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Thermodynamics and weak cosmic censorship conjecture of charged AdS black hole in the Rastall gravity with pressure*

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Treating the cosmological constant as a dynamical variable, we investigate the thermodynamics and weak cosmic censorship conjecture (WCCC) of a charged AdS black hole (BH) in the Rastall gravity. We determine the energy momentum relation of charged fermion at the horizon of the BH using the Dirac equation. Based on this relation, it is shown that the first law of thermodynamics still holds as a fermion is absorbed by the BH. However, the entropy of both the extremal and near-extremal BH decreases in the irreversible process, which means that the second law of thermodynamics is violated. Furthermore, we verify the validity of the WCCC by the minimum values of the metric function $h(r)$ at its final state. For the extremal charged AdS BH in the Rastall gravity, we find that the WCCC is always valid since the BH is extreme. While for the case of near-extremal BH, we find that the WCCC could be violable in the extended phase space (EPS), depending on the value of the parameters of the BH and their variations.

Keywords: Rastall gravity, extended phase space, black hole thermodynamics, weak cosmic censorship conjecture

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1. Introduction

An important feature for a black hole (BH) is that it owns an event horizon, which is a boundary where any particle, even light, cannot escape from it. The event horizon also plays a critical role in hiding the inner structure of BHs, including the curvature singularity. Near the singularity, the curvature of spacetime tends to diverge and the laws of physics are broken. To avoid these phenomena, Penrose proposed the weak cosmic censorship conjecture (WCCC), according to which the singularities resulting from gravitational collapse are hidden by the BH event horizon for the observers located at infinity.[1,2] However, there has been no universal proof of this conjecture up to date, so we should check it one by one in each model of gravity.

To test this conjecture, Wald devised an ideal experiment, where the charged spinning test particles are dropped into the extremal Kerr–Newman BH.[3] It was shown that the test particle could not be captured by the extremal Kerr–Newman BH, implying that the singularity can be hidden, which is consistent with the WCCC. Late on, the relevant research was extended to the scalar and electromagnetic test fields.[4–7] Recently, the WCCC was actively studied in many BHs via various methods.[8–33] Especially, Gwak investigated the thermodynamics and WCCC of a charged AdS Reissner–Nordström BH with pressure and volume.[34] He found that the second law of thermodynamics (SLT) would be violated as the pressure was considered although the first law and the WCCC still hold, which is different from the normal phase space without the pressure and volume contributions. Thereafter, using the test particle model of Gwak, thermodynamics and WCCC of a series BHs have been investigated in the extended phase space (EPS).[35–40] Importantly, their results showed that the WCCC is valid for the extremal BHs in the EPS since the configurations of the BHs are not changed as the particle is adsorbed. However, the WCCC of the near-extremal BHs is violable in the EPS.[41–44]

In this paper, we investigate the thermodynamics and WCCC in generalized Rastall theories of gravity by making use of the test fermion model in the EPS. We want to explore whether the thermodynamics and WCCC will be valid in Rastall theories of gravity. The Rastall theory contains richer physics in comparison with the Einstein theory. In general relativity, the geometry and matter fields are coupled minimally which results in the covariant conservation relation of the energy-momentum tensor. However, this relationship is only verified in the Minkowaski flat or weak field regime of gravity. Based on this limitation, the revised theory of Einstein's gravity theory was proposed by Rastall, in which the covariant derivative of the energy momentum tensor is no longer zero in curved spacetime. The thermodynamic behavior of BHs in the Rastall gravity is more general, and can better reflect the nature of gravity as a thermodynamic system. Therefore, we want to explore the thermodynamic laws and the WCCC

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in this context. As a result, the first law of thermodynamics (FLT) is recovered by the absorptions of the fermion. However, the entropy of BH decrease when a fermion drops into the extremal and near-extremal BH, which violates the SLT. Fortunately, the result shows that the event horizon still exists when a fermion is swallowed by the extremal BH. That is, the extremal BH in generalized Rastall theories of gravity cannot be overcharged, and the WCCC is valid. Different from the case of extremal, the WCCC for the near-extremal BH could be invalid, depending on the model parameters of the Rastall theories of gravity.

The outline of the paper is organized as follows. In Section 2, we review the BH solution and the thermodynamic quantities in the generalized Rastall theories of gravity. In Section 3, we achieve the energy-momentum relation of a charged fermion as it is swallowed by the BH. In Section 4, the first and second laws of thermodynamics are checked in the EPS. In Section 5, we concentrate on the WCCC under the fermion absorption with the contribution of the pressure. In Section 6, we present our conclusions.

2. A brief review about charged AdS black hole in the Rastall gravity

The covariant conservation of energy-momentum tensor is the backbone of the general relativity theory. Indeed, the limitation of such a theory is that conservation of energy-momentum tensor has been probed only in the flat or weak field cases of spacetime. Theoretically, in the strong domain of gravity, the actual nature of the spacetime geometry and the covariant conservation relation should be debated. In light of this fact, by relating $T^{\mu\nu}_{\ ;\mu}$ to the derivative of Ricci scalar, Rastall proposed a new formulation for gravity by adding some new terms to the Einstein equation.[45] Therefore, on the basis of Rastall theory of gravity, the ordinary energy-momentum conservation law is not always available in the curved spacetime and we should have[46]

$$T^{\mu\nu}_{\ ;\mu} = \lambda R^{,\nu}, \tag{1}$$

where R is the Ricci scalar of the spacetime, λ is the Rastall parameter. It leads to the modification of the Einstein field equations, which can be written as

$$G_{\mu\nu} + \kappa\lambda g_{\mu\nu} R = \kappa T_{\mu\nu}. \tag{2}$$

Here $G_{\mu\nu}$, $T_{\mu\nu}$, and κ are the Einstein tensor, energy-momentum tensor and coupling constant, respectively. For the case of $\lambda = 0$, the Einstein field equations can be recovered. In recent years, the Rastall theory has attracted a great deal of attention, and many works on various BH solutions and related thermodynamics have been investigated in the framework of Rastall theory.[46–55] For this theory, let us consider the field of matter consists only of electromagnetic field in the cosmological constant background, and the static spherically symmetric metric would be given by[56]

$$\mathrm{d}s^2 = -h(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{h(r)} + r^2\mathrm{d}\Omega^2_{d-2}, \tag{3}$$

$$h(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3-12\lambda}r^2, \tag{4}$$

where $h(r)$ is the metric function determined in terms of mass M and charge Q, and $\mathrm{d}\Omega^2_{d-2} = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$ is the volume of the unit $(d-2)$-sphere. The horizon radius is defined by $h(r) = 0$, and the thermal properties are defined on the horizon of the BH. The Hawking temperature T_{h}, entropy S_{h}, and electric potential Φ_{h} are obtained as follows:

$$T_{\mathrm{h}} = \frac{(1-4\lambda)Q^2 + (4\lambda-1)r_{\mathrm{h}}^2 + \Lambda r_{\mathrm{h}}^4}{4(4\lambda-1)\pi r_{\mathrm{h}}^3}, \tag{5}$$

$$S_{\mathrm{h}} = \pi {r_{\mathrm{h}}}^2, \tag{6}$$

$$\Phi_{\mathrm{h}} = -A_t(r_{\mathrm{h}}) = \frac{Q}{r_{\mathrm{h}}}. \tag{7}$$

The cosmological constant Λ is related to the AdS radius l, and both of them are related with each other by the pressure $P = -\frac{\Lambda}{8\pi} = \frac{(d-1)(d-2)}{16\pi l^2}$. In the normal phase space, the cosmological constant is fixed. However, many researchers have shown that the cosmological constant and its conjugate quantity can be identified as the thermodynamic pressure and volume.[57–60] Therefore, the thermodynamic volume can be obtained by

$$V_{\mathrm{h}} = \left(\frac{\partial M}{\partial P}\right)_{S,Q} = \frac{1}{1-4\lambda}\left(\frac{4\pi {r_{\mathrm{h}}}^3}{3}\right). \tag{8}$$

The first law of the BH thermodynamics is now written as

$$\mathrm{d}M = T_h\mathrm{d}S_h + \Phi_h\mathrm{d}Q + V_h\mathrm{d}P, \tag{9}$$

and the corresponding Smarr relation is also satisfied,

$$M = 2(T_{\mathrm{h}}S_{\mathrm{h}} - PV_{\mathrm{h}}) + \Phi_{\mathrm{h}}Q. \tag{10}$$

When the cosmological constant is treated as a constant, Eq. (9) is reduced to $\mathrm{d}M = T_{\mathrm{h}}\mathrm{d}S_{\mathrm{h}} + \Phi_h\mathrm{d}Q$. In this paper, we are interested in the EPS, that is, Eq. (9) is our concern. In this case, the mass M is not the internal energy but enthalpy, which is related to the internal energy and pressure,

$$M = U_{\mathrm{h}} + PV_{\mathrm{h}}, \tag{11}$$

where U_{h} is the internal energy. Next, we investigate whether the first law can be recovered under the absorption of fermion.

3. Charged fermion absorption

When the BH absorbs charged fermions, the conserved quantities of the BH will change. In this process, the motion of a charged fermion follows the Dirac equation[27]

$$\mathrm{i}\gamma^\mu\left(\partial_\mu + \Omega_\mu - \frac{\mathrm{i}}{\hbar}eA_\mu\right)\Psi + \frac{\mu_m}{\hbar}\Psi = 0, \tag{12}$$

$$\Omega_\mu \equiv \frac{\mathrm{i}}{2}\omega_\mu{}^{\alpha\beta}\Sigma_{\alpha\beta}, \tag{13}$$

$$\Sigma_{\alpha\beta} = \frac{\mathrm{i}}{4}\left[\gamma^\alpha, \gamma^\beta\right], \quad \left\{\gamma^\alpha, \gamma^\beta\right\} = 2\eta^{\alpha\beta}, \tag{14}$$

where μ_m and e are the mass and charge of the fermion, respectively. For the matrices γ_μ, there are many different choices and we can set

$$\gamma^t = \frac{1}{\sqrt{h(r)}}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \quad \gamma^\theta = r\begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix},$$
$$\gamma^r = \frac{1}{\sqrt{h(r)}}\begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}, \quad \gamma^\varphi = r\sin\theta\begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}. \tag{15}$$

In the above equation, we can employ the Pauli sigma matrices as follows:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{16}$$

For the fermion with spin 1/2, there are two different states: spin up ($\uparrow$) and spin down ($\downarrow$). Due to the fact that the spin down case is similar to the spin up case,[61–64] we choose the wave functions with spin up ($\uparrow$) thereafter. This leads to

$$\Psi_{(\uparrow)} = \begin{pmatrix} A \\ 0 \\ B \\ 0 \end{pmatrix} \mathrm{e}^{\left(\frac{\mathrm{i}}{\hbar}I_{(\uparrow)}(t,r,\theta,\phi)\right)}, \tag{17}$$

where A, B, I are functions of t, r, θ, ϕ. By combining Eqs. (29) and (12), we can obtain the motion equations to leading order of $\hbar$,

$$\begin{aligned}
&-\mathrm{i}\frac{A}{\sqrt{h(r)}}\left(\partial_t I_{(\uparrow)} - \mathrm{e}A_t\right) - B\sqrt{h(r)}\partial_r I_{(\uparrow)} + A\mu_m = 0,\\
&-\mathrm{i}\frac{B}{\sqrt{h(r)}}\left(\partial_t I_{(\uparrow)} - \mathrm{e}A_t\right) - A\sqrt{h(r)}\partial_r I_{(\uparrow)} + B\mu_m = 0,\\
&A\left(r\partial_\theta I_{(\uparrow)} + \mathrm{i}r\sin\theta\partial_\phi I_{(\uparrow)}\right) = 0,\\
&B\left(r\partial_\theta I_{(\uparrow)} + \mathrm{i}r\sin\theta\partial_\phi I_{(\uparrow)}\right) = 0.
\end{aligned} \tag{18}$$

We will pay more attention to the first and second expressions in Eq. (18), since the radial action is determined by them. To solve these equations, we should separate the action $I_{(\uparrow)}$, it can be expressed as

$$I_{(\uparrow)} = -\omega t + R(r) + \Theta(\theta,\phi), \tag{19}$$

where ω is the energy of fermion which is near the horizon. Putting Eq. (19) into Eq. (18) and canceling A and B, we can obtain the radial momentum we are most interested in, that is,

$$p^r \equiv g^{rr}p_r = g^{rr}\partial_r I_{(\uparrow)} = \sqrt{(\omega + eA_t)^2 + \mu_m{}^2 h(r)}. \tag{20}$$

The charged fermion will be absorbed by the BH when it drops into the BH horizon. In addition, near the event horizon, we have $h(r_\mathrm{h}) \to 0$, which means

$$\omega = \Phi_\mathrm{h} e + p^r_\mathrm{h}, \tag{21}$$

in which p^r_h is the radial momentum of the ingoing fermion near the event horizon of BH. Equation (21) is a relation among the momentum, energy and charge of the ingoing fermion. When $\omega < \Phi_\mathrm{h} e$, the energy of the BH flows out the horizon, which leads to the supperradiation occurring.[27] However, when the charged fermion comes into the BH in the positive flow of time, the energy of the charged fermion should be defined as a positive value. Therefore, we will choose the positive sign in front of p^r_h thereafter as carried out in Ref. [65] in order to ensure a positive time direction.

4. Thermodynamics with pressure and volume of the BH under charged fermion absorption

In this section, we use the relation (21) to study the thermodynamics of the BH. Upon a charged fermion is absorbed by the BH, the change of BH is infinitesimally due to energy and momenta of the fermion entering into the event horizon of the BH, so that the final state of the BH is represented by $(M + \mathrm{d}M, Q + \mathrm{d}Q, r_\mathrm{h} + \mathrm{d}r_\mathrm{h}, l + \mathrm{d}l)$, and $\mathrm{d}M$, $\mathrm{d}Q$, $\mathrm{d}r_\mathrm{h}$, $\mathrm{d}l$ denote the increases of the mass, charge and radius, respectively. In the process of absorption, the corresponding BH mass and electric charge infinitesimally change by the ingoing fermion, which is

$$\omega = \mathrm{d}U_\mathrm{h} = \mathrm{d}(M - PV_\mathrm{h}), \quad e = \mathrm{d}Q. \tag{22}$$

The relation (22) is different from the form in the normal phase space where the internal energy is related to the mass of the BH. Then, the energy momentum relation in Eq. (21) becomes

$$\mathrm{d}(M - PV_\mathrm{h}) = \Phi_\mathrm{h}\mathrm{d}Q + p^r_\mathrm{h}. \tag{23}$$

To rewrite Eq. (23) to the FLT, we first use Eq. (6), yielding the change of entropy as follows:

$$\mathrm{d}S_\mathrm{h} = 2\pi r_\mathrm{h}\mathrm{d}r_\mathrm{h}. \tag{24}$$

The variation of the event horizon $\mathrm{d}r_\mathrm{h}$ determined by the charge, energy, and radial momentum of the absorbed fermion will directly contribute to the changes of $h(r + r_\mathrm{h})$. That is, the change of $h(r)$ could lead to the shift of event horizon $(r_\mathrm{h} + \mathrm{d}r_\mathrm{h})$. Even so, the change of the horizon should satisfy

$$\begin{aligned}
&\mathrm{d}h_r = \frac{\partial h_r}{\partial M}\mathrm{d}M + \frac{\partial h_r}{\partial Q}\mathrm{d}Q + \frac{\partial h_r}{\partial r_\mathrm{h}}\mathrm{d}r_\mathrm{h} = 0,\\
&h_r = h(M, Q, l, r_\mathrm{h}),
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
&\frac{\partial h_r}{\partial M}\Big|_{r=r_\mathrm{h}} = -\frac{2}{r_\mathrm{h}},\\
&\frac{\partial h_r}{\partial Q}\Big|_{r=r_\mathrm{h}} = \frac{2Q}{{r_\mathrm{h}}^2},\\
&\frac{\partial h_r}{\partial l}\Big|_{r=r_\mathrm{h}} = -\frac{2{r_\mathrm{h}}^2}{l^3(1-4\lambda)},
\end{aligned}$$

$$\frac{\partial h_r}{\partial r}|_{r=r_{\rm h}} = -\frac{2Q^2}{{r_{\rm h}}^3} + \frac{2M}{{r_{\rm h}}^2} + \frac{2r_{\rm h}}{l^2(1-4\lambda)}. \tag{26}$$

With the help of Eqs. (23) and (25), we can obtain

$$\mathrm{d}r_{\rm h} = \frac{2l^2{r_{\rm h}}^2(-1+4\lambda)p_{\rm h}^r}{{r_{\rm h}}^4 - 2l^2(Q^2 - Mr_{\rm h})(-1+4\lambda)}. \tag{27}$$

Subsequently, with the energy relation, the variations of entropy, and thermodynamic volume of the BH can be written as

$$\mathrm{d}S_{\rm h} = \frac{4l^2\pi{r_{\rm h}}^3(-1+4\lambda)p_{\rm h}^r}{{r_{\rm h}}^4 - 2l^2(Q^2 - Mr)(-1+4\lambda)}, \tag{28}$$

$$\mathrm{d}V_{\rm h} = -\frac{8l^2\pi\,{r_{\rm h}}^4 p_{\rm h}^r}{{r_{\rm h}}^4 - 2l^2(Q^2 - Mr_{\rm h})(-1+4\lambda)}. \tag{29}$$

Making use of Eqs. (5), (28) and (29), we can find a relation

$$T_{\rm h}\mathrm{d}S_{\rm h} - P\mathrm{d}V_{\rm h} = p_{\rm h}^r. \tag{30}$$

Thus, the expression of the internal energy in Eq. (23) becomes

$$\mathrm{d}M = \Phi_{\rm h}\mathrm{d}Q + T_{\rm h}\mathrm{d}S_{\rm h} + V_{\rm h}\mathrm{d}P. \tag{31}$$

Therefore, it is worth noting that the coincidence between the variation of the charged BH in the Rastall gravity and the FLT under the fermion absorption, where the cosmological constant is treated as a dynamical variable.

With Eq. (28), we can investigate the SLT in the generalized Rastall theories. For the extremal BHs where $T_{\rm h} = 0$, we can get

$$\mathrm{d}S_{\rm extremal} = \frac{4l^2\pi(-1+4\lambda)p_{\rm h}^r}{3r_{\rm h}}. \tag{32}$$

In Ref. [56], for the case of $\Lambda < 0$, $\lambda < 1/4$ corresponds to the Anti-de Sitter spacetime, so we only talk about $\lambda < 1/4$ in the following discussion. Hence, there is $-1+4\lambda < 0$ in Eq. (32), which means that we can reach $\mathrm{d}S_{\rm extremal} < 0$. That is to say, the SLT can be violated at least for the extremal case under the charged fermion absorption.

Next, we turn to the non-extremal BH. To gain an intuitive understanding, we plot $\mathrm{d}S_{\rm h}$ for different λ and M in Fig. 1. From this figure, we find that there is always a divergent point which divides $\mathrm{d}S_{\rm h}$ into positive and negative regions when we choose different values of M and λ. Obviously, the negative region means that the change of entropy is less than zero ($\mathrm{d}S_{\rm h} < 0$), and the decrease of the entropy appears in ranges close to the extremal BH. Therefore, the SLT is invalid under charged fermion absorption, and the range of the violation appears under the extremal and near-extremal conditions. In addition, we also find that the violations of the second law depend on the model parameters M, l, Q, λ, that is, the magnitudes of the violations are related to these parameters.

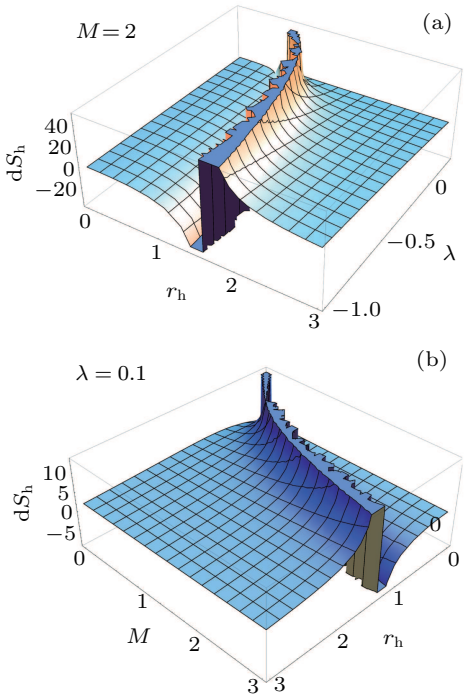


Fig. 1. The figures of critical $\mathrm{d}S_{\rm h}$ for different λ (a) and M (b) when $p_{\rm h}^r = 1$.

5. Weak cosmic censorship conjecture with pressure and volume

When the absorbed charge is enough more, the BH is overcharged and the WCCC is violated. In this section, we study the validity of the WCCC in the process that the fermions are swallowed by BHs. Therefore, we need check whether the event horizon will be destroyed at the final state. If the event horizon exists, the metric function $h(r)$ of the BH will have $h(r_{\rm min}) \leq 0$, where $r_{\rm min}$ is a minimum point of the function. At $r_{\rm min}$, the following relations are satisfied:

$$h(r)|_{r=r_{\rm min}} \equiv h_{\rm min} = \delta \leq 0, \quad \partial_r h(r)|_{r=r_{\rm min}} \equiv h'_{\rm min} = 0,$$
$$(\partial_r)^2 h(r)|_{r=r_{\rm min}} > 0. \tag{33}$$

When the minimum value satisfies the condition $\delta = 0$, the BH is an extremal BH. Here $h(r)$ is a function of (M, Q, l), and it becomes $h(M+\mathrm{d}M, Q+\mathrm{d}Q, l+\mathrm{d}l)$ during the process that a fermion drops into the event horizon. Correspondingly, the position of the minimum point and event horizon should change into $r_{\rm min} \to r_{\rm min} + \mathrm{d}r_{\rm min}, r_{\rm h} \to r_{\rm h} + \mathrm{d}r_{\rm h}$. Then, $h(r)$ also has a small shift which is denoted by $\mathrm{d}h_{\rm min}$. At the new lowest point, we have

$$\partial_r h|_{r=r_{\rm min}+\mathrm{d}r_{\rm min}} = h'_{\rm min} + \mathrm{d}h'_{\rm min} = 0. \tag{34}$$

$$\mathrm{d}h'_{\rm min} = \frac{\partial h'_{\rm min}}{\partial M}\mathrm{d}M + \frac{\partial h'_{\rm min}}{\partial Q}\mathrm{d}Q + \frac{\partial h'_{\rm min}}{\partial r_{\rm min}}\mathrm{d}r_{\rm min} + \frac{\partial h'_{\rm min}}{\partial l}\mathrm{d}l = 0. \tag{35}$$

Then, at the new minimum point, we can obtain

$$h(r_{\rm min} + \mathrm{d}r_{\rm min}) = h_{\rm min} + \mathrm{d}h_{\rm min}, \tag{36}$$

where

$$\mathrm{d}h_{\min} = \frac{\partial h_{\min}}{\partial M}\mathrm{d}M + \frac{\partial h_{\min}}{\partial Q}\mathrm{d}Q + \frac{\partial h_{\min}}{\partial l}\mathrm{d}l. \tag{37}$$

Inserting the condition (33) into Eq. (37), and combining with Eq. (23) we obtain

$$\mathrm{d}h_{\min} = 0. \tag{38}$$

For the extremal BH where we have $h_{\min} = \delta = 0$, the transformation of $h(r_{\min} + \mathrm{d}r_{\min})$ is expressed as

$$h_{\min} + \mathrm{d}h_{\min} = 0. \tag{39}$$

This means that the event horizon always exists, which depicts that the WCCC is valid in the EPS. It is interesting to note that the extremal BH keeps its configuration after the absorption. In other words, the extremal BH is still extremal, that is, the fermion with sufficient momentum and charge would not overcharge extremal BH in the Rastall gravity.

For the near-extremal BH, the energy of the fermion in Eq. (23) cannot be used because it is just applicable at the event horizon. Hence, we can expand Eq. (23) at $r_{\min} + \varepsilon$, which is

$$\begin{aligned}\mathrm{d}M = {} & \frac{2r_{\min}\left(r_{\min}{}^4\mathrm{d}l + l^3Q(4\lambda-1)\mathrm{d}Q\right) - \left(3lr_{\min}{}^4 + l^3\left(Q^2 - r_{\min}{}^2\right)(4\lambda-1)\right)\mathrm{d}r_{\min}}{2l^3r_{\min}{}^2(4\lambda-1)} \\ & + \frac{\left(3r_{\min}{}^5\mathrm{d}l + l^3Qr_{\min}(1-4\lambda)\mathrm{d}Q + \left(l^3Q^2(4\lambda-1) - 3lr_{\min}{}^4\right)\mathrm{d}r_{\min}\right)\varepsilon}{l^3r_{\min}{}^3(4\lambda-1)} + O(\varepsilon)^2.\end{aligned} \tag{40}$$

By combining Eqs. (40) and (37) we have

$$\begin{aligned}\mathrm{d}h_{\min} = {} & \frac{\left(3r_{\min}{}^4 + l^2\left(Q^2 - r_{\min}{}^2\right)(4\lambda-1)\right)\mathrm{d}r_{\min}}{l^2r_{\min}{}^3(4\lambda-1)} \\ & - \frac{2\left(3r_{\min}{}^5\mathrm{d}l + l^3Qr_{\min}(1-4\lambda)\mathrm{d}Q + \left(-3lr_{\min}^4 + l^3Q^2(-1+4\lambda)\right)\mathrm{d}r_{\min}\right)\varepsilon}{l^3r_{\min}{}^4(4\lambda-1)} + O(\varepsilon)^2.\end{aligned} \tag{41}$$

Meanwhile, we can reach the expression of l with the help of $h'(r_{\mathrm{h}}) = 0$, which is

$$l = \frac{\sqrt{3}r_{\min}{}^2}{\sqrt{\left(-Q^2 + r_{\min}{}^2\right)(4\lambda-1)}}, \tag{42}$$

and

$$\mathrm{d}l = \frac{\sqrt{3}r_{\min}\left(Qr_{\min}\mathrm{d}Q + \mathrm{d}r_{\min}\left(-2Q^2 + r_{\min}{}^2\right)\right)(4\lambda-1)}{\left(\left(-Q^2 + r_{\min}{}^2\right)(4\lambda-1)\right)^{3/2}}. \tag{43}$$

Using Eqs. (41)–(43), we can obtain

$$\mathrm{d}h_{\min} = O(\varepsilon)^2. \tag{44}$$

In the EPS, the minimum value of the near-extremal BH is

$$h_{\min} + \mathrm{d}h_{\min} = \delta_\varepsilon + O(\varepsilon)^2. \tag{45}$$

Naturally, once again we can obtain $h_{\min} + \mathrm{d}h_{\min} = 0$ when $\delta_\varepsilon \to 0$, $\varepsilon \to 0$, such that the accuracy of Eq. (39) is verified. However, both δ_ε and $O(\varepsilon)^2$ are small quantities about ε, and we cannot directly judge the size of these small quantities. Therefore, we want to find the specific expression of $O(\varepsilon)^2$ and δ_ε to discuss which is smaller. For the near-extremal BH, we should perform higher-order expansion, namely,

$$\mathrm{d}h_M = \frac{\left(6Q^2\mathrm{d}r_{\min} - 4Qr_{\min}\mathrm{d}Q - r_{\min}{}^2\mathrm{d}r_{\min}\right)\varepsilon^2}{r_{\min}{}^5} + O(\varepsilon)^3. \tag{46}$$

For δ_ε, we also have

$$\delta_\varepsilon = \frac{\left(-3r_{\min}{}^4 + l^2Q^2(4\lambda-1)\right)\varepsilon^2}{l^2r_{\min}{}^4(4\lambda-1)} + O(\varepsilon)^3. \tag{47}$$

Thus, we can combine Eqs. (46) and (47) and define

$$\mathcal{H}_{\mathrm{E}} = \frac{\delta_\varepsilon + \mathrm{d}h_M}{\varepsilon^2}. \tag{48}$$

The next step is to analyze the sign of Eq. (48). If the WCCC of the near-extermal BH is valid in the EPS, the value of Eq. (48) should be negative ($\mathcal{H}_{\mathrm{E}} < 0$). Apparently, the value of $\mathcal{H}_{\mathrm{E}}$ depends on the model parameters $(r_{\min}, Q, l, \lambda, \mathrm{d}r_{\min})$. As an attempt, we set $Q = 2$, $l = 1$, $\mathrm{d}Q = 0.5$, $p_{\mathrm{h}}^r = 1$, and plot the figure of Eq. (48), which makes the result more intuitive.

In Fig. 2, we take $\lambda = 0.1$ (left) and $\mathrm{d}r_{\min} = 0.5$ (right) respectively, it is worthwhile to point that there is a region where $\mathcal{H}_{\mathrm{E}} > 0$ when we choose different values of $\mathrm{d}r_{\min}$ or λ. Since the value of $\mathcal{H}_{\mathrm{E}}$ could be positive ($\mathcal{H}_{\mathrm{E}} > 0$), which states that the function $\mathcal{H}_{\mathrm{E}} = h(r_{\min} + \mathrm{d}r_{\min})$ has no real root. In this case, the WCCC could be invalid for the near-extermal BH, and the near-extremal BH could be overcharged in the process. It is also obvious from Fig. 2 that the magnitude of the violation is related to the parameters of the BH, that is, the magnitude of the violation depends on the parameters λ, $\mathrm{d}r_{\min}$, $r_{\min}$, which is quite different from the case of the extremal BH.

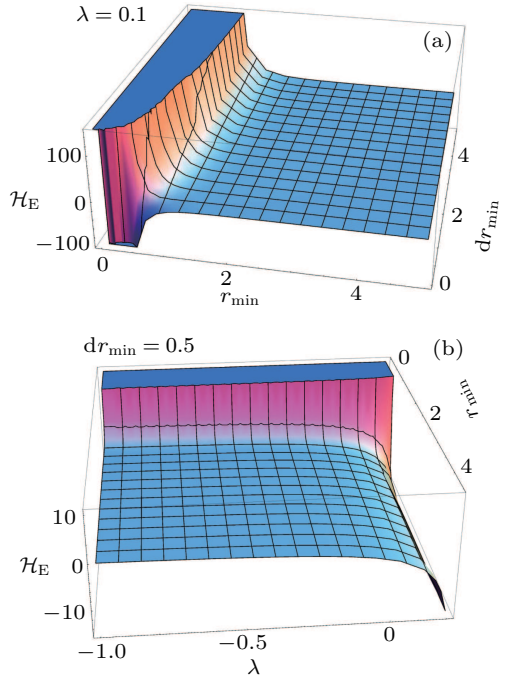


Fig. 2. The value of $\mathcal{H}_E$ for different dr_{min} (a) and λ (b) when $p_h^r = 1$.

6. Discussion and conclusion

The thermodynamic laws and the WCCC for a BH in Rastall gravity are discussed in the EPS where the cosmological constant is treated as a dynamical variable. We first obtain the energy-momentum relation of the particle which is captured by BHs in the Rastall gravity. Using this relation, we further study the thermodynamic laws and WCCC of BH. Fortunately, the FLT is found to be valid in the EPS under the fermion absorption. However, the change in the entropy of extremal BH is found to be less than zero $(dS_{extremal} < 0)$, implying that the SLT of extremal BH is invalid in the extended phase space. For the non-extremal BH, we find that the SLT may be violated for near-extremal BHs, and the magnitudes of the violation rely on the choice of the parameter model. This conclusion is consistent with the results in Einstein gravity.[34]

It is well known that the Bekenstein–Hawking entropy of the BH is related to the event horizon, that is, the thermodynamics of a BH is related to the stability of its horizon. Since there is a violation of the SLT of BH, it is necessary to judge the existence of the horizon and further to check the WCCC. To determine whether the event horizon still exists when a fermion falls in the extremal BH, the most effective method is to check the minimum value of function $h(r)$. Interestingly, we find that the minimum value of the function is not changed for the extremal BH, that is, $(dh_{min} = 0)$, so that $h_{min} + dh_{min} = 0$, implying that the extremal BH cannot be overcharged in the process due to its minimum value. The extremal BH is still extreme. This ensures the stability of the horizon under the charged particle absorption. Therefore, the conjecture shows its validity for the extremal BH in the Rastall gravity by adding a fermion. However, there is a situation where the minimum value of the function is greater than zero for the near-extremal BH $(\mathcal{H}_E > 0)$, as shown in Fig. 2. Clearly, for the case of the near-extremal BH, the WCCC could be violable. Similarly, the magnitudes of the violation depend on the values of related parameters.

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