \sum_{k=0}^{\infty} \frac{2^{2k+1}}{(2k+1)!} \frac{(fQ+gP)^{2k+1}}{\left(\sqrt{f^2+g^2}\right)^{2k+1}} \Gamma\left(k+\frac{1}{2}\right) : .
\end{aligned} \tag{13}$$

Here, $\xi = \frac{\eta}{\sqrt{f^2+g^2}}$ and $\Gamma(x) = 2\int_0^{\infty} \mathrm{d}\xi\, \mathrm{e}^{-\xi^2} \xi^{2x-1}$ with $\mathrm{Re}(x) > 0$ is the Gamma function.[33] Then using the rearranging double summation $\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} A_k B_n = \sum_{m=0}^{\infty}\sum_{n=0}^{m} A_{m-n}B_n$, the following well-known combinatorial formula[34]

$$\sum_{m=0}^{n} (-)^m \frac{x}{m+x}\binom{n}{m} = \left[\binom{n+x}{n}\right]^{-1},$$
$$\Gamma\left(k+\frac{1}{2}\right) = \sqrt{\pi}\frac{(2k-1)!!}{2^k},$$
$$\mathrm{e}^{\frac{-1}{f^2+g^2}(fQ+gP)^2} = \sum\nolimits_{l=0}^{\infty} \frac{(-)^l}{l!} \frac{(fQ+gP)^{2l}}{(f^2+g^2)^l},$$

we can rewrite Eq. (13) as

$$\begin{aligned}
&\frac{1}{(fQ+gP)} \\
&= : \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} (-)^l \frac{2}{l!k!(2k+1)} \frac{(fQ+gP)^{2l+2k+1}}{(f^2+g^2)^{l+k+1}} : \\
&= : \sum_{m=0}^{\infty}\sum_{k=0}^{m} (-)^{m-k} \frac{2}{(m-k)!k!(2k+1)} \frac{(fQ+gP)^{2m+1}}{(f^2+g^2)^{m+1}} : \\
&= : 2\sum_{m=0}^{\infty} (-)^m \sum_{k=0}^{m} (-)^k \frac{1/2}{k+1/2}\binom{m}{k} \frac{1}{m!} \frac{(fQ+gP)^{2m+1}}{(f^2+g^2)^{m+1}} : \\
&= : 2\sum_{m=0}^{\infty} \frac{1}{\binom{m+1/2}{m}} \frac{(-)^m}{m!} \frac{(fQ+gP)^{2m+1}}{(f^2+g^2)^{m+1}} : \\
&= \sum_{m=0}^{\infty} \frac{(-)^m\sqrt{\pi}}{\Gamma(m+3/2)} \frac{1}{(f^2+g^2)^{m+1}} : (fQ+gP)^{2m+1} : .
\end{aligned} \tag{14}$$

This is just the normal ordering of $(fQ+gP)^{-1}$. In particular, in Eq. (14) for $f=1$ and $g=0$ we have

$$\frac{1}{Q} = \sum_{m=0}^{\infty} \frac{(-)^m\sqrt{\pi}}{\Gamma(m+3/2)} : Q^{2m+1} :, \tag{15}$$

which is just the normal ordering of the inverse of the coordinate operator. Similarly, for $f=0$ and $g=1$, we obtain

$$\frac{1}{P} = \sum_{m=0}^{\infty} \frac{(-)^m\sqrt{\pi}}{\Gamma(m+3/2)} : P^{2m+1} :, \tag{16}$$

which is just the normal ordering of the inverse of the momentum operator.

Moreover, by using the mutual transformation rule of the normal and anti-normal orderings of operators derived in Eq. (10), we can obtain the anti-normal ordering of $(fQ+gP)^{-1}$ as follows:

$$\begin{aligned}
\frac{1}{(fQ+gP)} &= :\mathrm{e}^{-\frac{1}{2}\left(\frac{\partial^2}{\partial Q^2}+\frac{\partial^2}{\partial P^2}\right)} \sum_{m=0}^{\infty} \frac{(-)^m\sqrt{\pi}}{\Gamma(m+3/2)} \frac{1}{(f^2+g^2)^{m+1}} (fQ+gP)^{2m+1} : \\
&= \sum_{m=0}^{\infty} \frac{(-)^m\sqrt{\pi}}{\Gamma(m+3/2)} \frac{1}{(f^2+g^2)^{m+1}} :\mathrm{e}^{-\frac{1}{2}\left(\frac{\partial^2}{\partial Q^2}+\frac{\partial^2}{\partial P^2}\right)} (fQ+gP)^{2m+1} :
\end{aligned}$$

$$= \sum_{m=0}^{\infty} \frac{(-)^m \sqrt{\pi}}{\Gamma(m+3/2)} \frac{1}{(f^2+g^2)^{m+1}} \vdots \mathrm{e}^{-\frac{f^2+g^2}{2}\frac{\partial^2}{\partial(fQ+gP)^2}} (fQ+gP)^{2m+1} \vdots$$

$$= \sqrt{\frac{\pi}{2(f^2+g^2)}} \sum_{m=0}^{\infty} \frac{(-)^m}{2^m \Gamma(m+3/2)} \vdots H_{2m+1}\left(\frac{fQ+gP}{\sqrt{2(f^2+g^2)}}\right) \vdots . \tag{17}$$

Hence we can get the P-representation of $(fQ+gP)^{-1}$ immediately, i.e.,

$$P(z=q+\mathrm{i}p) = \sqrt{\frac{\pi}{2(f^2+g^2)}} \sum_{m=0}^{\infty} \frac{(-)^m}{2^m \Gamma(m+3/2)} \times H_{2m+1}\left(\frac{fq+gp}{\sqrt{2(f^2+g^2)}}\right), \tag{18}$$

as was pointed out in our introduction.

3.2. Normal ordering and anti-normal ordering of $(fQ+gP)^{-n}$

Now we discuss how to transform the negative exponential operator function $(fQ+gP)^{-n} \equiv \frac{1}{(fQ+gP)^n}$ with $n = 2,3,4,\ldots$ into its normal and anti-normal orderings. To this end, we shall make full use of the completeness relations of the intermediate coordinate–momentum representation $|\eta\rangle_{f,g}$,

$$\frac{1}{(fQ+gP)^n} = \frac{(-)^{n-1}}{(n-1)!} \frac{\partial^{n-1}}{\partial (fQ+gP)^{n-1}} \frac{1}{(fQ+gP)},$$
$$\frac{1}{(fQ+gP)} |\eta\rangle_{f,g} = \frac{1}{\eta} |\eta\rangle_{f,g}.$$

For convenience, we will utilize the parameter tracing method,

$$\frac{1}{(fQ+gP)^n} = \frac{(-)^{n-1}}{(n-1)!} \frac{\partial^{n-1}}{\partial (fQ+gP)^{n-1}} \frac{1}{(fQ+gP)} = \frac{(-)^{n-1}}{(n-1)!} \frac{\partial^{n-1}}{\partial t^{n-1}} \frac{1}{(fQ+gP)+t}\bigg|_{t=0}.$$

The motivation to do this lies in: once $(fQ+gP)^{-1}$ has acted on the state vector $|\eta\rangle_{f,g}$, the variable η will come in the denominator ($\frac{1}{\eta}|\eta\rangle_{f,g}$). Correspondingly, $\frac{\partial^{n-1}}{\partial(fQ+gP)^{n-1}}$ should become $\frac{\partial^{n-1}}{\partial \eta^{n-1}}$, where the η in $\frac{\partial^{n-1}}{\partial \eta^{n-1}}$ is identical to the one in the denominator of $\frac{1}{\eta}|\eta\rangle_{f,g}$. However, there exists η in the numerator of $\frac{1}{\eta}|\eta\rangle_{f,g}$, which confuses the calculations. For clarity, we will utilize the parameter tracing method as follows:

$$\frac{1}{(fQ+gP)^n} = \frac{(-)^{n-1}}{(n-1)!} \frac{\partial^{n-1}}{\partial (fQ+gP)^{n-1}} \frac{1}{(fQ+gP)} = \frac{(-)^{n-1}}{(n-1)!} \frac{\partial^{n-1}}{\partial t^{n-1}} \frac{1}{(fQ+gP)+t}\bigg|_{t=0} = \frac{(-)^{n-1}}{(n-1)!} \frac{\partial^{n-1}}{\partial t^{n-1}} \int_{-\infty}^{\infty} \mathrm{d}\eta \frac{1}{\eta+t} |\eta\rangle_{\mathrm{f,gf,g}} \langle\eta| \bigg|_{t=0}$$

$$= \frac{1}{\sqrt{\pi(f^2+g^2)}} \frac{(-)^{n-1}}{(n-1)!} \frac{\partial^{n-1}}{\partial t^{n-1}} : \int_{-\infty}^{\infty} \mathrm{d}\eta \frac{1}{\eta+t} \exp\left\{-\frac{1}{f^2+g^2}[\eta-(fQ+gP)]^2\right\}\bigg|_{t=0} : . \tag{19}$$

By setting $\eta+t=\eta'$, the integration in Eq. (19) then becomes

$$: \int_{-\infty}^{\infty} \mathrm{d}\eta \frac{1}{\eta+t} \exp\left\{-\frac{1}{f^2+g^2}[\eta-(fQ+gP)]^2\right\} : = : \int_{-\infty}^{\infty} \mathrm{d}\eta' \frac{1}{\eta'} \exp\left\{-\frac{1}{f^2+g^2}\left[\eta'-t-(fQ+gP)\right]^2\right\} : .$$

Substituting it into Eq. (19) yields

$$\begin{aligned}
&\frac{1}{(fQ+gP)^n} \\
&= \frac{1}{\sqrt{\pi(f^2+g^2)}} \frac{(-)^{n-1}}{(n-1)!} \frac{\partial^{n-1}}{\partial t^{n-1}} : \int_{-\infty}^{\infty} \mathrm{d}\eta' \frac{1}{\eta'} \exp\left\{-\frac{1}{f^2+g^2}\left[\eta'-t-(fQ+gP)\right]^2\right\}\bigg|_{t=0} : \\
&= \frac{1}{\sqrt{\pi(f^2+g^2)}} \frac{(-)^{n-1}}{(n-1)!} \frac{\partial^{n-1}}{\partial (fQ+gP)^{n-1}} : \int_{-\infty}^{\infty} \mathrm{d}\eta' \frac{1}{\eta'} \exp\left\{-\frac{1}{f^2+g^2}\left[\eta'-(fQ+gP)\right]^2\right\} : \\
&= \frac{(-)^{n-1}}{(n-1)!} \frac{\partial^{n-1}}{\partial (fQ+gP)^{n-1}} \times \sum_{m=0}^{\infty} \frac{(-)^m \sqrt{\pi}}{\Gamma(m+3/2)} \frac{1}{(f^2+g^2)^{m+1}} : (fQ+gP)^{2m+1} : . \\
&= \frac{(-)^{n-1}}{(n-1)!} \sum_{m=\left[\frac{n-1}{2}\right]}^{\infty} \frac{(-)^m \sqrt{\pi}}{\Gamma(m+3/2)} \frac{1}{(f^2+g^2)^{m+1}} \times \frac{(2m+1)!}{(2m-n+2)!} : (fQ+gP)^{2\,m-n+2} : .
\end{aligned} \tag{20}$$

Further, letting $m=l-1$, we can further simplify Eq. (20) as

$$(fQ+gP)^{-n} = \frac{(-)^n}{(n-1)!} \sum_{l=\left[\frac{n+1}{2}\right]}^{\infty} \frac{(-)^l \sqrt{\pi}}{\Gamma(l+1/2)} \times \frac{1}{(f^2+g^2)^l} \frac{(2l-1)!}{(2l-n)!} : (fQ+gP)^{2l-n} :, \tag{21}$$

which is just the normally ordered expansion of $(fQ+gP)^{-n}$ with $n=1,2,3,\ldots$. In the above calculations, we have substituted the normal ordering of $(fQ+gP)^{-1}$ in Eq. (14) into Eq. (20).

Similarly, by means of the mutual transformation formula between the normal and anti-normal orderings in

Eq. (10), we can obtain the anti-normally ordered expression of $(fQ+gP)^{-n}$ as

$$\begin{aligned}
&(fQ+gP)^{-n}\\
&=\frac{(-)^n}{(n-1)!}\vdots \mathrm{e}^{-\frac{1}{2}\left(\frac{\partial^2}{\partial Q^2}+\frac{\partial^2}{\partial P^2}\right)}\sum_{l=\left[\frac{n+1}{2}\right]}^{\infty}\frac{(-)^l\sqrt{\pi}}{\Gamma(l+1/2)}\\
&=\left(\sqrt{\frac{2}{f^2+g^2}}\right)^n\frac{(-)^n}{(n-1)!}\\
&\times\vdots\sum_{l=\left[\frac{n+1}{2}\right]}^{\infty}\frac{(-)^l\sqrt{\pi}}{\Gamma(l+1/2)}\frac{1}{2^l}\frac{(2l-1)!}{(2l-n)!}H_{2l-n}\left(\frac{fQ+gP}{\sqrt{2\left(f^2+g^2\right)}}\right)\vdots. \qquad (22)
\end{aligned}$$

In particular, taking $f=0$ and $g=1$ in Eqs. (21) and (22), we obtain the normal ordering and anti-normal ordering of the momentum operator P^{-n}, i.e.,

$$P^{-n}=\frac{(-)^n}{(n-1)!}:\sum_{l=\left[\frac{n+1}{2}\right]}^{\infty}\frac{(-)^l\sqrt{\pi}}{\Gamma(l+1/2)}\frac{(2l-1)!}{(2l-n)!}P^{2l-n}:, \qquad (23)$$

$$\begin{aligned}
P^{-n}&=\frac{\left(-\sqrt{2}\right)^n}{(n-1)!}\\
&\times\vdots\sum_{l=\left[\frac{n+1}{2}\right]}^{\infty}\frac{(-)^l\sqrt{\pi}}{\Gamma(l+1/2)}\frac{1}{2^l}\frac{(2l-1)!}{(2l-n)!}H_{2l-n}\left(\frac{P}{\sqrt{2}}\right)\vdots. \qquad (24)
\end{aligned}$$

For the case of $f=1$ and $g=0$, we similarly have

$$Q^{-n}=\frac{2(-)^{n-1}}{(n-1)!}:\sum_{l=\left[\frac{n+1}{2}\right]}^{\infty}\frac{(-)^l\sqrt{\pi}}{\Gamma(l+1/2)}\frac{(2l-1)!}{(2l-n)!}Q^{2l-n}:, \qquad (25)$$

$$\begin{aligned}
Q^{-n}&=\frac{\left(-\sqrt{2}\right)^n}{(n-1)!}\\
&\times\vdots\sum_{l=\left[\frac{n+1}{2}\right]}^{\infty}\frac{(-)^l\sqrt{\pi}}{\Gamma(l+1/2)}\frac{1}{2^l}\frac{(2l-1)!}{(2l-n)!}H_{2l-n}\left(\frac{Q}{\sqrt{2}}\right)\vdots. \qquad (26)
\end{aligned}$$

4. Applications

As some applications of the above newly-derived operator identities, we now present some useful mathematical integral formulas and some new operator differential relations.

Consider the anti-normally ordered expansion of the density operator[35–37]

$$\rho=\int\frac{\mathrm{d}^2z}{\pi}\vdots\langle -z|\rho|z\rangle\exp(zz^*+z^*a-za^*+a^\dagger a)\vdots, \qquad (27)$$

which tells us that once the coherent state matrix element $\langle -z|\rho|z\rangle$ is known, the anti-normal ordering of the density operator ρ can be obtained by integrating directly over all values of z in the whole space within the symbol $\vdots\ \vdots$. By substituting Eq. (3) into Eq. (27) and using Eq. (5), we have

$$\begin{aligned}
(fQ+gP)^n&=\left(\frac{\sqrt{f^2+g^2}}{2\mathrm{i}}\right)^n\int\frac{\mathrm{d}^2z}{\pi}\vdots H_n\left(\mathrm{i}\frac{(f-\mathrm{i}g)z-(f+\mathrm{i}g)z^*}{\sqrt{2(f^2+g^2)}}\right)\exp(-zz^*+z^*a-za^\dagger+a^\dagger a)\vdots\\
&=\left(\frac{\sqrt{f^2+g^2}}{2}\right)^n\vdots H_n\left(\frac{(f-\mathrm{i}g)\hat{a}+(f+\mathrm{i}g)\hat{a}^\dagger}{\sqrt{2(f^2+g^2)}}\right)\vdots. \qquad (28)
\end{aligned}$$

Since both sides of Eq. (28) are in anti-normal ordering, replacing $a\to x$ and $a^\dagger\to y$ in Eq. (28) yields directly

$$\begin{aligned}
&\int\frac{\mathrm{d}^2z}{\pi}H_n(\eta z+\eta^*z^*)\exp\left(-zz^*+z^*x-zy\right)\\
&=\mathrm{i}^nH_n(-\mathrm{i}\eta x+\mathrm{i}\eta^*y)\,\mathrm{e}^{-xy}. \qquad (29)
\end{aligned}$$

This is a new integral formula, and it is not easy to get with the usual mathematical integral method. However, we can adopt the quantum optics approach to present the result without really performing this integration. On the other hand, considering the P-representation in the coherent state $|z\rangle$ basis, and using Eqs. (3) and (5), we can derive the following identity:

$$\begin{aligned}
&(fQ+gP)^n\\
&=\left(\frac{\sqrt{f^2+g^2}}{2}\right)^n\int\frac{\mathrm{d}^2z}{\pi}H_n\left(\frac{f\frac{z+z^*}{\sqrt{2}}+g\frac{z-z^*}{\mathrm{i}\sqrt{2}}}{\sqrt{f^2+g^2}}\right)|z\rangle\langle z|\\
&=\left(\frac{\sqrt{f^2+g^2}}{2}\right)^n\int\frac{\mathrm{d}^2z}{\pi}H_n\left(\frac{(f-\mathrm{i}g)z+(f+\mathrm{i}g)z^*}{\sqrt{2(f^2+g^2)}}\right)\\
&\times:\exp\left(-zz^*+z^*a+za^\dagger-a^\dagger a\right):\\
&=\left(\frac{\sqrt{f^2+g^2}}{2\mathrm{i}}\right)^n:H_n\left[\mathrm{i}\frac{(f-\mathrm{i}g)\hat{a}+(f+\mathrm{i}g)\hat{a}^\dagger}{\sqrt{2(f^2+g^2)}}\right]:, \qquad (30)
\end{aligned}$$

which implies another integration formula

$$\begin{aligned}
&\int\frac{\mathrm{d}^2z}{\pi}H_n\left(\zeta^*z+\zeta z^*\right)\exp\left(-zz^*+z^*x+zy\right)\\
&=\mathrm{i}^{-n}H_n\left[\mathrm{i}(\zeta^*x+\zeta y)\right]\mathrm{e}^{xy}. \qquad (31)
\end{aligned}$$

As another application, we now derive the relation between the Hermite polynomial and the Laguerre polynomial. By virtue of Eq. (8), we know

$$L_n(Q)=:\mathrm{e}^{\frac{1}{4}\frac{\partial^2}{\partial Q^2}}L_n(Q):. \qquad (32)$$

Using the definition of the Laguerre polynomial

$$L_n(x)=\sum_{l=0}^{n}(-1)^lC_n^l\frac{n!}{l!}x^l \qquad (33)$$

in the right side of Eq. (32) within the symbol : :, we can rewrite Eq. (32) as

$$L_n(Q) = \sum_{l=0}^{n}(-1)^l C_n^l \frac{n!}{l!} : \mathrm{e}^{\frac{1}{4}\frac{\partial^2}{\partial Q^2}} Q^l : = \sum_{l=0}^{n} \mathrm{i}^l C_n^l \frac{n!}{2^l l!} : H_l(\mathrm{i}Q) : . \tag{34}$$

Comparing Eq. (32) with Eq. (34), we have

$$: \mathrm{e}^{\frac{1}{4}\frac{\partial^2}{\partial Q^2}} L_n(Q) := \sum_{l=0}^{n} \mathrm{i}^l C_n^l \frac{n!}{2^l l!} : H_l(\mathrm{i}Q) : . \tag{35}$$

Noticing that both sides are in normal ordering, so replacing $Q \to x$ in Eq. (35), we find the differential relation between the Hermite polynomial and the Laguerre polynomial, that is,

$$\mathrm{e}^{\frac{1}{4}\frac{\partial^2}{\partial x^2}} L_n(x) = \sum_{l=0}^{n} \mathrm{i}^l C_n^l \frac{n!}{2^l l!} H_l(\mathrm{i}x), \tag{36}$$

which is actually related to the exponential differential operator $\mathrm{e}^{\frac{1}{4}\frac{\partial^2}{\partial x^2}}$.

5. Conclusion

In summary, we have introduced a differential method for calculating the operator ordering and used it to present the normal and anti-normal ordering products of combination of coordinate and momentum operators. Compared to the existed ways, our method is neater and simpler in deriving the above operator ordering. As their important byproducts, the mutual transformation formulas between the normal ordering and anti-normal ordering are also obtained, which have good universality. Furthermore, based on these operator identities, some useful mathematical integral formulas and some new differential relations are given without really performing these integrations. These new formulas are useful in operator-ordering theory and special function theory.

Appendix A: Derivation of Eq. (2)

Proof Recall that the generating function of single-variable Hermite polynomial $H_n(q)$[38,39]

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(q) = \mathrm{e}^{-t^2+2qt}, \quad H_n(q) = \frac{\partial^n}{\partial t^n} \mathrm{e}^{-t^2+2qt}\bigg|_{t=0}. \tag{A1}$$

We now rewrite $H_n(q)$ as

$$\begin{aligned} H_n(q) &= \frac{\partial^n}{\partial t^n} \mathrm{e}^{-t^2} \mathrm{e}^{2qt}\bigg|_{t=0} = \frac{\partial^n}{\partial t^n} \mathrm{e}^{-\frac{1}{4}\frac{\partial^2}{\partial q^2}} \mathrm{e}^{2qt}\bigg|_{t=0} \\ &= \mathrm{e}^{-\frac{1}{4}\frac{\partial^2}{\partial q^2}} \frac{\partial^n}{\partial t^n} \mathrm{e}^{2qt}\bigg|_{t=0} = \mathrm{e}^{-\frac{1}{4}\frac{\partial^2}{\partial q^2}} (2q)^n \mathrm{e}^{2qt}\bigg|_{t=0} \\ &= \mathrm{e}^{-\frac{1}{4}\frac{\partial^2}{\partial q^2}} (2q)^n, \end{aligned} \tag{A2}$$

which is just the differential expression of the single-variable Hermite polynomial in Eq. (2).

Appendix B: Proof of Eq. (10)

Proof Using the property of the δ function, we have

$$\begin{aligned} : F\left(a, a^\dagger\right) : &=: \int \mathrm{d}^2 z F(z, z^*)\, \delta\left(a^\dagger - z^*\right) \delta(a - z) : \\ &=: \int \mathrm{d}^2 z F(z, z^*) \int \frac{\mathrm{d}^2\xi}{\pi^2} \mathrm{e}^{-\left(a^\dagger - z^*\right)\xi + (a-z)\xi^*} : \\ &= \int \mathrm{d}^2 z F(z, z^*) \int \frac{\mathrm{d}^2\xi}{\pi^2} \mathrm{e}^{-\left(a^\dagger - z^*\right)\xi} \mathrm{e}^{(a-z)\xi^*}. \end{aligned} \tag{B1}$$

According to the Baker–Housdorff formula, equation (B1) can be rewritten as

$$\begin{aligned} : F\left(a, a^\dagger\right) : &= \int \mathrm{d}^2 z F(z, z^*) \int \frac{\mathrm{d}^2\xi}{\pi^2} \mathrm{e}^{(a-z)\xi^*} \mathrm{e}^{-\left(a^\dagger - z^*\right)\xi} \mathrm{e}^{\xi\xi^*} \\ &= \vdots \int \mathrm{d}^2 z F(z, z^*) \int \frac{\mathrm{d}^2\xi}{\pi^2} \mathrm{e}^{\xi\xi^*} \mathrm{e}^{(a-z)\xi^* - \left(a^\dagger - z^*\right)\xi} \vdots \\ &= \vdots \int \mathrm{d}^2 z F(z, z^*) \int \frac{\mathrm{d}^2\xi}{\pi^2} \mathrm{e}^{-\frac{\partial^2}{\partial a \partial a^\dagger}} \mathrm{e}^{(a-z)\xi^* - \left(a^\dagger - z^*\right)\xi} \vdots \\ &= \vdots \mathrm{e}^{-\frac{\partial^2}{\partial a \partial a^\dagger}} \int \mathrm{d}^2 z F(z, z^*) \int \frac{\mathrm{d}^2\xi}{\pi^2} \mathrm{e}^{(a-z)\xi^* - \left(a^\dagger - z^*\right)\xi} \vdots \\ &= \vdots \mathrm{e}^{-\frac{\partial^2}{\partial a \partial a^\dagger}} \int \mathrm{d}^2 z F(z, z^*)\, \delta\left(a^\dagger - z^*\right) \delta(a - z) \vdots \\ &= \vdots \mathrm{e}^{-\frac{\partial^2}{\partial a \partial a^\dagger}} F(a a^\dagger) \vdots . \end{aligned} \tag{B2}$$

Further, using the identities $\frac{\partial}{\partial a} = \frac{1}{\sqrt{2}}\frac{\partial}{\partial Q} + \frac{1}{\mathrm{i}\sqrt{2}}\frac{\partial}{\partial P}$ and $\frac{\partial}{\partial a^\dagger} = \frac{1}{\sqrt{2}}\frac{\partial}{\partial Q} - \frac{1}{\mathrm{i}\sqrt{2}}\frac{\partial}{\partial P}$, equation (B2) can be rewritten as Eq. (10). This completes the proof.

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