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Growth induced buckling of morphoelastic rod in viscous medium*

Yitong Zhang(张一桐), Shuai Zhang(张帅), and Peng Wang(王鹏)†

School of Civil Engineering and Architecture, University of Jinan, Jinan 250022, *China*

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Biological growth is a common phenomenon in nature, and some organisms such as DNA molecules and bacterial filaments grow in viscous media. The growth induced instability of morphoelastic rod in a viscous medium is studied in this paper. Based on the Kirchhoff kinetic analogy method, the mechanical model for growing elastic thin rod in the viscous medium is established. A perturbation analysis is used to analyze the stability of the growing elastic rod in the viscous medium. We apply the results into planar growing ring and get its criterion of instability. Take the criterion into DNA ring to discuss the influence of viscous resistance on its instability.

Keywords: growth, morphoelastic rod, Kirchhoff rod, buckling, perturbation

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1. Introduction

Growth is a common phenomenon in natural world. Growth means changing in mass in biology. Because growth can induce stress and stress inversely can affect growth, so during process of growth except mass varying structure and material character also along with varying. More evidence indicates that mechanical factor plays an important role in pattern formation of biological body,[1] such as many structures across scales from cell to tissue like neuron to brain.[2–5] Growth induced deformation is equivalent to constraint induced deformation. For example, convolution formation during developing of brain is affected by the skeletal constraint or different growth.[6]

Elastic rod or filament plays an important role in engineering and biological field. It can model many structures like undersea cable, sperm, bacteria, fungi, hair and so on.[7–10] In recently past years, the elastic rod model has attracted a great deal of attention from researchers since elastic rod has successfully modeled the helical structure of DNA.[11] Since DNA structures can be seen as an elastic rod model, quantitative methods can be applied to analyze its helical structures and dynamic behavior related to its function.[12,13] Based on the Kirchhoff elastic rod model, Goriely and Tabor developed a new perturbation scheme that is suitable for analyzing dynamic instability of filament.[14,15] Liu and Xue[11,16] studied thin elastic rod nonlinear mechanics of the DNA model. Xue *et al.*[17,18] developed the analytical mechanic theory of elastic rods. Furrer *et al.*[19] studied the relationship between the intrinsic shape and the existence of multiple stable equilibria within the context of DNA rings based on the Kirchhoff elastic rod model. Shi *et al.*[20] derived a generalized one-dimensional time-independent nonlinear Schrö dinger equation for the stationary state configurations of supercoiled DNA. Wang *et al.* studied symmetries and conserved quantities of Kirchhoff and Cosserat elastic rods.[21–23] Since many living things such as cell, sperm, and DNA are in fluid environment, viscous action should be taken into consideration in research of their mechanical behavior. Wolgemuth *et al.*[8] studied dynamic supercoiling bifurcation of Bacillus subtilis in fluid using the filament model to explain complex supercoiled structures during cells growth. Liu and Sheng[24,25] investigated the stability of elastic rod without growth in a viscous medium. Wang *et al.*[26] used the perturbation method to study the influence of viscous action on the dynamic instability of the elastic rod. The study on instability of growing elastic rod in viscous media is an alternative way to understand how the growth and viscous resistant couple to affect structures of living things in biology.

In this paper, we adopt the mophoelastic rod model. The arrangement of this paper is as follows. In Section 2, a brief review of a new perturbation method proposed by Goriely is introduced. In Section 3, we extend the growing elastic rod model into the case of considering the viscous resistance, and derive its equilibrium equation. In Section 4, a system of first order expanding differential equations which can determine threshold of instability is derived. In Section 5, a linear analysis is applied to growing elastic ring to get its instability criterion, and mode selection as well as affections of viscous resistance to growing elastic ring beyond instability are discussed. Finally, we give conclusions.

2. A brief review of the new perturbation scheme

As discussed in Ref. [15], usual methods can result in an inconsistent perturbation scheme, so we will follow the method in Ref. [15]. The main idea of the coordinate base

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†Corresponding author. E-mail: sdpengwang@163.com

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vector perturbation scheme is to perturb the principle axis coordinate base vectors and maintain orthogonality of new base vectors in order to make arc length conservative. That is, for any order perturbation ε^n, satisfying $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$. Thus the new basis vector can be expressed in the unperturbed basis as

$$\boldsymbol{e}_i = \boldsymbol{e}_i^{(0)} + \varepsilon \sum_{j=1}^{3} A_{ij} \boldsymbol{e}_j^{(0)} + O(\varepsilon^2), \quad (i = 1,2,3), \tag{1}$$

where $\boldsymbol{A}$ is an antisymmetric matrix and it can be written as

$$\boldsymbol{A} = \begin{pmatrix} 0 & c_3 & -c_2 \\ -c_3 & 0 & c_1 \\ c_2 & -c_1 & 0 \end{pmatrix}. \tag{2}$$

Hence, the basis director takes the form

$$\boldsymbol{e}_i = \boldsymbol{e}_i^{(0)} + \varepsilon \left(\boldsymbol{c} \times \boldsymbol{e}_i^{(0)} \right) + O\left(\varepsilon^2\right), \quad (i = 1,2,3), \tag{3}$$

where $\boldsymbol{c}$ is an arbitrary zero-order vector and it can be written as $\boldsymbol{c} = c_1 \boldsymbol{e}_1^{(0)} + c_2 \boldsymbol{e}_2^{(0)} + c_3 \boldsymbol{e}_3^{(0)}$. It can be verified that Eqs. (1) and (3) are equivalent. The first-order perturbation base vectors in Eq. (3) can be expressed as

$$\begin{aligned} \boldsymbol{e}^{(1)} &= \left(\boldsymbol{e}_1^{(1)}, \boldsymbol{e}_2^{(1)}, \boldsymbol{e}_3^{(1)} \right)^{\mathrm{T}} \\ &= \left(\boldsymbol{c} \times \boldsymbol{e}_1^{(0)}, \boldsymbol{c} \times \boldsymbol{e}_2^{(0)}, \boldsymbol{c} \times \boldsymbol{e}_3^{(0)} \right)^{\mathrm{T}} \\ &= \begin{pmatrix} c_3 \boldsymbol{e}_2^{(0)} - c_2 \boldsymbol{e}_3^{(0)} \\ -c_3 \boldsymbol{e}_1^{(0)} + c_1 \boldsymbol{e}_3^{(0)} \\ c_2 \boldsymbol{e}_1^{(0)} - c_1 \boldsymbol{e}_2^{(0)} \end{pmatrix} \\ &= \begin{pmatrix} 0 & c_3 & -c_2 \\ -c_3 & 0 & c_1 \\ c_2 & -c_1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{e}_1^{(0)} \\ \boldsymbol{e}_2^{(0)} \\ \boldsymbol{e}_3^{(0)} \end{pmatrix}. \end{aligned} \tag{4}$$

Thus, the first-order perturbation base vectors can be written as

$$\boldsymbol{e}_i^{(1)} = \sum_{j=1}^{3} A_{ij} \boldsymbol{e}_j^{(0)}, \tag{5}$$

where A_{ij} is the element in matrix $\boldsymbol{A}$.

For a vector $\boldsymbol{V} = \sum_{i=1}^{3} v_i \boldsymbol{e}_i$, it can be expanded as

$$\begin{aligned} \boldsymbol{V} &= \boldsymbol{V}^{(0)} + \varepsilon \boldsymbol{V}^{(1)} + O\left(\varepsilon^2\right) \\ &= \sum_{i=1}^{3} v_i^{(0)} \boldsymbol{e}_i^{(0)} + \varepsilon \sum_{i=1}^{3} \left[v_i^{(1)} + \left(\boldsymbol{c} \times \boldsymbol{v}^{(0)} \right)_i \right] \boldsymbol{e}_i^{(0)} \\ &\quad + O\left(\varepsilon^2\right), \end{aligned} \tag{6}$$

where $v_i^{(1)}$ and $(\boldsymbol{c} \times \boldsymbol{v}^{(0)})_i$ represent the projection of vectors $\boldsymbol{v}^{(1)}$ and $\boldsymbol{c} \times \boldsymbol{v}^{(0)}$ in the $\boldsymbol{e}_i^{(0)}$ direction, respectively. The specific deductive procedure of Eq. (6) is shown in Appendix A.

3. Growing elastic rod

3.1. The three configurations

We introduce three configurations for different states of the growing elastic rod.[27] Initial configuration is the unstressed state at the initial time $t = 0$ and the superscript 0 indicates the quantities of this state. For example, the arc length of rod centerline can be written as s^0 in initial configuration. Reference configuration is the state that the rod grows without constraints, so that at any given time t, it is in an unstressed state. This state is not realized in the experiment, so it is also called the virtual configuration and the superscript g is used to indicate the quantities of this state. The arc length of rod centerline can be written as s^g in reference configuration. Current configuration is the true state of the growing rod at time t considering exterior load, body force and boundary conditions affections. The quantities in current configuration has no superscript. The arc length of rod centerline can be written as s in current configuration.

We define growth stretch as

$$\gamma = \frac{\partial s^{\mathrm{g}}}{\partial s^0} = \varepsilon^{\mathrm{g}} + 1, \tag{7}$$

where γ describes the local change of length of material point s^0 at time t, which is caused by growth. Here ε^{g} is the growth strain. Note that s^{g} is the function of t and s^0 and can be written as $s^{\mathrm{g}} = s^{\mathrm{g}}(s^0, t)$. The elastic stretch from the reference configuration to the current configuration can be expressed as

$$\alpha = \frac{\partial s}{\partial s^{\mathrm{g}}} = \varepsilon^{\mathrm{e}} + 1, \tag{8}$$

where ε^{e} is elastic strain. The total amount of stretch can be written as

$$\begin{aligned} \lambda &= \frac{\partial s}{\partial s^0} = \frac{\partial s}{\partial s^{\mathrm{g}}} \frac{\partial s^{\mathrm{g}}}{\partial s^0} = \alpha\gamma \\ &= \varepsilon^{\mathrm{e}} + \varepsilon^{\mathrm{e}} \varepsilon^{\mathrm{g}} + \varepsilon^{\mathrm{g}} + 1. \end{aligned} \tag{9}$$

3.2. Growing rod

Due to the slow-growth process, it is assumed that the growing rod in the unstressed state is always in equilibrium. In the reference configuration, the force and moment equilibrium equations of the elastic rod can be written as

$$\begin{aligned} &\frac{\partial \boldsymbol{F}}{\partial s^{\mathrm{g}}} + \boldsymbol{f}_{\mathrm{exp}} = 0, \\ &\frac{\partial \boldsymbol{M}}{\partial s^{\mathrm{g}}} + \frac{\partial \boldsymbol{r}}{\partial s^{\mathrm{g}}} \times \boldsymbol{F} + \boldsymbol{m}_{\mathrm{exp}} = 0, \end{aligned} \tag{10}$$

where $\boldsymbol{F}$ and $\boldsymbol{M}$ are the resultant vector and resultant moment of the internal force, respectively; $\boldsymbol{f}_{\mathrm{exp}}$ and $\boldsymbol{m}_{\mathrm{exp}}$ are the body force and couple per unit reference length applied on the cross section at s^{g}. The equilibrium equations of the initial configuration and the current configuration can be obtained by transforming the independent variables. According to Eqs. (7)–(9), the equilibrium equations of the current configuration can be obtained as follows:

$$\begin{aligned} &\frac{\partial \boldsymbol{F}}{\partial s} + \alpha^{-1} \boldsymbol{f}_{\mathrm{exp}} = 0, \\ &\frac{\partial \boldsymbol{M}}{\partial s} + \frac{\partial \boldsymbol{r}}{\partial s} \times \boldsymbol{F} + \alpha^{-1} \boldsymbol{m}_{\mathrm{exp}} = 0. \end{aligned} \tag{11}$$

The equilibrium equation of the initial configuration can be written as

$$\begin{aligned}&\frac{\partial \boldsymbol{F}}{\partial s^0}+\gamma \boldsymbol{f}_{\exp}=0,\\&\frac{\partial \boldsymbol{M}}{\partial s^0}+\frac{\partial \boldsymbol{r}}{\partial s^0}\times F+\gamma \boldsymbol{m}_{\exp}=0.\end{aligned}\tag{12}$$

In order to close the system of Eq. (10), the linear elastic constitutive relation is introduced,

$$\begin{aligned}\boldsymbol{M} =\ & EI_x(\omega_1-\hat{\omega}_1)\boldsymbol{e}_1+EI_y(\omega_2-\hat{\omega}_2)\boldsymbol{e}_2\\&+GJ(\omega_3-\hat{\omega}_3)\boldsymbol{e}_3,\end{aligned}\tag{13}$$

where E and G are Young's modulus and shear modulus of the rod, respectively. I_x and I_y are the second moments of area and J is polar moment of the cross section. For the case of rod with uniform circular cross section, these parameters have relations $I_x=I_y=J/2$. Here $\boldsymbol{\omega}$ is the curvature-twisting vector at time t and $\hat{\boldsymbol{\omega}}$ is the intrinsic curvature-twisting vector of the rod in the reference configuration at time t_0. The curvature-twisting vector satisfies the equation

$$\frac{\partial \boldsymbol{e}_i}{\partial s^{\mathrm{g}}}=\boldsymbol{\omega}\times\boldsymbol{e}_i,\quad i=1,2,3,\tag{14}$$

where $\boldsymbol{e}_i$ $(i=1,2,3)$ are the base vectors of the principal axis coordinate and $\boldsymbol{e}_3$ is the vector along the tangent direction of the centerline. Axial stretch and axial stress should satisfy the equation

$$F_3=(\alpha-1)EA,\tag{15}$$

where A is the area of cross section.

3.3. Growing rod in viscous medium

Consider that the position of a material point on the centerline is s_0^0 at the time $t=0$, and the position of this point becomes s_0^{g} due to the growth at given time t; $\boldsymbol{r}$ is the position vector of this point at time t. The base vector $\boldsymbol{e}_3$ satisfies $\boldsymbol{e}_3=\partial\boldsymbol{r}/\partial s$ and s denotes the position of material point in current configuration. The growth velocity of a material point can be written as[8]

$$\boldsymbol{v}=\left.\frac{\partial \boldsymbol{r}}{\partial t}\right|_{s_0^g}+\left.\frac{\partial s^{\mathrm{g}}}{\partial t}\right|_{s_0^0}\left.\frac{\partial \boldsymbol{r}}{\partial s^{\mathrm{g}}}\right|_t.\tag{16}$$

It can be seen from the above equation that different growth modes result in different growth velocities of the material point. The angular velocity $\boldsymbol{\Omega}$ of the rod cross section deduced by bending and torsion of the rod satisfies

$$\frac{\partial \boldsymbol{e}_i}{\partial t}=\boldsymbol{\Omega}\times\boldsymbol{e}_i,\quad i=1,2,3.\tag{17}$$

According to the viscous dynamics theory of elastic rods,[28,29] the viscous resistance and viscous resistance moment of the rod in the viscous medium can be expressed as

$$\boldsymbol{f} =-\varsigma_\perp(\boldsymbol{v}-\boldsymbol{e}_3\boldsymbol{e}_3\cdot\boldsymbol{v})-\varsigma_{\|}\boldsymbol{e}_3\boldsymbol{e}_3\cdot\boldsymbol{v},\tag{18}$$

$$\boldsymbol{m} =-\varsigma_{\mathrm{R}}\boldsymbol{e}_3\boldsymbol{e}_3\cdot\boldsymbol{\Omega},\tag{19}$$

where $\boldsymbol{e}_3\boldsymbol{e}_3$ is dyad; $\varsigma_\perp$ and $\varsigma_{\|}$ are transverse and longitudinal viscous drag coefficients respectively, and ς_{R} is the viscous drag coefficient of torque. For a thin rod, when the length L of the rod is much larger than the radius a of cross section, we ignore heterogeneity of the viscous medium letting $\varsigma=\varsigma_\perp=\varsigma_{\|}$, and the following approximate formulae may be applied in calculation of viscous drag coefficients:[30]

$$\varsigma\approx 4\pi\mu/\ln(L/2a),\tag{20a}$$

$$\varsigma_{\mathrm{R}}\approx 4\pi\mu a^2,\tag{20b}$$

where μ is the viscosity of the medium. Therefore we have

$$\boldsymbol{f}=-\varsigma\boldsymbol{v}\tag{21}$$

and

$$\frac{\partial \boldsymbol{f}}{\partial s^{\mathrm{g}}}=-\varsigma\frac{\partial}{\partial s^{\mathrm{g}}}\left(\left.\frac{\partial \boldsymbol{r}}{\partial t}\right|_{s_0^g}\right)-\varsigma\frac{\partial}{\partial s^{\mathrm{g}}}\left(\left.\frac{\partial s^{\mathrm{g}}}{\partial t}\right|_{s_0^0}\left.\frac{\partial \boldsymbol{r}}{\partial s^{\mathrm{g}}}\right|_t\right).\tag{22}$$

We suppose the growing rod always being in equilibrium state during stressless growing process, so $\partial\boldsymbol{r}/\partial t=0$. Equation (22) can be rewritten as

$$\frac{\partial \boldsymbol{f}}{\partial s^{\mathrm{g}}}=-\varsigma\left(\alpha\frac{\partial \boldsymbol{e}_3}{\partial t}+\boldsymbol{e}_3\frac{\partial \alpha}{\partial t}\right).\tag{23}$$

For an inextensible rod we have $\alpha=1$, and for an extensible rod $\alpha=\alpha(s^{\mathrm{g}},t)$. Without considering other external forces, in viscous fluids with zero Reynolds number, the body force and moment per unit length of the elastic growing rod are respectively in balance with viscous resistance and viscous resistance moment. In the case of inextensible rod, the growing elastic rod equations in the viscous medium can be expressed as

$$\begin{aligned}&\frac{\partial^2 \boldsymbol{F}}{\partial (s^{\mathrm{g}})^2}-\varsigma\alpha\frac{\partial \boldsymbol{e}_3}{\partial t}=0,\\&\frac{\partial \boldsymbol{M}}{\partial s^{\mathrm{g}}}+\alpha\boldsymbol{e}_3\times\boldsymbol{F}-\varsigma_{\mathrm{R}}\boldsymbol{e}_3\boldsymbol{e}_3\cdot\boldsymbol{\Omega}=0.\end{aligned}\tag{24}$$

3.4. Dimensionless form

We suppose that only the length of rod varies during growth, but maintains the cross section being constant, so we have the following dimensionless form:

$$\begin{aligned}&M\to ME\sqrt{AI},\quad F\to AEF,\\&\varsigma\to\varsigma A\sqrt{\rho AE/I},\quad \varsigma_{\mathrm{R}}\to\varsigma_{\mathrm{R}}\sqrt{\rho AEI},\\&\omega\to\omega\sqrt{A/I},\quad \Omega\to\Omega\sqrt{AE/(I\rho)},\\&s\to s\sqrt{I/A},\quad s^{\mathrm{g}}\to s^{\mathrm{g}}\sqrt{I/A},\\&t\to t\sqrt{I\rho/(AE)},\end{aligned}$$

where ρ is the mass density of the rod. Then equations (13) and (15) can be rewritten as

$$\boldsymbol{M}=(\omega_1-\hat{\omega}_1)\boldsymbol{e}_1+(\omega_2-\hat{\omega}_2)\boldsymbol{e}_2+\Gamma(\omega_3-\hat{\omega}_3)\boldsymbol{e}_3,\tag{25}$$

$$F_3 = \alpha - 1, \tag{26}$$

where $\Gamma = 2\,G/E$. Equation (24) can be rewritten as

$$\frac{\partial^2 \boldsymbol{F}}{\partial (s^{\mathrm{g}})^2} - \varsigma\alpha\frac{\partial \boldsymbol{e}_3}{\partial t} = 0, \tag{27a}$$

$$\frac{\partial \boldsymbol{M}}{\partial s^{\mathrm{g}}} + \alpha \boldsymbol{e}_3 \times \boldsymbol{F} - \varsigma_{\mathrm{R}}\boldsymbol{e}_3\boldsymbol{e}_3 \cdot \boldsymbol{\Omega} = 0. \tag{27b}$$

4. First order perturbation

Expanding Eq. (26) in the base vector perturbation scheme we can obtain the zero-order perturbation equation

$$F_3^{(0)} = \alpha^{(0)} - 1. \tag{28}$$

The first order perturbation equation is

$$\alpha^{(1)} = F_3^{(1)}. \tag{29}$$

The first order form of Eq. (25) is

$$\sum_{i=1}^{3} M_i^{(1)} e_i^{(0)} = \omega_1^{(1)} \boldsymbol{e}_1^{(0)} + \omega_2^{(1)} \boldsymbol{e}_2^{(0)} + \Gamma\omega_3^{(1)} \boldsymbol{e}_3^{(0)}. \tag{30}$$

The first order perturbation expanding form of curvature-twisting vector is

$$\boldsymbol{\omega}^{(1)} = \sum_{i=1}^{3} [\omega_i^{(1)} + (c \times \boldsymbol{\omega}^{(0)})_i] \boldsymbol{e}_i^{(0)} = \boldsymbol{\omega}_0^{(1)} + \boldsymbol{c} \times \boldsymbol{\omega}^{(0)}, \tag{31}$$

where $\boldsymbol{\omega}_0^{(1)} = \sum_{i=1}^{3} \omega_i^{(1)} \boldsymbol{e}_i^{(0)}$ represents the expression of the first-order curvature-twisting vector in unperturbed basis. The first-order curvature-twisting vector can also be written as

$$\boldsymbol{\omega}^{(1)} = \boldsymbol{c}' - (\boldsymbol{\omega}^{(0)} \times \boldsymbol{c}) = \sum_{i=1}^{3} c_i' \boldsymbol{e}_i^{(0)}. \tag{32}$$

Substituting Eq. (32) into Eq. (31) yields

$$\boldsymbol{\omega}_0^{(1)} = \sum_{i=1}^{3} c_i' \boldsymbol{e}_i^{(0)} + \boldsymbol{\omega}^{(0)} \times \boldsymbol{c}. \tag{33}$$

We have the similar expression of first order angular velocity. As the unperturbed solution is stationary, i.e., $\boldsymbol{\Omega}^{(0)} = 0$, we have

$$\boldsymbol{\Omega}_0^{(1)} = \sum_{i=1}^{3} \dot{c}_i \boldsymbol{e}_i^{(0)}. \tag{34}$$

Expanding Eq. (27) by using Eq. (6), and substituting Eq. (29) and Eqs. (32)–(34) into the result, we can obtain the first order perturbation form:

$$\begin{aligned}
0 = {} & -\varsigma\alpha^{(0)}\dot{c}_2 + (F_1^{(1)})'' - 2(F_2^{(1)})'\omega_3^{(0)} + 2(F_3^{(1)})'\omega_2^{(0)} - F_1^{(1)}[(\omega_2^{(0)})^2 + (\omega_3^{(0)})^2] + F_2^{(1)}[\omega_2^{(0)}\omega_1^{(0)} - \omega_3^{(0)\prime}] \\
& + F_1^{(0)}\left\{\omega_2^{(0)}\omega_1^{(0)}c_3 - \omega_3^{(0)}\omega_1^{(0)}c_2 - 2\omega_2^{(0)}c_2' - 2\omega_3^{(0)}c_3' - \omega_2^{(0)\prime}c_2 - \omega_3^{(0)\prime}c_3\right\} \\
& + F_2^{(0)}\left\{\omega_3^{(0)}\omega_1^{(0)}c_1 + c_3\left[\left(\omega_2^{(0)}\right)^2 + \left(\omega_3^{(0)}\right)^2\right] + c_1\omega_2^{(0)\prime} + 2c_1'\omega_2^{(0)} - c_3''\right\} \\
& + F_3^{(0)}\left\{-\omega_2^{(0)}\omega_1^{(0)}c_1 - c_2\left[\left(\omega_2^{(0)}\right)^2 + \left(\omega_3^{(0)}\right)^2\right] + c_1\omega_3^{(0)\prime} + 2c_1'\omega_3^{(0)} + c_2''\right\} \\
& - 2\left(F_1^{(0)}\right)'\left[\omega_2^{(0)}c_2 + \omega_3^{(0)}c_3\right] + 2\left(F_2^{(0)}\right)'\left[\omega_2^{(0)}c_1 - c_3'\right] \\
& + F_3^{(1)}\left[\omega_3^{(0)}\omega_1^{(0)} + \omega_2^{(0)\prime}\right] + 2\left(F_3^{(0)}\right)'\left[\omega_3^{(0)}c_1 + c_2'\right] - (F_2^{(0)})''c_3 + (F_3^{(0)})''c_2,
\end{aligned} \tag{35}$$

$$\begin{aligned}
0 = {} & \varsigma\alpha^{(0)}\dot{c}_1 + (F_2^{(1)})'' - 2(F_3^{(1)})'\omega_1^{(0)} + 2(F_1^{(1)})'\omega_3^{(0)} - F_2^{(1)}[(\omega_1^{(0)})^2 + (\omega_3^{(0)})^2] + F_3^{(1)}[\omega_3^{(0)}\omega_2^{(0)} - \omega_1^{(0)\prime}] \\
& + F_2^{(0)}\left\{\omega_3^{(0)}\omega_2^{(0)}c_1 - \omega_1^{(0)}\omega_2^{(0)}c_3 - 2\omega_3^{(0)}c_3' - 2\omega_1^{(0)}c_1' - \omega_3^{(0)\prime}c_3 - \omega_1^{(0)\prime}c_1\right\} \\
& + F_3^{(0)}\left\{\omega_1^{(0)}\omega_2^{(0)}c_2 + c_1\left[\left(\omega_1^{(0)}\right)^2 + \left(\omega_3^{(0)}\right)^2\right] + c_2\omega_3^{(0)\prime} + 2c_2'\omega_3^{(0)} - c_1''\right\} \\
& + F_1^{(0)}\left\{-\omega_3^{(0)}\omega_2^{(0)}c_2 - c_3\left[\left(\omega_1^{(0)}\right)^2 + \left(\omega_3^{(0)}\right)^2\right] + c_2\omega_1^{(0)\prime} + 2c_2'\omega_1^{(0)} + c_3''\right\} \\
& - 2\left(F_2^{(0)}\right)'\left[\omega_1^{(0)}c_1 + \omega_3^{(0)}c_3\right] + 2\left(F_3^{(0)}\right)'\left[\omega_3^{(0)}c_2 - c_1'\right] \\
& + F_1^{(1)}\left[\omega_1^{(0)}\omega_2^{(0)} + \omega_3^{(0)\prime}\right] + 2\left(F_1^{(0)}\right)'\left[\omega_1^{(0)}c_2 + c_3'\right] - (F_3^{(0)})''c_1 + (F_1^{(0)})''c_3,
\end{aligned} \tag{36}$$

$$\begin{aligned}
0 = {} & \left(F_3^{(1)}\right)'' - 2(F_1^{(1)})'\omega_2^{(0)} + 2(F_2^{(1)})'\omega_1^{(0)} - F_3^{(1)}\left[\left(\omega_1^{(0)}\right)^2 + \left(\omega_2^{(0)}\right)^2\right] + F_1^{(1)}\left[\omega_1^{(0)}\omega_3^{(0)} - \omega_2^{(0)\prime}\right] + F_2^{(1)}\left[\omega_1^{(0)\prime} + \omega_2^{(0)}\omega_3^{(0)}\right] \\
& + F_3^{(0)}\left\{\omega_1^{(0)}\omega_3^{(0)}c_2 - \omega_2^{(0)}\omega_3^{(0)}c_1 - 2\omega_1^{(0)}c_1' - 2\omega_2^{(0)}c_2' - \omega_1^{(0)\prime}c_1 - \omega_2^{(0)\prime}c_2\right\} \\
& + F_1^{(0)}\left\{\omega_2^{(0)}\omega_3^{(0)}c_3 + c_2\left[\left(\omega_1^{(0)}\right)^2 + \left(\omega_2^{(0)}\right)^2\right] + c_3\omega_1^{(0)\prime} + 2c_3'\omega_1^{(0)} - c_2''\right\} \\
& + F_2^{(0)}\left\{-\omega_1^{(0)}\omega_3^{(0)}c_3 - c_1\left[\left(\omega_1^{(0)}\right)^2 + \left(\omega_2^{(0)}\right)^2\right] + c_3\omega_2^{(0)\prime} + 2c_3'\omega_2^{(0)} + c_1''\right\} \\
& - 2\left(F_3^{(0)}\right)'\left[\omega_1^{(0)}c_1 + \omega_2^{(0)}c_2\right] + 2\left(F_1^{(0)}\right)'\left[\omega_1^{(0)}c_3 - c_2'\right] + 2\left(F_2^{(0)}\right)'\left[\omega_2^{(0)}c_3 + c_1'\right] + (F_2^{(0)})''c_1 - (F_1^{(0)})''c_2,
\end{aligned} \tag{37}$$

$$0 = c_1'' - c_1(1-\Gamma)\left(\omega_3^{(0)}\right)^2 + c_1\left[(\hat{\omega}_3)^2 - \Gamma\hat{\omega}_2\omega_2^{(0)}\right] - c_2(1-\Gamma)\left[\left(\omega_3^{(0)}\right)' + \hat{\omega}_2\omega_1^{(0)}\right] + c_3\left(\hat{\omega}_2' - \omega_2^{(0)\prime}\right) + c_2'(\Gamma-2)\omega_3^{(0)} + \Gamma c_3'\left(\omega_2^{(0)} + \Gamma\hat{\omega}_2 - \hat{\omega}_2\right) - \alpha^{(0)}F_2^{(1)} - \alpha^{(0)}c_3F_1^{(0)} - \alpha^{(1)}F_2^{(0)}, \tag{38}$$

$$0 = c_2'' + c_1(1-\Gamma)\left(\omega_3^{(0)}\right)' + c_1(\Gamma\omega_2^{(0)}\omega_1^{(0)} - \omega_1^{(0)}\hat{\omega}_2) + c_3\left(\omega_3^{(0)}\right)\left(\hat{\omega}_2 + \omega_2^{(0)}\right) - c_2(1-\Gamma)\left[\left(\omega_1^{(0)}\right)^2 - \left(\omega_3^{(0)}\right)^2\right] - (\Gamma-2)\omega_3^{(0)}c_1' - \Gamma c_3'\omega_1^{(0)} + \alpha^{(0)}F_1^{(1)} - \alpha^{(0)}c_3F_2^{(0)} + \alpha^{(1)}F_1^{(0)}, \tag{39}$$

$$0 = \Gamma c_3'' + c_1(1-\Gamma)\omega_1^{(0)}\omega_3^{(0)} + c_1\left[\hat{\omega}_2' - \Gamma\omega_2^{(0)\prime}\right] + c_2(1-\Gamma)\left(\hat{\omega}_2\omega_3^{(0)} - \omega_1^{(0)\prime}\right) + c_3\left(\hat{\omega}_2^2 - \hat{\omega}_2\omega_2^{(0)}\right) + \Gamma\left(c_2'\omega_1^{(0)} - c_1'\omega_2^{(0)}\right) + \alpha^{(0)}c_1F_1^{(0)} + \alpha^{(0)}c_2F_2^{(0)} - \varsigma_R\dot{c}_3, \tag{40}$$

where the primes represent the partial derivatives to the arc length s^g in virtual configuration, the overdots represent the partial derivatives to time t. Equations (35)–(40) are the first-order perturbation equations of the growing elastic rod in the viscous medium, which are the second-order partial differential equations with six variables $F_i^{(1)}$, c_i $(i=1,2,3)$. The system of equations has non-trivial solution which determines the critical growth value $\gamma=\gamma^*$.

5. Growing ring

A classic problem in the theory of elastic rod is the buckling of a planar twisted ring to a non-planar configuration which was first considered in Refs. [31,32]. We take this problem as an example. Consider an growing elastic ring with radius 1 and cross-sectional radius a in the initial unstressed state. It is assumed that it grows linearly in time meaning $\gamma=1+mt$ and keeps all other material properties unchanged. In initial configuration, the initial length of the ring is $L=2\pi\gamma$, where $\gamma=1$ and initial curvature-twisting vector is $\hat{\omega}=(0,1,0)$.

We consider the buckling of the ring at time t. The radius of the ring is γ and the curvature-twisting vector is $\omega^{(0)}=(0,\frac{1}{\gamma},0)$ at this time t. This state is maintained by a bending moment $M=\frac{1}{\gamma}-1$ generated by the periodic boundary condition. From Eq. (28), we can know $\alpha^{(0)}=1$. Set $s_0^0=1$, $m=1$ and let the ring be situated in the x–y plane, and then from Eqs. (35)–(40) we can obtain the following equations:

$$0 = -\varsigma\dot{c}_2 + \left(F_1^{(1)}\right)'' + 2\frac{1}{\gamma}\left(F_3^{(1)}\right)' - F_1^{(1)}\left(\frac{1}{\gamma}\right)^2, \tag{41}$$

$$0 = \varsigma\dot{c}_1 + \left(F_2^{(1)}\right)'', \tag{42}$$

$$0 = \frac{1}{\gamma}\varsigma c_2 + (F_3^{(1)})'' - 2\left(F_1^{(1)}\right)' - F_3^{(1)}\left(\frac{1}{\gamma}\right)^2, \tag{43}$$

$$0 = c_1'' + c_1\left[\frac{(1-\gamma)^2-(1-\gamma)\Gamma}{\gamma^2}\right] + c_3'\left[\frac{(1-\gamma)(\Gamma-1)+1}{\gamma}\right] - F_2^{(1)}, \tag{44}$$

$$0 = c_2'' + F_1^{(1)}, \tag{45}$$

$$0 = \Gamma c_3'' + c_3\left[\frac{(1-\gamma)^2-(1-\gamma)}{\gamma^2}\right] - \varsigma_R\dot{c}_3 - c_1'\frac{1}{\gamma}\Gamma. \tag{46}$$

Assume that the above equations have a periodic solution

$$u_j = x_j\exp\left(\sigma t + \frac{Ins}{\gamma}\right), \quad j=1,2,3,4,5,6, \tag{47}$$

where I is an imaginary unit; $u_j=F_j^{(1)}$ when $j=1,2,3$, and $u_j=c_{j-3}$ when $j=4,5,6$. Substituting them into Eqs. (41)–(46), we can obtain a system of equations

$$\boldsymbol{L}\cdot\boldsymbol{x}=0, \tag{48}$$

where $\boldsymbol{L}$ is the coefficient matrix and it can be written as

$$\boldsymbol{L} = \begin{pmatrix} -\frac{n^2+1}{\gamma^2} & 0 & \frac{2In}{\gamma^2} & 0 & -\varsigma\sigma & 0 \\ 0 & -\frac{n^2}{\gamma^2} & 0 & \varsigma\sigma & 0 & 0 \\ -\frac{2In}{\gamma} & 0 & -\frac{n^2+1}{\gamma^2} & 0 & \frac{\varsigma}{\gamma} & 0 \\ 0 & -1 & 0 & \frac{-n^2+(1-\gamma)^2-\Gamma(1-\gamma)}{\gamma^2} & 0 & \frac{In+(\Gamma-1)I(1-\gamma)n}{\gamma^2} \\ 1 & 0 & 0 & 0 & -\frac{n^2}{\gamma^2} & 0 \\ 0 & 0 & 0 & -\frac{In\Gamma}{\gamma^2} & 0 & \frac{\gamma^2-\gamma-n^2\Gamma}{\gamma^2} - \varsigma_R\sigma \end{pmatrix}. \tag{49}$$

Equation (48) will have nonzero solutions if the determinant of matrix $\boldsymbol{L}$ is equal to zero,

$$\Delta = \mathrm{Det}[\boldsymbol{L}] = 0. \tag{50}$$

The characteristic equation contains two pending parameters σ and n. The mode number n must take an integer due to the periodic boundary condition. The stability of the growth ring solution can be determined by the sign of the real part of the eigenvalue σ. It is stable when $\mathrm{Re}(\sigma) < 0$ and instable when $\mathrm{Re}(\sigma) > 0$, and $\sigma = 0$ corresponds to the critical state.[15] The determinant of matrix $\boldsymbol{L}$ can be expanded as

$$\begin{aligned}
\Delta \;=\; & \frac{n^4}{\gamma^{12}}\left(n^2-1\right)\left[n^2-(\gamma-1)^2\right]\left(-\gamma^2+\gamma-\gamma\Gamma+\Gamma n^2\right) \\
& +\frac{n^2\sigma}{\gamma^{10}}\left\{(\gamma-1)\gamma^3\varsigma\left[\gamma^2+\gamma(\Gamma-2)-\Gamma\right]+n^8\varsigma_{\mathrm{R}}\right\} \\
& +\frac{n^4\sigma}{\gamma^{10}}(\gamma-1)\varsigma_{\mathrm{R}}+\frac{n^2\sigma}{\gamma^{10}}\left\{\gamma^3\varsigma\left[\gamma^2+r(\Gamma-2)-2\Gamma+2\right]\right. \\
& \left.-(\gamma+\Gamma-1)\right\} \\
& +\frac{n^8\sigma}{\gamma^{10}}\left\{2\Gamma\gamma^2\varsigma-\varsigma_{\mathrm{R}}3\left[\gamma^2+\gamma(\Gamma-2)-\Gamma+3\right]\right\} \\
& +\frac{n^6\sigma}{\gamma^{10}}\left\{\left[2\gamma^2+2\gamma(\Gamma-2)-2\Gamma+3\right]\varsigma_{\mathrm{R}}\right. \\
& \left.+\gamma^2\varsigma\left[\gamma^2(\Gamma+2)+\gamma(\Gamma+2)-2\Gamma\right]\right\} \\
& +\frac{\sigma^6\varsigma}{\gamma^6}\left[2n^6\varsigma_{\mathrm{R}}-\gamma^3(\gamma-1)\varsigma\right] \\
& +\frac{n^4\sigma^2\varsigma}{\gamma^6}\left\{-\left[\gamma^2+r(\Gamma-2)-\Gamma+2\right]\varsigma_{\mathrm{R}}+\gamma^2\Gamma\varsigma\right\} \\
& +\frac{n^4\sigma^2\varsigma}{\gamma^6}\left[-\gamma^4\varsigma+\gamma^3\varsigma+\gamma^2\left(\Gamma\varsigma-\varsigma_{\mathrm{R}}\right)\right. \\
& \left.-(\Gamma-2)\gamma\varsigma_{\mathrm{R}}+\Gamma\varsigma_{\mathrm{R}}\right]+\frac{\left(n^2+1\right)\left(\sigma^3\varsigma^2\varsigma_{\mathrm{R}}\right)}{\gamma^2}.
\end{aligned} \tag{51}$$

For $\Delta(\sigma = 0) = 0$ corresponding to the critical state of buckling of the growing ring, we can obtain

$$\gamma(n) = \frac{-\Gamma+\sqrt{\Gamma^2+\Gamma\left(4n^2-2\right)+1}+1}{2}. \tag{52}$$

When $\gamma > 1$, the growing ring deviates from the initial configuration. According to the periodic boundary condition, we obtain that growing ring is in a stable state for the mode $n = 0$ or 1 and the threshold of instability is $n \geq 2$. This result is consistent with the growing ring without the viscous medium[10] and it indicates that the effect of the viscous medium on the threshold of instability of the growing ring is negligible.

We apply the result into a DNA ring. The classical parameter value of length L and radius a of cross section of the DNA ring are $L \sim 5 \times 10^{-6}$ cm and $a \sim 10^{-7}$ cm.[33] The viscosity of the nucleoplasm is measured to be $25 \sim 1000$ Pa·s.[34] Thus we can calculate the value range of viscous drag coefficients in nucleoplasm to be $\varsigma \sim [1.2, 96]$, $\varsigma_{\mathrm{R}} \sim [15, 1200]$. As we take $\gamma = 2$ and $\Gamma = 1$, it is unstable state, and relations of σ^2 and mode number n with different viscous resistance coefficients are shown in Fig. 1. When we vary the value of viscous drag coefficient, the change trend of the $\sigma^2 - n$ curve is almost unchanged. However, the value of σ^2 changes notably at the same mode number when the viscous drag coefficients change, which means that viscous resistance on instability of growing rod is remarkable. It can be clearly seen from Fig. 1 that when the viscous drag coefficient increases, the value of σ^2 decreases. This illustrates that the greater the viscous resistance, the smaller the amplitude of the instable rod is. However, the exact amplitude only can be determined by nonlinear analysis which we will discuss in the future.

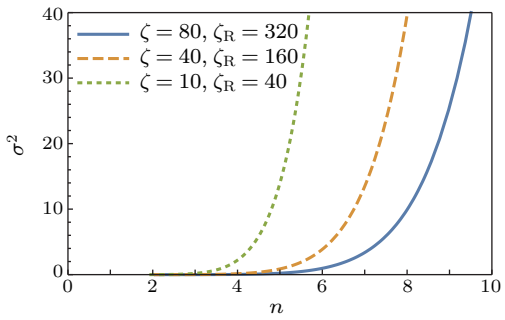


Fig. 1. The σ^2–n curve with different viscous drag coefficients for $\gamma = 2$ and $\Gamma = 1$.

6. Conclusions

Since most living things live in viscous medium environment, instability of growing rod in a viscous medium has been analyzed in the present paper. The dynamics model of growing rod in the viscous medium and its dynamics equations are constructed. A system of linear differential equations is deduced by a new perturbation scheme to analyze stability of growing rod in the viscous medium. Taking planar growing ring as an example, we obtain its bucking criterion of growth stretch. We can conclude that viscous resistance has no effect on stability of growing ring but has remarkable effect on instability of growing ring. Applying the results into a DNA ring, we obtain a sketch about growth ratio σ and mode number n, from which can read the affection of viscous resistance on instability of growing DNA ring.

In the near future, we will investigate how the geometrical structure of growing ring in the viscous medium changes beyond buckling and analyze the exact geometrical structure by nonlinear analysis and numerical simulation.

Appendix A

For a vector $\boldsymbol{V} = \sum_{i=1}^{3} v_i \boldsymbol{e}_i$, we can expand it by coordinate base vector perturbation scheme to the first order,

$$\begin{aligned}
\boldsymbol{V} &= \boldsymbol{V}^{(0)} + \varepsilon \boldsymbol{V}^{(1)} + O\left(\varepsilon^2\right) \\
&= \sum_{i=1}^{3} v_i^{(0)} \boldsymbol{e}_i^{(0)} + \varepsilon \sum_{i=1}^{3} \left(v_i^{(0)} \boldsymbol{e}_i^{(1)} + v_i^{(1)} \boldsymbol{e}_i^{(0)}\right) + O\left(\varepsilon^2\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{3} v_i^{(0)} \boldsymbol{e}_i^{(0)} + \varepsilon \sum_{i=1}^{3} (v_i^{(0)} (\boldsymbol{c} \times \boldsymbol{e}_i^{(0)}) + v_i^{(1)} \boldsymbol{e}_i^{(0)}) \\
&= \sum_{i=1}^{3} v_i^{(0)} \boldsymbol{e}_i^{(0)} + \varepsilon (v_1^{(0)} (c_3 \boldsymbol{e}_2^{(0)} - c_2 \boldsymbol{e}_3^{(0)}) \\
&\quad + v_2^{(0)} (-c_3 \boldsymbol{e}_1^{(0)} + c_1 \boldsymbol{e}_3^{(0)}) + v_3^{(0)} (c_2 \boldsymbol{e}_1^{(0)} - c_1 \boldsymbol{e}_2^{(0)})) \\
&\quad + \varepsilon \sum_{i=1}^{3} v_i^{(1)} \boldsymbol{e}_i^{(0)} + O(\varepsilon^2) \\
&= \sum_{i=1}^{3} v_i^{(0)} \boldsymbol{e}_i^{(0)} + \varepsilon ((v_3^{(0)} c_2 - v_2^{(0)} c_3) \boldsymbol{e}_1^{(0)} \\
&\quad + (v_1^{(0)} c_3 - v_3^{(0)} c_1) \boldsymbol{e}_2^{(0)} + (v_2^{(0)} c_1 - v_1^{(0)} c_2) \boldsymbol{e}_3^{(0)}) \\
&\quad + \varepsilon \sum_{i=1}^{3} v_i^{(1)} \boldsymbol{e}_i^{(0)} + O(\varepsilon^2) \\
&= \sum_{i=1}^{3} v_i^{(0)} \boldsymbol{e}_i^{(0)} + \varepsilon \sum_{i=1}^{3} \left[v_i^{(1)} + \left(\boldsymbol{c} \times \boldsymbol{v}^{(0)} \right)_i \right] \boldsymbol{e}_i^{(0)} + O(\varepsilon^2),
\end{aligned}
$$

where $v_i^{(1)}$ and $(\boldsymbol{c} \times \boldsymbol{v}^{(0)})_i$ represent the projection of vectors $\boldsymbol{v}^{(1)}$ and $\boldsymbol{c} \times \boldsymbol{v}^{(0)}$ in the $\boldsymbol{e}_i^{(0)}$ directions, respectively.

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