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Monogamy and polygamy relations of multiqubit entanglement based on unified entropy*

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Monogamy relation is one of the essential properties of quantum entanglement, which characterizes the distribution of entanglement in a multipartite system. By virtual of the unified-(q,s) entropy, we obtain some novel monogamy and polygamy inequalities in general class of entanglement measures. For the multiqubit system, a class of tighter monogamy relations are established in term of the α-th power of unified-(q,s) entanglement for $\alpha \geq 1$. We also obtain a class of tighter polygamy relations in the β-th ($0 \leq \beta \leq 1$) power of unified-(q,s) entanglement of assistance. Applying these results to specific quantum correlations, e.g., entanglement of formation, Renyi-q entanglement of assistance, and Tsallis-q entanglement of assistance, we obtain the corresponding monogamy and polygamy relations. Typical examples are presented for illustration. Furthermore, the complementary monogamy and polygamy relations are investigated for the α-th ($0 \leq \alpha \leq 1$) and β-th ($\beta \geq 1$) powers of unified entropy, respectively, and the corresponding monogamy and polygamy inequalities are obtained.

Keywords: quantum monogamy, quantum polygamy, unified entropy

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1. Introduction

Quantum entanglement[1–8] plays an important role in quantum information processing, which is considered as one of the most significant resources of quantum communication. It has been proved to be completely safe for any interior or exterior eavesdropper. In the past decades, the bipartite entanglement system has been investigated extensively, and abundant precious results are obtained. The usual monogamy inequality[9,10] may write as $\mathcal{E}(\rho_{A|BC}) \geq \mathcal{E}(\rho_{AB}) + \mathcal{E}(\rho_{AC})$, where $\rho_{AB} = \mathrm{tr}_C(\rho_{ABC})$ and $\rho_{AC} = \mathrm{tr}_B(\rho_{ABC})$. The $\mathcal{E}$ here stands for one kind of entanglement measure, which is essentially different from the classical correlation. The monogamy relation indicates that the entanglement between subsystems constraints each other. In Ref. [11], the authors found that the monogamy relation may play an important role in quantum key distribution security.

To understand more about the relations among quantum entanglements of multipartite systems is crucial for their application in quantum processing. The application of monogamy relation performed in quantum information theory may be helpful in this aim. The monogamy inequality, which is also called CKW inequality,[9] was first proposed by Coffman–Kundu–Wootters based on the concurrence of the qubit system.[12] Later on, many other forms of monogamy relations were established, including those for multiqubit and high-dimensional systems.[13–22] Recently, in Refs. [23,24], the authors gave an alternative definition of the monogamy relation with no inequality employed.

Qualitatively, for a composite quantum system, the monogamy inequality implies a trade-off relation of entanglement among subsystems, i.e., the more two subsystems are entangled the less this pair will interfere with the rest of others within the system. Different from the CKW inequality fulfilled by the concurrence, the assisted entanglement E_a[25] satisfies polygamy inequality $E_a(\rho_{A|BC}) \leq E_a(\rho_{AB}) + E_a(\rho_{AC})$ for tripartite state ρ_{ABC}, which is the dual property of the monogamy relation in the multipartite system. For a three-qubit system, reference [25] described a class of polygamy of bipartite entanglement in term of tangle of assistance, which was later generalized to the multipartite system based on different kinds of assisted entanglements.[26–28] Generally, the entanglement of assistance does not satisfy the usual monogamy relations. In Refs. [16,20], the authors found special classes of quantum states satisfying the monogamy relation for both concurrence of assistance and negativity of assistance. It was found that for the multiqubit system, the monogamy relations for the x-th power of entanglement of formation and concurrence are satisfied for $x \geq \sqrt{2}$ and $x \geq 2$, respectively.[20,21] Recently, various monogamy and polygamy inequalities are obtained under the non-negative power in the multipartite system.[17,18,20,22]

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In this paper, using the unified entropy,[29,30] we give new monogamy and polygamy inequalities, which are tighter than the existing ones in some classes of quantum states. Applying these results to the quantum correlations such as entanglement of formation, Renyi-q entanglement of assistance, and Tsallis-q entanglement of assistance, we then obtain the corresponding monogamy and polygamy relations. By using the two-qubit entanglement of individual subsystems, a class of tighter monogamy relations are established for multiqubit entanglement, in term of the α-th power of unified-(q,s) entanglement for $\alpha \geq 1$. We present also a class of tighter polygamy relations for the multiqubit entanglement, in term of the β-th $(0 \leq \beta \leq 1)$ power of unified-(q,s) entanglement of assistance. The yet unknown complementary monogamy and polygamy relations are investigated for the α-th $(0 \leq \alpha \leq 1)$ and β-th $(\beta \geq 1)$ powers of unified entropy, respectively. Several illustrating examples are given.

2. Monogamy of multiqubit relations for unified entropy

For any quantum state $\rho \in \mathcal{H}$, its unified-(q,s) entropy is defined as[29,30]

$$S_{q,s}(\rho) = \frac{1}{(1-q)s}\left((\mathrm{Tr}\rho^q)^s - 1\right), \tag{1}$$

with $q,s \geq 0$, $q \neq 1$, and $s \neq 0$.

For a bipartite pure state $|\psi\rangle_{AB} \in \mathcal{H}_{AB}$ with its reduced density matrix $\rho_A = \mathrm{Tr}_B|\psi\rangle_{AB}\langle\psi|$ onto subsystem A and each $q,s \geq 0$, its unified-(q,s) entanglement (UE) is defined as[27]

$$E_{q,s}(|\psi\rangle_{AB}) = S_{q,s}(\rho_A). \tag{2}$$

For a bipartite mixed state $\rho_{AB} \in \mathcal{H}_{AB}$, we define its UE via convex-roof extension

$$E_{q,s}(\rho_{AB}) = \min_{\{p_i,|\psi_i\rangle_{AB}\}} \sum_i p_i E_{q,s}(|\psi_i\rangle_{AB}), \tag{3}$$

where the minimum is taken over all possible pure state decompositions of $\rho_{AB} = \sum_i p_i|\psi_i\rangle_{AB}\langle\psi_i|$ and $\sum_i p_i = 1$. Since the UE in Eq. (3) is continuous with respect to the parameters q and s, UE reduces to Renyi-q entanglement (RE)[31,32] as s tends to 0, and converges to Tsallis-q entanglement (TE)[28,33] when s tends to 1. With any nonnegative s, UE converges to the entanglement of formation (EoF) as q tends to 1,

$$\lim_{q\to 1} E_{q,s}(\rho_{AB}) = E_{\mathrm{f}}(\rho_{AB}). \tag{4}$$

Therefore, UE is one of the most general classes of bipartite entanglement measures, with RE, TE, and EoF as special classes of UE.[27]

As a dual quantity to UE, unified-(q,s) entanglement of assistance (UEoA) is defined as[34]

$$E^{\mathrm{a}}_{q,s}(\rho_{AB}) = \max_{\{p_i,|\psi_i\rangle_{AB}\}} \sum_i p_i E_{q,s}(|\psi_i\rangle_{AB}) \tag{5}$$

for each $q,s \geq 0$, where the maximum is taken over all possible pure state decompositions of $\rho_{AB} = \sum_i p_i|\psi\rangle_{AB}\langle\psi|$ and $\sum_i p_i = 1$.

Similarly, as UEoA in Eq. (5) is continuous for the parameters q and s, it assures that UEoA reduces to Renyi-q entanglement of assistance (REoA)[32] and Tsallis-q entanglement of assistance (TEoA)[28] when s tends to 0 or 1, respectively. For any nonnegative s, as q tends to 1, UEoA reduces to the entanglement of assistance (EoA),[34]

$$\lim_{q\to 1} E^{\mathrm{a}}_{q,s}(\rho_{AB}) = E^{\mathrm{a}}(\rho_{AB}). \tag{6}$$

By using UE in Eq. (3) to quantify the bipartite quantum entanglement, the monogamy inequality is established in multiqubit systems; for any N-qubit state $\rho_{A_1A_2\cdots A_N}$ and its two-qubit reduced density matrices $\rho_{A_1A_i}$ with $i = 2,\ldots,N$, we have

$$E_{q,s}(\rho_{A_1|A_2\cdots A_N}) \geq \sum_{i=2}^{N} E_{q,s}(\rho_{A_1A_i}) \tag{7}$$

for $q \geq 2$, $0 \leq s \leq 1$, and $qs \leq 3$.[27]

It is also shown that UEoA can be used to characterize the polygamy of multiqubit entanglement as[34]

$$E^{\mathrm{a}}_{q,s}(\rho_{A_1|A_2\cdots A_N}) \leq \sum_{i=2}^{N} E^{\mathrm{a}}_{q,s}(\rho_{A_1A_i}) \tag{8}$$

for $1 \leq q \leq 2$ and $-q^2+4q-3 \leq s \leq 1$. It is further improved that, when $q \geq 2$, $0 \leq s \leq 1$, and $qs \leq 3$,[18]

$$E^{\alpha}_{q,s}(\rho_{A|B_1\cdots B_{N-1}}) \geq \sum_{j=1}^{N-1} \alpha^{w_{\mathrm{H}}(j)} E^{\alpha}_{q,s}(\rho_{AB_j}) \tag{9}$$

for any multiqubit state $\rho_{A_1A_2\cdots A_N}$ and $\alpha \geq 1$, where $w_{\mathrm{H}}(j)$ is the Hamming weight. For any non-negative integer j and its binary expansion $j = \sum_{i=0}^{n-1} j_i 2^i$, with $\log_2 j \leq n$ and $j_i \in \{0,\ 1\}$, $i = 0,1,\ldots,n-1$, we can always define a unique binary vector j associated with j, $j = (j_0, j_1, \ldots, j_{n-1})$. For the binary vector j defined above, the Hamming weight $w_{\mathrm{H}}(j)$ is defined by the number of 1's in $\{j_0, j_1, \ldots, j_{n-1}\}$. Thus, one has $w_{\mathrm{H}}(j) \leq \log_2 j \leq j$.

3. Tighter monogamy relations of multiqubit for unified entropy

Monogamy of entanglement is an intriguing feature of quantum entanglement, which characterizes the distributions of quantum entanglement among multipartite quantum systems, and becomes one of the hot issues in the study of quantum information theory in recent years. And the optimized monogamy relations give rise to finer characterizations of the entanglement distributions, which are tightly related to the security of quantum cryptographic protocols based on entanglement[11] (it limits the amount of correlations that an

eavesdropper can have with the honest parties). Therefore, obtaining tighter monogamy relations of entanglement is necessary to understand the whole picture of quantum entanglement. Here, in term of the α-th power of UE, we provide a class of tighter monogamy relations for the multiqubit entanglement. First, we give two lemmas.

Lemma 1[22,35] For any real numbers x and t, $0 \le t \le 1$, $x \in [1,\infty)$, we have $(1+t)^x \ge 1+(2^x-1)t^x$.

Proof Let $f(x,y) = (1+y)^x - y^x$ with $x \ge 1$, $y \ge 1$. Then $\partial f/\partial y = x[(1+y)^{x-1} - y^{x-1}] \ge 0$. Therefore, $f(x,y)$ is an increasing function of y, i.e., $f(x,y) \ge f(x,1) = 2^x - 1$. Set $y = 1/t$, $0 < t \le 1$, we obtain $(1+t)^x \ge 1+(2^x-1)t^x$. When $t = 0$, the inequality is trivial.

Lemma 2 For any $2\otimes 2\otimes 2$ mixed state $\rho \in \mathbb{H}_A \otimes \mathbb{H}_B \otimes \mathbb{H}_C$, if $E_{q,s}(\rho_{AB}) \ge E_{q,s}(\rho_{AC})$, we have

$$E^{\alpha}_{q,s}(\rho_{A|BC}) \ge E^{\alpha}_{q,s}(\rho_{AB}) + (2^{\alpha}-1)E^{\alpha}_{q,s}(\rho_{AC}) \tag{10}$$

for $\alpha \ge 1$, $q \ge 2$, and $0 \le s \le 1$, $qs \le 3$.

Proof It has been shown that $E_{q,s}(\rho_{A|BC}) \ge E_{q,s}(\rho_{AB}) + E_{q,s}(\rho_{AC})$ for arbitrary $2\otimes 2\otimes 2$ tripartite state ρ_{ABC}.[27] Then, if $E_{q,s}(\rho_{AB}) \ge E_{q,s}(\rho_{AC})$, we have

$$\begin{aligned}
& E^{\alpha}_{q,s}(\rho_{A|BC}) \\
\ge\ & (E_{q,s}(\rho_{AB}) + E_{q,s}(\rho_{AC}))^{\alpha} \\
=\ & E^{\alpha}_{q,s}(\rho_{AB})\left(1+\frac{E_{q,s}(\rho_{AC})}{E_{q,s}(\rho_{AB})}\right)^{\alpha} \\
\ge\ & E^{\alpha}_{q,s}(\rho_{AB})\left[1+(2^{\alpha}-1)\left(\frac{E_{q,s}(\rho_{AC})}{E_{q,s}(\rho_{AB})}\right)^{\alpha}\right] \\
=\ & E^{\alpha}_{q,s}(\rho_{AB}) + (2^{\alpha}-1)E^{\alpha}_{q,s}(\rho_{AC}),
\end{aligned}$$

where the second inequality is due to Lemma 1.

In Lemma 2, without loss of generality, we have assumed that $E_{q,s}(\rho_{AB}) \ge E_{q,s}(\rho_{AC})$, since the subsystems A and B are equivalent. Moreover, in the proof of Lemma 2, we have assumed $E_{q,s}(\rho_{AB}) > 0$. If $E_{q,s}(\rho_{AB}) = 0$ and $E_{q,s}(\rho_{AB}) \ge E_{q,s}(\rho_{AC})$, then $E_{q,s}(\rho_{AB}) = E_{q,s}(\rho_{AC}) = 0$. The lower bound is trivially zero. For multipartite qubit systems, we have the following theorem.

Theorem 1 For any multiqubit state $\rho_{AB_1\cdots B_{N-1}}$, if $E_{q,s}(\rho_{AB_i}) \ge \sum_{j=i+1}^{N-1} E_{q,s}(\rho_{AB_j})$ for every $i = 1,\ldots,N-2$, we have

$$\begin{aligned}
& E^{\alpha}_{q,s}(\rho_{A|B_1B_2\cdots B_{N-1}}) \\
\ge\ & E^{\alpha}_{q,s}(\rho_{AB_1}) + \cdots + (2^{\alpha}-1)^{N-3}E^{\alpha}_{q,s}(\rho_{AB_{N-2}}) \\
& + (2^{\alpha}-1)^{N-2}E^{\alpha}_{q,s}(\rho_{AB_{N-1}})
\end{aligned} \tag{11}$$

for $\alpha \ge 1$, $q \ge 2$, and $0 \le s \le 1$, $qs \le 3$.

Proof From the inequality in Lemma 1, we have

$$\begin{aligned}
& E^{\alpha}_{q,s}(\rho_{A|B_1B_2\cdots B_{N-1}}) \\
\ge\ & E^{\alpha}_{q,s}(\rho_{AB_1}) + (2^{\alpha}-1)\left(\sum_{j=2}^{N-1} E_{q,s}(\rho_{AB_j})\right)^{\alpha} \\
\ge\ & E^{\alpha}_{q,s}(\rho_{AB_1}) + (2^{\alpha}-1)E^{\alpha}_{q,s}(\rho_{AB_2}) \\
& + (2^{\alpha}-1)^2\left(\sum_{j=3}^{N-1} E_{q,s}(\rho_{AB_j})\right)^{\alpha} \\
\ge\ & \cdots \\
\ge\ & E^{\alpha}_{q,s}(\rho_{AB_1}) + (2^{\alpha}-1)E^{\alpha}_{q,s}(\rho_{AB_2}) + \cdots \\
& + (2^{\alpha}-1)^{N-2}E^{\alpha}_{q,s}(\rho_{AB_{N-1}}).
\end{aligned} \tag{12}$$

Theorem 1 gives a new class of monogamy relations for multiqubit states, which includes inequality (7) as a special case. Inequality (11) reduces to inequality (7) when $\alpha = 1$. Recently, another class of multiqubit monogamy inequalities in terms of the α-th power of UE was provided.[18] For $\alpha \ge 1$, $q \ge 2$, $0 \le s \le 1$, $qs \le 3$ and any multiqubit state $\rho_{AB_1\cdots B_{N-1}}$, it was found

$$E^{\alpha}_{q,s}(\rho_{A|B_1\cdots B_{N-1}}) \ge \sum_{j=1}^{N-1} \alpha^{j-1} E^{\alpha}_{q,s}(\rho_{AB_j}), \tag{13}$$

with the constraint $E_{q,s}(\rho_{AB_i}) \ge \sum_{j=i+1}^{N-1} E_{q,s}(\rho_{AB_j})$ for each $i = 1,\ldots,N-2$. As for $2^{\alpha}-1 \ge \alpha$ with $\alpha \ge 1$, obviously, our formula (11) in Theorem 1 gives a tighter monogamy relation with larger lower bounds than Eq. (13) given in Ref. [18].

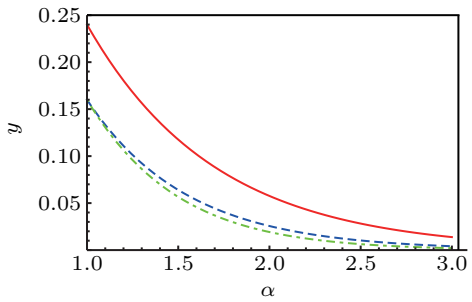


Fig. 1. The y represents the lower bounds of UE of $|\psi\rangle_{ABC}$, which are functions of α. The red solid line shows the values of UE of $|\psi\rangle_{ABC}$ in Example 1, the dashed line illustrates the lower bound in our calculation, the green dot-dashed line represents the lower bound given in Ref. [18].

Example 1 Let us consider the three-qubit state $|\psi\rangle_{ABC}$ in the generalized Schmidt decomposition form

$$\begin{aligned}
|\psi\rangle_{ABC} =\ & \lambda_0|000\rangle + \lambda_1 e^{i\varphi}|100\rangle + \lambda_2|101\rangle \\
& + \lambda_3|110\rangle + \lambda_4|111\rangle ,
\end{aligned} \tag{14}$$

where $\lambda_i \ge 0$, $i = 0,1,2,3,4$, and $\sum_{i=0}^{4} \lambda_i^2 = 1$. From the definition of UE in Eq. (3), when $q = 2$ and s tends to 1, we have

$$\begin{aligned}
& T_2(|\psi\rangle_{A|BC}) = 1 - \mathrm{Tr}\rho_A^2, \\
& T_2(\rho_{AB}) = \min_{\{p_i,|\psi_i\rangle_{AB}\}} \sum_i p_i T_2(|\psi_i\rangle_{AB}),
\end{aligned}$$

where the minimum is taken over all possible pure state decompositions of $\rho_{AB} = \sum_i p_i|\psi\rangle_{AB}\langle\psi|$ with $\sum_i p_i = 1$ and $\rho_A = \mathrm{Tr}_{BC}|\psi\rangle_{ABC}\langle\psi|$, $\rho_{AB} = \mathrm{Tr}_C|\psi\rangle_{ABC}\langle\psi|$. Then we get

$$T_2(|\psi\rangle_{A|BC}) = 2\lambda_0^2(\lambda_2^2 + \lambda_3^2 + \lambda_4^2),$$

$$T_2(\rho_{AB}) = 2\lambda_0^2\lambda_2^2, \quad T_2(\rho_{AC}) = 2\lambda_0^2\lambda_3^2.$$

Taking $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \sqrt{5}/5$, one can get

$$T_2(|\psi\rangle_{A|BC})^\alpha = \left(\frac{6}{25}\right)^\alpha,$$
$$T_2(\rho_{AB})^\alpha + \alpha T_2(\rho_{AC})^\alpha = (1+\alpha)\left(\frac{2}{25}\right)^\alpha,$$
$$T_2(\rho_{AB})^\alpha + (2^\alpha - 1)T_2(\rho_{AC})^\alpha = 2^\alpha\left(\frac{2}{25}\right)^\alpha.$$

Obviously, the new result is better than Eq. (13) for $\alpha \geq 1$, as shown in Fig. 1.

Generally, conditions for inequalities (11) and (13) are not always satisfied. In following, we give a universal monogamy inequality.

Theorem 2 For any multiqubit state $\rho_{AB_1\cdots B_{N-1}}$, if $E_{q,s}(\rho_{AB_i}) \geq \sum_{k=i+1}^{N-1} E_{q,s}(\rho_{AB_k})$ for $i = 1,2,\ldots,m$, and $E_{q,s}(\rho_{AB_j}) \leq \sum_{k=j+1}^{N-1} E_{q,s}(\rho_{AB_j})$ for $j = m+1,\ldots,N-2$, $\forall$ $1 \leq m \leq N-3$, $N \geq 4$, we have

$$\begin{aligned} & E_{q,s}^\alpha(\rho_{A|B_1B_2\cdots B_{N-1}}) \\ \geq & \sum_{i=1}^{m}(2^\alpha-1)^{i-1}E_{q,s}^\alpha(\rho_{AB_i}) + (2^\alpha-1)^{m+1}(E_{q,s}^\alpha(\rho_{AB_{m+1}}) \\ & + \cdots + E_{q,s}^\alpha(\rho_{AB_{N-2}})) + (2^\alpha-1)^m E_{q,s}^\alpha(\rho_{AB_{N-1}}) \end{aligned} \tag{15}$$

for $\alpha \geq 1$, $q \geq 2$, and $0 \leq s \leq 1$, $qs \leq 3$.

Proof From the inequality in Lemma 1, we have

$$\begin{aligned} & E_{q,s}^\alpha(\rho_{A|B_1B_2\cdots B_{N-1}}) \\ \geq & E_{q,s}^\alpha(\rho_{AB_1}) + (2^\alpha-1)\left(\sum_{i=2}^{N-1}E_{q,s}(\rho_{AB_i})\right)^\alpha \\ \geq & E_{q,s}^\alpha(\rho_{AB_1}) + (2^\alpha-1)E_{q,s}^\alpha(\rho_{AB_2}) \\ & + (2^\alpha-1)^2\left(\sum_{i=3}^{N-1}E_{q,s}(\rho_{AB_i})\right)^\alpha \\ \geq & \cdots \\ \geq & E_{q,s}^\alpha(\rho_{AB_1}) + \cdots + (2^\alpha-1)^{m-1}E_{q,s}^\alpha(\rho_{AB_m}) \\ & + (2^\alpha-1)^m\left(\sum_{i=m+1}^{N-1}E_{q,s}(\rho_{AB_i})\right)^\alpha. \end{aligned} \tag{16}$$

Similarly, as $E_{q,s}(\rho_{AB_j}) \leq \sum_{k=j+1}^{N-1} E_{q,s}(\rho_{AB_j})$ for $j = m+1,\ldots,N-2$, we get

$$\begin{aligned} & E_{q,s}^\alpha(\rho_{A|B_{m+1}\cdots B_{N-1}}) \\ \geq & (2^\alpha-1)E_{q,s}^\alpha(\rho_{AB_{m+1}}) + \left(\sum_{i=m+2}^{N-1}E_{q,s}(\rho_{AB_i})\right)^\alpha \\ \geq & (2^\alpha-1)(E_{q,s}^\alpha(\rho_{AB_{m+1}}) + \cdots + E_{q,s}^\alpha(\rho_{AB_{N-2}})) \\ & + E_{q,s}^\alpha(\rho_{AB_{N-1}}). \end{aligned} \tag{17}$$

Combining inequalities (16) and (17), we then have Theorem 2.

In a special case, inequality (15) reduces to Eq. (7) of Ref. [27] for $\alpha = 1$. For $\alpha > 1$, $(2^\alpha-1)^m > \alpha^m$, and all $1 \leq m \leq N-3$, formula (15) in Theorem 2 leads to a tighter monogamy relation with larger lower bounds than Eqs. (9) and (37) in Ref. [18]. Theorem 2 gives another monogamy relation based on the entanglement measure UE. Comparing inequality (11) in Theorem 1 with inequality (15) in Theorem 2, one may notice that inequality (11) is better than inequality (15). However, we note that for those classes of states that do not satisfy the conditions in Theorem 1, Theorem 2 is better.

Example 2 Consider the 4-qubit generlized W state

$$|W\rangle_{AB_1B_2B_3} = \frac{1}{2}(|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle). \tag{18}$$

Suppose s tends to 1 and $q = 2$, we then have $T_2(|W\rangle_{A|B_1B_2B_3}) = 3/8$, $T_2(\rho_{AB_i}) = 1/8$, $i = 1,2,3$, where $T_2(\rho_{AB})$ has the same definition in Example 1. It is easy to see that $T_2(\rho_{AB_i})$ does not satisfy the condition (11) of Theorem 1 and inequality (13) from Ref. [18]. From inequality (15) of Theorem 2, we have ${T_2}^\alpha(|W\rangle_{A|B_1B_2B_3}) \geq (2^\alpha-1)(1/8)^\alpha$, as shown in Fig. 2.

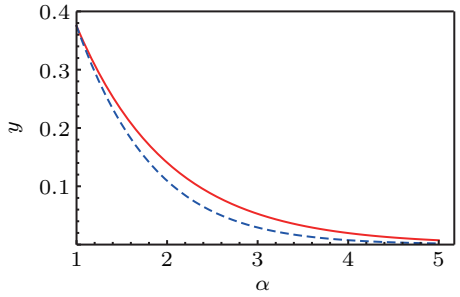


Fig. 2. The y represents the lower bound of UE of $|W\rangle_{AB_1B_2B_3}$, which is a function of α. The red solid line denotes the UE of $|W\rangle_{AB_1B_2B_3}$ in Example 2, and the blue dashed line represents the lower bound from inequality (15) of Theorem 2.

In Ref. [20], the authors presented $E_{A|B_1B_2\cdots B_{N-1}}^\alpha \geq E_{AB_1}^\alpha + \frac{\alpha}{\sqrt{2}}E_{AB_2}^\alpha + \cdots + \left(\frac{\alpha}{\sqrt{2}}\right)^{N-2}E_{AB_{N-1}}^\alpha$ for $\alpha \geq \sqrt{2}$ with the conditions $C_{AB_i} \geq C_{A|B_{i+1}\cdots B_{N-1}}, i = 1,2,\ldots,N-2$ for N-qubit states; in Ref. [21], the authors obtained $E_{A|B_1B_2\cdots B_{N-1}}^\alpha \geq E_{A|B_1}^\alpha + E_{A|B_2}^\alpha + \cdots + E_{AB_{N-1}}^\alpha$ for $\alpha \geq \sqrt{2}$. Because EoF coincides with concurrence for any $2 \otimes d$-dimensional pure state, we get $E_{A|B_1B_2\cdots B_{N-1}}^\alpha \geq E_{AB_1}^\alpha + (2^{\frac{\alpha}{\sqrt{2}}}-1)E_{AB_2}^\alpha + \cdots + \left(2^{\frac{\alpha}{\sqrt{2}}}-1\right)^{N-2}E_{AB_{N-1}}^\alpha$ for $\alpha \geq \sqrt{2}$ with the same conditions as q tends to 1 in Theorem 1. As $2^{\frac{\alpha}{\sqrt{2}}} - 1 \geq \alpha/\sqrt{2} \geq 1$ for $\alpha \geq \sqrt{2}$, obviously, our result is better than that in Refs. [20,21].

4. Tighter polygamy relations of multiqubit entanglement

Different from the property of inequality (11) of Theorem 1, we now provide a class of polygamy relations in multiqubit systems in terms of UEoA, which are tighter than the existing ones. For preparation, we first give two lemmas.

Lemma 3 For any real numbers x and t, $0 \leq t \leq 1$, $0 \leq x \leq 1$, we have $(1+t)^x \leq 1 + (2^x-1)t^x$.

Proof Let $f(x,y)=(1+y)^x-y^x$ with $0\le x\le 1$, $y\ge 1$. Then $\partial f/\partial y = x[(1+y)^{x-1}-y^{x-1}]\le 0$. Therefore, $f(x,y)$ is a decreasing function of y, i.e., $f(x,y)\le f(x,1)=2^x-1$. Set $y=1/t$, $0<t\le 1$, we obtain $(1+t)^x\le 1+(2^x-1)t^x$. When $t=0$, the inequality is trivial.

Lemma 4 For any $2\otimes 2\otimes 2$ mixed state $\rho\in\mathbb{H}_A\otimes\mathbb{H}_B\otimes\mathbb{H}_C$, if $E^{\rm a}_{q,s}(\rho_{AB})\ge E^{\rm a}_{q,s}(\rho_{AC})$, we have

$$(E^{\rm a}_{q,s}(\rho_{A|BC}))^\beta\le (E^{\rm a}_{q,s}(\rho_{AB}))^\beta+(2^\beta-1)(E^{\rm a}_{q,s}(\rho_{AC}))^\beta \quad (19)$$

for $0\le\beta\le 1$, $-q^2+4q-3\le s\le 1$, and $1\le q\le 2$.

Proof It has been shown that $E^{\rm a}_{q,s}(\rho_{A|BC})\le E^{\rm a}_{q,s}(\rho_{AB})+E^{\rm a}_{q,s}(\rho_{AC})$ for arbitrary $2\otimes 2\otimes 2$ tripartite state ρ_{ABC}.[34] Then, if $E^{\rm a}_{q,s}(\rho_{AB})\ge E^{\rm a}_{q,s}(\rho_{AC})$, we have

$$\begin{aligned}
&(E^{\rm a}_{q,s}(\rho_{A|BC}))^\beta\\
\le\ &(E^{\rm a}_{q,s}(\rho_{AB})+E^{\rm a}_{q,s}(\rho_{AC}))^\beta\\
=\ &(E^{\rm a}_{q,s}(\rho_{AB}))^\beta\left(1+\frac{E^{\rm a}_{q,s}(\rho_{AC})}{E^{\rm a}_{q,s}(\rho_{AB})}\right)^\beta\\
\le\ &(E^{\rm a}_{q,s}(\rho_{AB}))^\beta\left[1+(2^\beta-1)\left(\frac{E^{\rm a}_{q,s}(\rho_{AC})}{E^{\rm a}_{q,s}(\rho_{AB})}\right)^\beta\right]\\
=\ &(E^{\rm a}_{q,s}(\rho_{AB}))^\beta+(2^\beta-1)(E^{\rm a}_{q,s}(\rho_{AC}))^\beta\ ,
\end{aligned}$$

where the second inequality is due to Lemma 3. Here without loss of generality, we have assumed that $E^{\rm a}_{q,s}(\rho_{AB})\ge E^{\rm a}_{q,s}(\rho_{AC})$. Moreover, if $E^{\rm a}_{q,s}(\rho_{AB})=0$, we have $E^{\rm a}_{q,s}(\rho_{AB})=E^{\rm a}_{q,s}(\rho_{AC})=0$.

With preceding lemmas, we can then get the following Theorem for multiqubit systems.

Theorem 3 For any multiqubit state $\rho_{AB_1\cdots B_{N-1}}$ conditioned with $E^{\rm a}_{q,s}(\rho_{AB_i})\ge\sum_{j=i+1}^{N-1}E^{\rm a}_{q,s}(\rho_{AB_j})$ for $i=1,\ldots,N-2$, we have

$$(E^{\rm a}_{q,s}(\rho_{A|B_1\cdots B_{N-1}}))^\beta\le\sum_{j=1}^{N-1}(2^\beta-1)^{j-1}(E^{\rm a}_{q,s}(\rho_{AB_j}))^\beta \quad (20)$$

for $0\le\beta\le 1$, $-q^2+4q-3\le s\le 1$, and $1\le q\le 2$.

Proof From the inequality in Lemma 3,

$$\begin{aligned}
&(E^{\rm a}_{q,s}(\rho_{A|B_1\cdots B_{N-1}}))^\beta\\
\le\ &(E^{\rm a}_{q,s}(\rho_{AB_1}))^\beta+(2^\beta-1)\left(\sum_{j=2}^{N-1}E^{\rm a}_{q,s}(\rho_{AB_j})\right)^\beta\\
\le\ &(E^{\rm a}_{q,s}(\rho_{AB_1}))^\beta+(2^\beta-1)(E^{\rm a}_{q,s}(\rho_{AB_2}))^\beta\\
&+(2^\beta-1)^2\left(\sum_{j=3}^{N-1}E^{\rm a}_{q,s}(\rho_{AB_j})\right)^\beta\\
\le\ &\cdots\\
\le\ &(E^{\rm a}_{q,s}(\rho_{AB_1}))^\beta+(2^\beta-1)(E^{\rm a}_{q,s}(\rho_{AB_2}))^\beta+\cdots\\
&+(2^\beta-1)^{N-2}(E^{\rm a}_{q,s}(\rho_{AB_{N-1}}))^\beta.
\end{aligned}$$

Recently, another class of multiqubit polygamy inequalities in terms of the β-th power of concurrence was found.[18] That is, for $0\le\beta\le 1$, $-q^2+4q-3\le s\le 1$, $1\le q\le 2$, and any multiqubit state $\rho_{AB_1\cdots B_{N-1}}$,

$$(E^{\rm a}_{q,s}(\rho_{A|B_1\cdots B_{N-1}}))^\beta\le\sum_{j=1}^{N-1}\beta^{j-1}(E^{\rm a}_{q,s}(\rho_{AB_j}))^\beta \quad (21)$$

with constraint $E^{\rm a}_{q,s}(\rho_{AB_i})\ge\sum_{j=i+1}^{N-1}E^{\rm a}_{q,s}(\rho_{AB_j})$ for each $i=1,\ldots,N-2$. Since $2^\beta-1\le\beta$ with $0\le\beta\le 1$, formula (20) in Theorem 3 obviously yields a tighter polygamy relation with stringent upper bounds than inequality (21) of Ref. [18].

Example 3 Let us consider again the three-qubit state $|\psi\rangle_{ABC}$ of Eq. (14). From the definition of UEoA in Eq. (5), when $q=2$ and s tends to 1, we have

$$\begin{aligned}
&T_2(|\psi\rangle_{A|BC})=1-{\rm Tr}\rho_A^2,\\
&T_2(\rho_{AB})=\max_{\{p_i,|\psi_i\rangle_{AB}\}}\sum_i p_iT_2(|\psi_i\rangle_{AB}),
\end{aligned}$$

where the maximum is taken over all possible pure state decompositions of $\rho_{AB}=\sum_i p_i|\psi\rangle_{AB}\langle\psi|$, $\sum_i p_i=1$ and $\rho_A={\rm Tr}_{BC}|\psi\rangle_{ABC}\langle\psi|$, $\rho_{AB}={\rm Tr}_C|\psi\rangle_{ABC}\langle\psi|$. Then we have

$$\begin{aligned}
&T^{\rm a}_2(|\psi\rangle_{A|BC})=2\lambda_0^2(\lambda_2^2+\lambda_3^2+\lambda_4^2),\\
&T^{\rm a}_2(\rho_{AB})=2\lambda_0^2(\lambda_2^2+\lambda_4^2),\\
&T^{\rm a}_2(\rho_{AC})=2\lambda_0^2(\lambda_3^2+\lambda_4^2).
\end{aligned}$$

Let $\lambda_0=\lambda_1=\lambda_2=\lambda_3=\lambda_4=\sqrt{5}/5$, one readily gets

$$\begin{aligned}
&T^{\rm a}_2(|\psi\rangle_{A|BC})^\beta=\left(\frac{6}{25}\right)^\beta,\\
&T^{\rm a}_2(\rho_{AB})^\beta+\beta T^{\rm a}_2(\rho_{AC})^\beta=(1+\beta)\left(\frac{4}{25}\right)^\beta,\\
&T^{\rm a}_2(\rho_{AB})^\beta+(2^\beta-1)T^{\rm a}_2(\rho_{AC})^\beta=2^\beta\left(\frac{4}{25}\right)^\beta.
\end{aligned}$$

It is obvious that the result in this work is better than inequality (21) of Ref. [18] when $0\le\beta\le 1$, see Fig. 3.

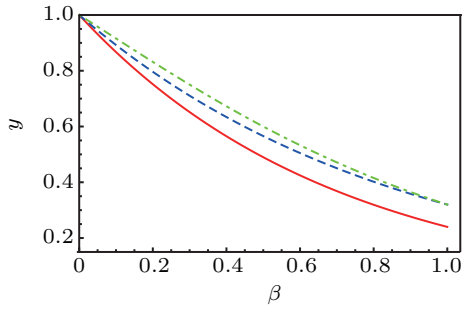


Fig. 3. The y denotes the upper bound of the UEoA of $|\psi\rangle_{ABC}$, which is a function of β. The red solid line represents the UEoA of $|\psi\rangle_{ABC}$ in Ref. (14), blue dashed line represents the upper bound of our result, green dot-dashed line represents the upper bound given in Ref. [18].

Note that the conditions of (20) in Theorem 3 and (21) in Ref. [18] are not generally satisfied. To get a widespread result, we have the following observation.

Theorem 4 For any multiqubit state $\rho_{AB_1\cdots B_{N-1}}$, if $E^{\rm a}_{q,s}(\rho_{AB_i})\ge\sum_{k=i+1}^{N-1}E^{\rm a}_{q,s}(\rho_{AB_k})$ for $i=1,\ldots,m$, and

$E^{\rm a}_{q,s}(\rho_{AB_j}) \le \sum_{k=j+1}^{N-1} E^{\rm a}_{q,s}(\rho_{AB_k})$ for $j = m+1,\ldots,N-2$, $\forall$ $1 \le m \le N-3$, $N \ge 4$, we have

$$\begin{aligned}
&(E^{\rm a}_{q,s}(\rho_{A|B_1\cdots B_{N-1}}))^\beta \\
&\le \sum_{i=1}^{m}(2^\beta-1)^{i-1}(E^{\rm a}_{q,s}(\rho_{AB_m}))^\beta \\
&\quad +(2^\beta-1)^{m+1}((E^{\rm a}_{q,s}(\rho_{AB_{m+1}}))^\beta+\cdots \\
&\quad +(E^{\rm a}_{q,s}(\rho_{AB_{N-2}}))^\beta) \\
&\quad +(2^\beta-1)^m(E^{\rm a}_{q,s}(\rho_{AB_{N-1}}))^\beta
\end{aligned} \tag{22}$$

for $0 \le \beta \le 1$, $-q^2+4q-3 \le s \le 1$, and $1 \le q \le 2$.

Proof From inequality (8) and Lemma 3, we have

$$\begin{aligned}
&(E^{\rm a}_{q,s}(\rho_{A|B_1\cdots B_{N-1}}))^\beta \\
&\le (E^{\rm a}_{q,s}(\rho_{AB_1}))^\beta+(2^\beta-1)\left(\sum_{j=2}^{N-1}E^{\rm a}_{q,s}(\rho_{AB_j})\right)^\beta \\
&\le (E^{\rm a}_{q,s}(\rho_{AB_1}))^\beta+(2^\beta-1)(E^{\rm a}_{q,s}(\rho_{AB_2}))^\beta \\
&\quad +(2^\beta-1)^2\left(\sum_{j=3}^{N-1}E^{\rm a}_{q,s}(\rho_{AB_j})\right)^\beta \\
&\le \cdots \\
&\le (E^{\rm a}_{q,s}(\rho_{AB_1}))^\beta+\cdots+(2^\beta-1)^{m-1}(E^{\rm a}_{q,s}(\rho_{AB_m}))^\beta \\
&\quad +(2^\beta-1)^m\left(\sum_{i=m+1}^{N-1}E^{\rm a}_{q,s}(\rho_{AB_i})\right)^\beta.
\end{aligned} \tag{23}$$

Similarly, as $E^{\rm a}_{q,s}(\rho_{AB_j}) \le \sum_{k=j+1}^{N-1} E^{\rm a}_{q,s}(\rho_{AB_k})$ for $j = m+1,\ldots,N-2$, we can get

$$\begin{aligned}
&\left(\sum_{i=m+1}^{N-1}E^{\rm a}_{q,s}(\rho_{AB_i})\right)^\beta \\
&\le (2^\beta-1)(E^{\rm a}_{q,s}(\rho_{AB_{m+1}}))^\beta+\left(\sum_{i=m+2}^{N-1}E^{\rm a}_{q,s}(\rho_{AB_i})\right)^\beta \\
&\le (2^\beta-1)((E^{\rm a}_{q,s}(\rho_{AB_{m+1}}))^\beta+\cdots+(E^{\rm a}_{q,s}(\rho_{AB_{N-2}}))^\beta) \\
&\quad +(E^{\rm a}_{q,s}(\rho_{AB_{N-1}}))^\beta.
\end{aligned} \tag{24}$$

Combining inequalities (23) and (24), we then obtain Theorem 4.

Theorem 4 gives another monogamy relation based on the entanglement measure UEoA. Comparing inequality (20) of Theorem 3 with inequality (22) of Theorem 4, inequality (20) is obviously tighter than inequality (22). Note that for those classes of states not satisfying the conditions of inequality (20) in Theorem 3 and inequality (21) in Ref. [18], the inequality (22) still holds.

Example 4 Consider the 4-qubit generlized W state

$$|W\rangle_{AB_1B_2B_3} = \frac{1}{2}(|1000\rangle+|0100\rangle+|0010\rangle+|0001\rangle). \tag{25}$$

When s tends to 1 and $q=2$, we have $T^{\rm a}_2(|W\rangle_{A|B_1B_2B_3}) = 3/8$, $T^{\rm a}_2(\rho_{AB_i}) = 1/8$ for $i = 1,2,3$, where $T^{\rm a}_2(\rho_{AB})$ has the same definition in Example 3. It is easy to see that $T^{\rm a}_2(\rho_{AB_i})$ does not satisfy the conditions (20) in Theorem 3 and inequality (21) in Ref. [18]. However, from inequality (22) of Theorem 4, we have $T^{{\rm a}\beta}_2(|W\rangle_{A|B_1B_2B_3}) \le (2^{\beta+1}-1)(\frac{1}{8})^\beta$, as shown in Fig. 4.

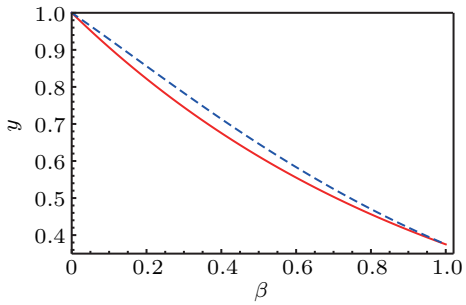


Fig. 4. The y denotes the upper bound of UEoA of $|W\rangle_{AB_1B_2B_3}$. Solid (red) line is exactly the UEoA of $|W\rangle_{AB_1B_2B_3}$, dashed (blue) line represents the upper bound of $T^{\rm a}_2(|W\rangle_{A|B_1B_2B_3})$ in inequality (22) for $0 \le \beta \le 1$.

5. Complementary monogamy and polygamy properties of multiqubit systems

In this section, we investigate monogamy and polygamy relations of multiqubit states. General monogamy and polygamy inequalities are given in the α-th ($0 \le \alpha \le 1$) power and the β-th ($\beta \ge 1$) power of unified entropy, respectively, which are complementary to the preceding two sections for different regions of parameter chosen in α and β.[35,36]

Lemma 5 For any real numbers x and t, $t \ge 1$, $0 \le x \le 1$, we have $(1+t)^x \ge 1+(2^x-1)t^x$.

Proof Let $f(x,y) = (1+y)^x - y^x$ with $0 \le x \le 1$, $0 < y \le 1$. Then $\partial f/\partial y = x[(1+y)^{x-1}-y^{x-1}] \le 0$. Therefore, $f(x,y)$ is a decreasing function of y, i.e., $f(x,y) \ge f(x,1) = 2^x-1$. Set $y = 1/t$, $t \ge 1$, we obtain $(1+t)^x \ge 1+(2^x-1)t^x$. When $t=0$, the inequality is trivial.

Lemma 6 For any $2\otimes 2\otimes 2$ mixed state $\rho \in \mathbb{H}_A\otimes\mathbb{H}_B\otimes\mathbb{H}_C$, if $E_{q,s}(\rho_{AB}) \le E_{q,s}(\rho_{AC})$, we have

$$E^\alpha_{q,s}(\rho_{A|BC}) \ge E^\alpha_{q,s}(\rho_{AB})+(2^\alpha-1)E^\alpha_{q,s}(\rho_{AC}) \tag{26}$$

for $0 \le \alpha \le 1$, $q \ge 2$, and $0 \le s \le 1$, $qs \le 3$.

Taking the similar method in proving Theorem 2, we get the following theorem.

Theorem 5 For any multiqubit state $\rho_{AB_1\cdots B_{N-1}}$, if $E_{q,s}(\rho_{AB_i}) \le \sum_{k=i+1}^{N-1} E_{q,s}(\rho_{AB_k})$ for $i = 1,2,\ldots,m$, and $E_{q,s}(\rho_{AB_j}) \ge \sum_{k=j+1}^{N-1} E_{q,s}(\rho_{AB_j})$ for $j = m+1,\ldots,N-2$, $\forall$ $1 \le m \le N-3$, $N \ge 4$, we have

$$\begin{aligned}
&E^\alpha_{q,s}(\rho_{A|B_1B_2\cdots B_{N-1}}) \\
&\ge \sum_{i=1}^{m}(2^\alpha-1)^{i-1}E^\alpha_{q,s}(\rho_{AB_i})+(2^\alpha-1)^{m+1}(E^\alpha_{q,s}(\rho_{AB_{m+1}}) \\
&\quad +\cdots+E^\alpha_{q,s}(\rho_{AB_{N-2}}))+(2\alpha-1)^mE^\alpha_{q,s}(\rho_{AB_{N-1}})
\end{aligned} \tag{27}$$

for $0 \le \alpha \le 1$, $q \ge 2$, and $0 \le s \le 1$, $qs \le 3$.

Theorem 5 provides a new monogamy relation based on UE. For $\alpha = 1$, (27) reduces to (7) as a special case.

For $0<\alpha<1$, inequality (27) in Theorem 5 gives a new class of monogamy inequality, which is complementary to inequality (15) of Theorem 2 in different region of parameter α. Theorem 5 assumes that $E_{q,s}(\rho_{AB_i}) \leq \sum_{k=i+1}^{N-1} E_{q,s}(\rho_{AB_k})$ and $E_{q,s}(\rho_{AB_j}) \geq \sum_{k=j+1}^{N-1} E_{q,s}(\rho_{AB_j})$ are partially satisfied for the $2\otimes 2\otimes\cdots\otimes 2$ mixed state $\rho \in \mathbb{H}_A \otimes \mathbb{H}_{B_1} \otimes \cdots \otimes \mathbb{H}_{B_{N-1}}$. If $E_{q,s}(\rho_{AB_i}) \leq \sum_{k=i+1}^{N-1} E_{q,s}(\rho_{AB_k})$ holds generally for $i = 1,2,\ldots,N-2$, then we have the following theorem.

Theorem 6 If $E_{q,s}(\rho_{AB_i}) \leq \sum_{k=i+1}^{N-1} E_{q,s}(\rho_{AB_k})$ for all $i = 1,2,\ldots,N-2$, then we have

$$\begin{aligned} & E^{\alpha}_{q,s}(\rho_{A|B_1B_2\cdots B_{N-1}}) \\ \geq\ & E^{\alpha}_{q,s}(\rho_{AB_1}) + \cdots + (2^{\alpha}-1)^{N-3} E^{\alpha}_{q,s}(\rho_{AB_{N-2}}) \\ & + (2^{\alpha}-1)^{N-2} E^{\alpha}_{q,s}(\rho_{AB_{N-1}}), \end{aligned} \tag{28}$$

for $0 \leq \alpha \leq 1$, $q \geq 2$, and $0 \leq s \leq 1$, $qs \leq 3$.

Example 5 Consider Eq. (14) again. From the definition of UE, when $q=2$ and s tends to 1, we have

$$\begin{aligned} & T_2(|\psi\rangle_{A|BC}) = 2\lambda_0^2(\lambda_2^2+\lambda_3^2+\lambda_4^2), \\ & T_2(\rho_{AB}) = 2\lambda_0^2\lambda_2^2, \quad T_2(\rho_{AC}) = 2\lambda_0^2\lambda_3^2. \end{aligned}$$

Set $\lambda_0=\lambda_1=\lambda_2=\lambda_3=\lambda_4=\sqrt{5}/5$, one gets

$$\begin{aligned} & {T_2}^{\alpha}(|\psi\rangle_{A|BC}) = \left(\frac{6}{25}\right)^{\alpha}, \\ & {T_2}^{\alpha}(\rho_{AB}) + (2^{\alpha}-1){T_2}^{\alpha}(\rho_{AC}) = 2^{\alpha}\left(\frac{2}{25}\right)^{\alpha}, \end{aligned}$$

as shown in Fig. 5.

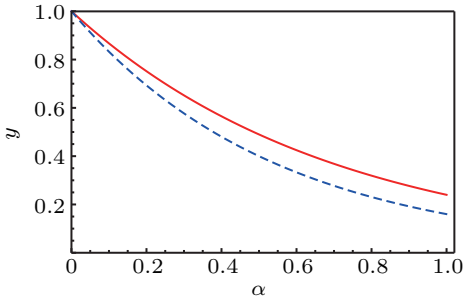


Fig. 5. The y denotes the lower bound of UE of $|\psi\rangle_{ABC}$, which is a function of α. The red solid line represents the UE of $|\psi\rangle_{ABC}$ in inequality (14), blue dashed line represents the lower bound of inequality (28).

By improving an inequality and using the β-th ($\beta \geq 1$) power of UEoA, we obtain a new class of weighted polygamy inequalities of multipartite entanglement in the multiqubit quantum system.

Lemma 7 For any real numbers x and t, $t \geq 1$, $x \geq 1$, we have $(1+t)^x \leq 1+(2^x-1)t^x$.

Proof Let $f(x,y) = (1+y)^x - y^x$ with $x \geq 1$, $0 < y \leq 1$, $\partial f/\partial y = x[(1+y)^{x-1} - y^{x-1}] \geq 0$. Therefore, $f(x,y)$ is an increasing function of y, i.e., $f(x,y) \leq f(x,1) = 2^x - 1$. Set $y = 1/t$, $t \geq 1$. We obtain $(1+t)^x \leq 1+(2^x-1)t^x$. Notice when $t=1$, the inequality is true.

Lemma 8 For any $2\otimes 2\otimes 2$ mixed state $\rho \in \mathbb{H}_A \otimes \mathbb{H}_B \otimes \mathbb{H}_C$, if $E^{\mathrm{a}}_{q,s}(\rho_{AB}) \leq E^{\mathrm{a}}_{q,s}(\rho_{AC})$, we have

$$(E^{\mathrm{a}}_{q,s}(\rho_{A|BC}))^{\beta} \leq (E^{\mathrm{a}}_{q,s}(\rho_{AB}))^{\beta} + (2^{\beta}-1)(E^{\mathrm{a}}_{q,s}(\rho_{AC}))^{\beta}$$

for $\beta \geq 1$, $-q^2+4q-3 \leq s \leq 1$, and $1 \leq q \leq 2$.

Using the similar method in Theorem 3, we have the following conclusion.

Theorem 7 For any multiqubit state $\rho_{AB_1\cdots B_{N-1}}$, if $E^{\mathrm{a}}_{q,s}(\rho_{AB_i}) \leq \sum_{k=i+1}^{N-1} E^{\mathrm{a}}_{q,s}(\rho_{AB_k})$ for $i = 1,2,\ldots,m$, and $E^{\mathrm{a}}_{q,s}(\rho_{AB_j}) \geq \sum_{k=j+1}^{N-1} E^{\mathrm{a}}_{q,s}(\rho_{AB_j})$ for $j = m+1,\ldots,N-2$, $\forall$ $1 \leq m \leq N-3$, $N \geq 4$, we have

$$\begin{aligned} & (E^{\mathrm{a}}_{q,s}(\rho_{A|B_1B_2\cdots B_{N-1}}))^{\beta} \\ \leq\ & \sum_{i=1}^{m} (2^{\beta}-1)^{i-1} (E^{\mathrm{a}}_{q,s}(\rho_{AB_i}))^{\beta} \\ & + (2^{\beta}-1)^{m+1} \left((E^{\mathrm{a}}_{q,s}(\rho_{AB_{m+1}}))^{\beta} + \cdots + (E^{\mathrm{a}}_{q,s}(\rho_{AB_{N-2}}))^{\beta} \right) \\ & + (2^{\beta}-1)^{m} (E^{\mathrm{a}}_{q,s}(\rho_{AB_{N-1}}))^{\beta} \end{aligned} \tag{29}$$

for $\beta \geq 1$, $-q^2+4q-3 \leq s \leq 1$, and $1 \leq q \leq 2$.

Theorem 7 gives a new polygamy relation based on UE and inequality (29) may reduce to inequality (8) for $\beta = 1$. For $\beta > 1$, inequality (29) in Theorem 7 presents a novel class of polygamy inequality, which is complementary to inequality (20) in Theorem 3 in different region of parameter β. Theorem 7 assumes that $E^{\mathrm{a}}_{q,s}(\rho_{AB_i}) \leq \sum_{k=i+1}^{N-1} E^{\mathrm{a}}_{q,s}(\rho_{AB_k})$ and $E^{\mathrm{a}}_{q,s}(\rho_{AB_j}) \geq \sum_{k=j+1}^{N-1} E^{\mathrm{a}}_{q,s}(\rho_{AB_j})$ partially hold for the $2\otimes 2\otimes\cdots\otimes 2$ mixed state $\rho \in \mathbb{H}_A \otimes \mathbb{H}_{B_1} \otimes \cdots \otimes \mathbb{H}_{B_{N-1}}$. In order $E^{\mathrm{a}}_{q,s}(\rho_{AB_i}) \leq \sum_{k=i+1}^{N-1} E^{\mathrm{a}}_{q,s}(\rho_{AB_k})$ to be general for $i = 1,2,\ldots,N-2$, we have the following theorem.

Theorem 8 If $E^{\mathrm{a}}_{q,s}(\rho_{AB_i}) \leq \sum_{k=i+1}^{N-1} E^{\mathrm{a}}_{q,s}(\rho_{AB_k})$ for all $i = 1,2,\ldots,N-2$, then we have

$$(E^{\mathrm{a}}_{q,s}(\rho_{A|B_1\cdots B_{N-1}}))^{\beta} \leq \sum_{j=1}^{N-1} (2^{\beta}-1)^{j-1} (E^{\mathrm{a}}_{q,s}(\rho_{AB_j}))^{\beta} \tag{30}$$

for $\beta \geq 1$, $-q^2+4q-3 \leq s \leq 1$, and $1 \leq q \leq 2$.

Example 6 Consider again the three-qubit state $|\psi\rangle_{ABC}$ (14). From the definition of UEoA, when $q=2$ and s tends to 1, we have

$$\begin{aligned} & T^{\mathrm{a}}_2(|\psi\rangle_{A|BC}) = 2\lambda_0^2(\lambda_2^2+\lambda_3^2+\lambda_4^2), \\ & T^{\mathrm{a}}_2(\rho_{AB}) = 2\lambda_0^2(\lambda_2^2+\lambda_4^2), \\ & T^{\mathrm{a}}_2(\rho_{AC}) = 2\lambda_0^2(\lambda_3^2+\lambda_4^2). \end{aligned}$$

Set $\lambda_0=\lambda_1=\lambda_2=\lambda_3=\lambda_4=\sqrt{5}/5$, one gets

$$\begin{aligned} & T^{\mathrm{a}\beta}_2(|\psi\rangle_{A|BC}) = \left(\frac{6}{25}\right)^{\beta}, \\ & T^{\mathrm{a}\beta}_2(\rho_{AB})^{+}(2^{\beta}-1)T^{\mathrm{a}\beta}_2(\rho_{AC}) = 2^{\beta}\left(\frac{4}{25}\right)^{\beta}, \end{aligned}$$

as shown in Fig. 6.

Monogamy and polygamy of entanglement can restrict the possible correlations between the authorized users and the

eavesdroppers, thus tightening the security bounds in quantum cryptography. The optimized monogamy and polygamy relations give rise to finer characterizations of the entanglement distributions. The complementary monogamy and polygamy relations may help to investigate the efficiency of entanglement used in quantum cryptography and characterizations of the entanglement distributions. These results may highlight future works on the study of quantum key distribution based on multipartite quantum entanglement distributions.

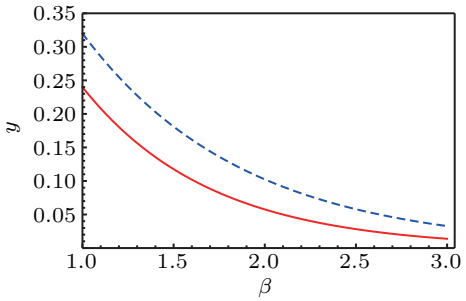


Fig. 6. The y denotes the upper bound of UEoA of $|\psi\rangle_{ABC}$, which is a function of β. The red solid line represents the UEoA of $|\psi\rangle_{ABC}$ in inequality (14), blue dashed line represents the upper bound of inequality (30).

6. Conclusion

Entanglement monogamy and polygamy are two fundamental properties of multipartite entangled states. We provide in this work a characterization of multiqubit entanglement constraints in terms of UE. Employing the two-qubit entanglement of individual subsystems, a class of tighter monogamy inequalities are established for multiqubit entanglement based on the α-th ($\alpha \geq 1$) power of UE. We also provide a class of tighter polygamy inequalities of multiqubit entanglement in terms of the β-th ($0 \leq \beta \leq 1$) power of UEoA. Applying these results to specific quantum correlations, such as entanglement of formation, Renyi-q entanglement of assistance, and Tsallis-q entanglement of assistance, we obtain the corresponding tighter monogamy or polygamy relation for some classes of quantum states. Moreover, we investigate the α-th ($0 \leq \alpha \leq 1$) and β-th ($\beta \geq 1$) power of monogamy and polygamy relations based on the UE and UEoA. These monogamy and polygamy relations are complementary to the existing ones in different regions of parameters α and β. Last, it is worth mentioning that the approach used in this work is applicable to the study of monogamy and polygamy natures in other entanglement measures, such as entanglement of formation, Renyi-q entanglement of assistance, and Tsallis-q entanglement of assistance, or other high-dimensional quantum systems.

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