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# Multiple Lagrange stability and Lyapunov asymptotical stability of delayed fractional-order Cohen–Grossberg neural networks*

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This paper addresses the coexistence and local stability of multiple equilibrium points for fractional-order Cohen–Grossberg neural networks (FOCGNNs) with time delays. Based on Brouwer's fixed point theorem, sufficient conditions are established to ensure the existence of $\prod_{i=1}^{n}(2K_i+1)$ equilibrium points for FOCGNNs. Through the use of Hardy inequality, fractional Halanay inequality, and Lyapunov theory, some criteria are established to ensure the local Lagrange stability and the local Lyapunov asymptotical stability of $\prod_{i=1}^{n}(K_i+1)$ equilibrium points for FOCGNNs. The obtained results encompass those of integer-order Hopfield neural networks with or without delay as special cases. The activation functions are nonlinear and nonmonotonic. There could be many corner points in this general class of activation functions. The structure of activation functions makes FOCGNNs could have a lot of stable equilibrium points. Coexistence of multiple stable equilibrium points is necessary when neural networks come to pattern recognition and associative memories. Finally, two numerical examples are provided to illustrate the effectiveness of the obtained results.

**Keywords:** fractional-order Cohen–Grossberg neural networks, multiple Lagrange stability, multiple Lyapunov asymptotical stability, time delays

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## 1. Introduction

Over the past decades, recurrent neural networks have attracted great attentions of many researchers because of the potential applications in pattern recognition, associative memory, parallel computation, classification, and optimization.[1–8] Cohen–Grossberg neural network (CGNN) was firstly introduced in 1983,[9] which includes many models such as neurobiology, population biology and evolutionary theory. The CGNN is typical, and popular.[10–12] Some well-known neural networks like Hopfield neural networks,[13] cellular neural networks,[14] and bidirectional associative memory neural networks[15] can also be seen as its special cases. In most applications of neural networks, the designed neural network is required to have the stable dynamics. For instance, in the applications of combinatorial optimization, the neural system needs to be globally stable.[16–20] And in the applications of associative memory, the neural system needs to be locally stable.[21–32] Becasuse of the finite speeds of the transmission and switching of signals in neural networks, time delays are unavoidable. It is necessary to take time delays into neural networks. Many researchers have been investigated extensively global stability and local stability of delayed CGNNs. Some criteria were obtained for global exponential stability of delayed CGNNs.[17] Multiple $\psi$-type stability of CGNNs with both time-varying discrete delays and distributed delays was discussed by authors in Ref. [33]. Stability results for delayed Takagi–Sugeno fuzzy CGNNs were presented in Ref. [34].

Recently, many researchers began to pay attention to dynamic behaviors of fractional-order differential systems. More than 300 years ago, fractional-order calculus was firstly introduced. Due to lacking of application background, physical and geometrical representation, solution methods, fractional-order calculus did not attract much attention for a long time. In fact, we can find many applications of fractional calculus, such as viscoelastic liquids,[35] electromagnetic waves,[36] electrode–electrolyte polarization,[37] heat conduction,[38] neural networks,[39] and so on. It is known that fractional derivatives are nonlocal and fractional derivatives have weakly singular kernels. The fractional-order model is more accurate for the description of memory and hereditary properties of various materials and processes compared with integer-order operators.[40] Therefore, integrating the fractional calculus into neural networks can bring about a significant improvement, which leads to fractional-order neural networks (FONNs). The investigation on FONNs has become a hotspot.[41] Global asymptotic $\omega$-periodicity was formulated for FONNs in Ref. [42]. Synchronization of delayed fractional-order memristive neural networks has been considered in Ref. [43]. Multistability in Mittag–Leffler sense of FONNs has been investigated in Refs. [30] and [41]. For more

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references, refer to Refs. [44–47].

In many applications of recurrent neural networks, such as associative memory, pattern recognition, classification, *etc*., coexistence of multiple stable equilibrium points is a necessary feature. Each stable equilibrium point can be designed as an individual pattern. Recently, multistability of recurrent neural networks has been investigated in depth. Based on neural networks with concave–convex characteristics, the existence of a large number of stable equilibrium points was discussed in Ref. [48]. The boundaries of attractive basins of stable equilibria of two-neuron neural networks by studying stable manifolds of unstable equilibria in Ref. [24]. In Ref. [32], multiple Lagrange stability under perturbation for recurrent neural networks was addressed. For more references, refer to Refs. [49–52]. However, most of the results focus on the multistability of integer-order recurrent neural networks. There are very few studies on the multistability of fractional-order neural networks. The authors investigated the existence and local stability of multiple equilibria of fractional-order recurrent neural networks in Refs. [30] and [41]. To the best of our knowledge, there is less consideration for local stability of fractional-order Cohen–Grossberg neural networks (FOCGNNs). In fact, FOCGNNs include a lot of models from practical problems, and have properties of infinity memory and hereditary, which make them more effective and accurate than integral-order ones in the modeling of physical phenomena.[53] As a result, it is necessary to investigate the multistability of FOCGNNs.

Based on the above discussion, the main goal of this paper is to study the multistability of FOCGNNs. The main results are presented as follows. (i) Sufficient conditions are derived to ensure the existence of $\prod_{i=1}^{n}(2K_i+1)$ equilibrium points based on Brouwer's fixed point theorem. (ii) By using Hardy inequality, fractional Halanay inequality, local Lagrange stability of $\prod_{i=1}^{n}(K_i+1)$ equilibrium points is addressed. (iii) Local Lyapunov asymptotical stability of $\prod_{i=1}^{n}(K_i+1)$ equilibrium points is discussed based on Lyapunov theory. The difficulties in this paper are described as follows. i) Because of nonlinear activation functions and coupling parameters in FOCGNNs, it is necessary to establish some conditions to ensure the existence of equilibrium points. ii) The existing methods for studying the multistability of integer-order CGNNs are not suitable for the discussion of the multistability of FOCGNNs. It is necessary to introduce new methods to discuss the stability of FOCGNNs. iii) It is also difficult to study the stability of equilibrium points of FOCGNNs because of the existence of time delays.

The reminder part of the paper is organized as follows. In Section 2, some problem formulations and several lemmas are described. In Section 3, the existence, local Lagrange stability and local Lyapunov asymptotical stability of multiple equilibrium points of delayed FOCGNNs are stated. Two numerical examples are given to illustrate the effectiveness of the proposed results in Section 4. In Section 5, the conclusion is drawn.

## 2. Problem description and preliminaries

Consider a class of delayed FOCGNNs described by the following differential equations for $i=1,2,\ldots,n$:

$$\begin{aligned} D^{\alpha}x_i(t) \;=\; & -a_i(x_i(t))[b_i(x_i(t)) - \sum_{j=1}^{n} c_{ij}f_j(x_j(t)) \\ & - \sum_{j=1}^{n} d_{ij}f_j(x_j(t-\tau)) - I_i], \end{aligned} \tag{1}$$

where $0<\alpha<1$; $x_i(t)$ is the state of the $i$-th unit at time $t$; $a_i(x_i(t))$ and $b_i(x_i(t))$ are the amplification and the self-signal functions, respectively; $c_{ij}$ and $d_{ij}$ represent the connection weight of the $j$-th unit on the $i$-th unit; $f_i(\cdot)$ is the activation function; $I_i$ corresponds to the external input; and $\tau$ stands for the transmission delay. The operation $D^{\alpha}x(t)$ consists of the Caputo's derivative of $x(t)$, defined by

$$D^{\alpha}x(t) = \frac{1}{\Gamma(m-\alpha)}\int_{t_0}^{t}(t-\tau)^{m-\alpha-1}x^{(m)}(\tau)\mathrm{d}\tau,$$

where $m$ is the integer satisfying $m-1<\alpha<m$, and $\Gamma(\cdot)$ is the Gamma function, $\Gamma(s)=\int_0^{\infty}t^{s-1}\mathrm{e}^{-t}\mathrm{d}t$.

Assume that parameters of FOCGNNs (1) satisfy the following properties:

**Assumption 1** $a_i(u)$ is locally Lipschitz continuous for $i=1,2,\ldots,n$. And $a_i(u)\in[\underline{a}_i,\overline{a}_i]$, $0<\underline{a}_i,\overline{a}_i<+\infty$ for $u\in\mathcal{R}$.

**Assumption 2** $b_i(u)$ is locally Lipschiz continuous for $i=1,2,\ldots,n$. And

$$|b_i(u+v)-b_i(v)|\geq l_i|u|,\ u,v\in\mathcal{R},\ l_i>0.$$

Consider the activation functions in Ref. [30] which satisfy the following assumptions.

**Assumption 3** $f_i(u)$ is continuous for $i=1,2,\ldots,n$. And

$$m_i\leq f_i(u)\leq M_i,\ u\in\mathcal{R},$$

where $-\infty<m_i<M_i<+\infty$.

**Assumption 4** There exist some constants $-\infty=q_i^{(0)}<p_i^{(0)}<q_i^{(1)}<p_i^{(1)}<\cdots<q_i^{(K_i-1)}<p_i^{(K_i-1)}<q_i^{(K_i)}<p_i^{(K_i)}=+\infty$ such that

$$\begin{aligned} &\lambda_i^{(k)}\leq\frac{f_i(u)-f_i(v)}{u-v}\leq\eta_i^{(k)}, u,v\in(q_i^{(k)},p_i^{(k)}),\\ &k=0,1,\ldots,K_i,\\ &\mu_i^{(k)}\leq\frac{f_i(u)-f_i(v)}{u-v}\leq\upsilon_i^{(k)}, u,v\in[p_i^{(k-1)},q_i^{(k)}],\\ &k=1,2,\ldots,K_i, \end{aligned}$$

where $\lambda_i^{(k)}$, $\eta_i^{(k)}$, $\mu_i^{(k)}$, $\nu_i^{(k)}\in\mathcal{R}$, $K_i$ is a non-negative integer.

Let

$$L_i=\max_{0\leq k\leq K_i}\{|\lambda_i^{(k)}|,|\eta_i^{(k)}|\}.$$

**Definition 1**[54] FOCGNN (1) is uniformly stable in Lagrange sense, if there is a position constant $K = K(\alpha) > 0$ such that $\| x(t,\psi) \| < K$ for any $\alpha > 0$,

$$\psi \in \{\psi \in C([-\tau,0],\mathcal{R}^n) | \ \| \psi \| < \alpha\}, \ t \geq 0.$$

**Lemma 1**[30] Let $\alpha \in (0,1)$, $t_0 \geq 0$ and $g \in \mathcal{C}([t_0,+\infty),\mathcal{R})$ is differentiable. If there is $t_1$ with $t_1 > t_0$ such that $g(t_1) = 0$ and $g(t) < 0$ $(g(t) \leq 0)$ for $t_0 \leq t < t_1$, then $D^\alpha g(t_1) > 0$ $(D^\alpha g(t_1) \geq 0)$.

**Lemma 2** (Hardy inequality[55]) Assume there exist constants $e_k \geq 0$, $p_k > 0$, then

$$\left(\prod_{k=1}^{M+1} e_k^{p_k}\right)^{\frac{1}{S_{M+1}}} \leq \left(\sum_{k=1}^{M+1} p_k e_k^r\right)^{1/r} S_{M+1}^{-1/r}, \tag{2}$$

where $r > 0$, $S_{M+1} = \sum_{k=1}^{M+1} p_k$, $k = 1,2,\ldots,M+1$.

Specially, if $p_{M+1} = 1$, $r = S_{M+1}$, then the following inequality holds,

$$\left(\prod_{k=1}^{M} e_k^{p_k}\right) e_{M+1} \leq \frac{1}{r}\left(\sum_{k=1}^{M} p_k e_k^r\right) + \frac{1}{r} e_{M+1}^r.$$

**Lemma 3** (Fractional Halanay Inequality[56]) Assume $g(t) \geq 0$ is a continuous function, and

$$D^\alpha g(t) \leq s_0 + s_3 g(t) + s_4 \sup_{t-\tau(t)\leq\theta\leq t} g(\theta), \ t \in [0,+\infty),$$
$$g(t) = |\varphi(t)|, \ t \in (-\infty,0],$$

where $s_0 \geq 0$, $s_4 \geq 0$, $s_3 < 0$, $\varphi(t)$ is bounded and continuous, $-\sigma \leq t - \tau(t) \leq t$. Assume $g_0 = |\varphi(0)|$ and $M_0 = \sup_{-\sigma\leq\theta\leq 0} |\varphi(\theta)|$. When $s_3 + s_4 < 0$, then

$$g(t) \leq \frac{s_3}{s_3+s_4} g_0 - \frac{s_0}{s_3+s_4} + M_0, \ t \geq 0.$$

In addition, when $\lim_{t\to+\infty}(t - \tau(t)) = +\infty$, then there is one point $t_*$ with $t_*(M_0,\varepsilon) > 0$ for any given $\varepsilon > 0$ such that

$$g(t) \leq -\frac{s_0}{s_3+s_4} + \varepsilon, \ t \leq t_*.$$

**Lemma**[57] Let $0 < \alpha < 1$, all the eigenvalues $\lambda$ of matrix $-A + B \in \mathcal{R}^{n\times n}$ satisfy the inequality $|\arg(\lambda)| > \pi/2$. There is no pure imaginary roots for the characteristic equation $\det(\Delta(s)) = \det(s^\gamma I + A - \mathrm{e}^{s\tau} B) = 0$, where $\tau > 0$, then the zero solution of system

$$D^\alpha x(t) = -Ax(t) + Bx(t-\tau)$$

is globally Lyapunov asymptotically stable.

**Lemma 5**[58] Consider delayed fractional-order differential inequality as follows:

$$D^\alpha x(t) \leq -ax(t) + bx(t-\tau), \ q \in (0,1),$$
$$x(t) = h(t), \ t \in [-\tau,0].$$

Consider the following delayed fractional-order differential systems

$$D^\alpha y(t) = -ay(t) + by(t-\tau), \ q \in (0,1),$$
$$y(t) = h(t), \ t \in [-\tau,0],$$

where $x(t) > 0$, $y(t) > 0$ are continuous, $a > 0$, $b > 0$ and $h(t) > 0, t \in [-\tau,0]$. Then the following inequality holds:

$$x(t) \leq y(t), \ t \in [0,+\infty).$$

Denote

$$\begin{aligned}
(q_i^{(0)}, p_i^{(0)}) =& (q_i^{(0)}, p_i^{(0)})^1 \cup [p_i^{(0)}, q_i^{(1)}]^0 \cup (q_i^{(1)}, p_i^{(1)})^0 \cup [p_i^{(1)}, q_i^{(2)}]^0 \\
&\cup \cdots \cup [p_i^{(K_i-1)}, q_i^{(K_i)}]^0 \cup (q_i^{(K_i)}, p_i^{(K_i)})^0, \\
[p_i^{(0)}, q_i^{(0)}] =& (q_i^{(0)}, p_i^{(0)})^0 \cup [p_i^{(0)}, q_i^{(1)}]^1 \cup (q_i^{(1)}, p_i^{(1)})^0 \cup [p_i^{(1)}, q_i^{(2)}]^0 \\
&\cup \cdots \cup [p_i^{(K_i-1)}, q_i^{(K_i)}]^0 \cup (q_i^{(K_i)}, p_i^{(K_i)})^0, \\
&\cdots \\
(q_i^{(K_i)}, p_i^{(K_i)}) =& (q_i^{(0)}, p_i^{(0)})^0 \cup [p_i^{(0)}, q_i^{(1)}]^0 \\
&\cup (q_i^{(1)}, p_i^{(1)})^0 \cup [p_i^{(1)}, q_i^{(2)}]^0 \\
&\cup \cdots \cup [p_i^{(K_i-1)}, q_i^{(K_i)}]^0 \cup (q_i^{(K_i)}, p_i^{(K_i)})^1, \\
\Theta =& \prod_{i=1}^{n} (q_i^{(0)}, p_i^{(0)})^{\delta_i^{(0)}} \cup [p_i^{(0)}, q_i^{(1)}]^{\delta_i^{(1)}} \\
&\cup (q_i^{(1)}, p_i^{(1)})^{\delta_i^{(2)}} \cup [p_i^{(1)}, q_i^{(2)}]^{\delta_i^{(3)}} \\
&\cup \cdots \cup [p_i^{(K_i-1)}, q_i^{(K_i)}]^{\delta_i^{(2K_i)}} \cup (q_i^{(K_i)}, p_i^{(K_i)})^{\delta_i^{(2K_i+1)}},
\end{aligned}$$

where $(\delta_i^{(0)}, \delta_i^{(1)}, \ldots, \delta_i^{(2K_i+1)}) = (1,0,\ldots,0)$ or $(0,1,\ldots,0)$ or $\cdots$ or $(0,0,\ldots,1)$. Denote

$$\begin{aligned}
\Theta^{(1)} =& \prod_{i=1}^{n} (q_i^{(0)}, p_i^{(0)})^{\delta_i^{(0)}} \cup [p_i^{(0)}, q_i^{(1)}]^0 \\
&\cup (q_i^{(1)}, p_i^{(1)})^{\delta_i^{(1)}} \cup [p_i^{(1)}, q_i^{(2)}]^0 \\
&\cup \cdots \cup [p_i^{(K_i-1)}, q_i^{(K_i)}]^0 \cup (q_i^{(K_i)}, p_i^{(K_i)})^{\delta_i^{(K_i)}},
\end{aligned}$$

where $(\delta_i^{(0)}, \delta_i^{(1)}, \ldots, \delta_i^{(K_i)}) = (1,0,\ldots,0)$ or $(0,1,\ldots,0)$ or $\cdots$ or $(0,0,\ldots,1)$. Denote $\Theta^{(2)} = \Theta - \Theta^{(1)}$. It is easy to see that $\Theta$, $\Theta^{(1)}$, $\Theta^{(2)}$ have $\prod_{i=1}^n (2K_i+1)$, $\prod_{i=1}^n (K_i+1)$ and $\prod_{i=1}^n (2K_i+1) - \prod_{i=1}^n (K_i+1)$ elements, respectively.

Define the bounding functions as follows for $i = 1,2,\ldots,n$:

$$\begin{aligned}
F_i^-(x) =& -b_i(x) + (c_{ii} + d_{ii}) f_i(x) + \sum_{j=1, j\neq i}^{n} \min\{c_{ij} m_j, c_{ij} M_j\} \\
&+ \sum_{j=1, j\neq i}^{n} \min\{d_{ij} m_j, d_{ij} M_j\} + I_i, \\
F_i^+(x) =& -b_i(x) + (c_{ii} + d_{ii}) f_i(x) + \sum_{j=1, j\neq i}^{n} \max\{c_{ij} m_j, c_{ij} M_j\} \\
&+ \sum_{j=1, j\neq i}^{n} \max\{d_{ij} m_j, d_{ij} M_j\} + I_i,
\end{aligned}$$

We can obtain that $a_i(x_i(t))F_i^-(x_i(t)) \le D^\alpha x_i(t) \le a_i(x_i(t))F_i^+(x_i(t))$. Define the following function:

$$F_i(x) = -b_i(x)+(c_{ii}+d_{ii})f_i(x)+\sum_{j=1,j\neq i}^{n} c_{ij}f_j(x_j) + \sum_{j=1,j\neq i}^{n} d_{ij}f_j(x_j)+I_i.$$

Then $F_i^-(\cdot) \le F_i(\cdot) \le F_i^+(\cdot)$.

## 3. Main results

Now, some sufficient conditions will be presented to ensure the existence and stability of multiple equilibrium points of FOCGNN (1) with time delays.

**Lemma 6** Assume Assumption 1–Assumption 4 hold. If

$$\begin{aligned} &F_i^-(q_i^{(k)}) > 0,\ k = 1,2,\ldots,K_i,\\ &F_i^+(p_i^{(k)}) < 0,\ k = 0,1,2,\ldots,K_i-1 \end{aligned} \tag{3}$$

for $i = 1,2,\ldots,n$, then there exist at least $\prod_{i=1}^n (2K_i+1)$ equilibrium points for FOCGNN (1).

**Proof** Let $\Theta_\delta$ be an arbitrary subregion located in $\Theta$. It can be derived from Assumption 2 that

$$u[b_i(u)-b_i(0)] \ge l_i u^2.$$

Therefore,

$$\begin{aligned} &b_i(u) \le l_i u + b_i(0) \ \text{ for } \ u < 0,\\ &b_i(u) \ge l_i u + b_i(0) \ \text{for} \ u \ge 0. \end{aligned}$$

Then

$$\begin{aligned} &\lim_{u\to-\infty} b_i(u) = -\infty, \quad \lim_{u\to+\infty} b_i(u) = +\infty,\\ &\lim_{x\to-\infty} F_i^-(x) = +\infty, \quad \lim_{x\to+\infty} F_i^+(x) = -\infty. \end{aligned}$$

From inequalities (3), we have

$$\begin{aligned} &F_i(q_i^{(k)}) > 0,\ k = 1,2,\ldots,K_i,\\ &F_i(p_i^{(k)}) < 0,\ k = 0,1,2,\ldots,K_i-1. \end{aligned} \tag{4}$$

Denote $\overline{\Theta}_\delta$ as the closure of $\Theta_\delta$. We could assume $\overline{\Theta}_\delta = \prod_{i=1}^n [\alpha_i,\beta_i]$. From expressions (4), we can obtain $F_i(\alpha_i)F_i(\beta_i) < 0$. Hence, there exists $\hat{x}_i \in (\alpha_i,\beta_i)$ such that $F_i(\hat{x}_i) = 0$. $\overline{\Theta}_\delta$ is a compact convex set. Construct a continuous mapping $\mathcal{G} : \overline{\Theta}_\delta \to \overline{\Theta}_\delta$ such that $\mathcal{G}(x_1,x_2,\ldots,x_n) = (\hat{x}_1,\hat{x}_2,\ldots,\hat{x}_n)$. According to Brouwer's fixed point theorem, there exists at least one fixed point $x^*$ such that $\mathcal{G}(x^*) = x^*$ with $x^* = (x_1^*,x_2^*,\ldots,x_n^*)^{\mathrm{T}}$. Obviously, $x^*$ is the equilibrium point of FOCGNN (1) with time delays located in $\Theta_\delta$. Because of the arbitrariness of $\Theta_\delta$, there exist at least $\prod_{i=1}^n (2K_i+1)$ equilibrium points for FOCGNN (1) with time delays.

**Lemma 7** Assume Assumption 1–Assumption 4 and inequalities (3) hold. Any subregion of $\Theta^{(1)}$ is positively invariant.

**Proof** Let $\Theta_\kappa$ be an arbitrary subregion located in $\Theta^{(1)}$. We could assume $\Theta_\kappa = \prod_{i=1}^n (\rho_i,\chi_i)$. We state that the corresponding solution $x_i(t)$ $(t \ge 0)$ of FOCGNN (1) will stay in $(\rho_i,\chi_i)$ for the initial condition $\phi_i(s) \in (\rho_i,\chi_i)$, $s \in (-\infty,0]$. If this is not true, there exists some time $t_1 > 0$ such that $x_i(t_1)$ escapes from $(\rho_i,\chi_i)$. Without loss of generality, suppose that $x_i(t) \ge \chi_i$. Then

$$\begin{aligned} &x_i(t_1) = \chi_i,\\ &x_i(t) \in (\rho_i,\chi_i),\ t \in [0,t_1). \end{aligned}$$

According to Lemma 1, $D^\alpha x_i(t_1) > 0$. On the other hand,

$$\begin{aligned} D^\alpha x_i(t_1) &= a_i(x_i(t_1))F_i(x_i(t_1))\\ &\le a_i(\chi_i)F_i^+(\chi_i)\\ &< 0. \end{aligned}$$

This yields a contradiction to $D^\alpha x_i(t_1) > 0$. Similarly, the other case is either not true. Therefore, $x_i(t)$ will stay in $(\rho_i,\chi_i)$ for all $t > 0$. That is $\Theta_\kappa$ is positively invariant.

Next multiple Lagrange stability and Lyapunov asymptotical stability of FOCGNN (1) with time delays will be investigated.

**Theorem 1** Assume Assumption 1–Assumption 4 and expressions (3) hold. There exist $\prod_{i=1}^n (K_i+1)$ locally Lagrange stable equilibrium points for FOCGNN (1), if there exist constants $\omega_k > 0$ $(k = 1,2,\ldots,N_1)$, $\gamma_k > 0$ $(k = 1,2,\ldots,N_2)$, $\xi_{ij},\xi_{ij}^*,\delta_{ij},\delta_{ij}^*,\theta_{ij},\theta_{ij}^*,\varepsilon_{ij},\varepsilon_{ij}^* \in \mathcal{R}$ such that

$$\sigma_1 > \sigma_2 \tag{5}$$

holds, where

$$\begin{aligned} \sigma_1 = \min_{1\le i\le n}\Bigg\{ &r\underline{a}_i l_i - \overline{a}_i \sum_{j=1}^{n}\sum_{k=1}^{N_1} \omega_k |c_{ij}|^{r\xi_{ij}/\omega_k} L_j^{r\delta_{ij}/\omega_k}\\ &- \sum_{j=1}^{n} \overline{a}_j |c_{ji}|^{r\xi_{ji}^*} L_i^{r\delta_{ji}^*}\\ &- \overline{a}_i \sum_{j=1}^{n}\sum_{k=1}^{N_2} \gamma_k |d_{ij}|^{r\theta_{ij}/\gamma_k} L_j^{r\varepsilon_{ij}/\gamma_k} \Bigg\}, \end{aligned}$$

$\sigma_2 = \max_{1\le i\le n}\{\sum_{j=1}^n \overline{a}_j |d_{ji}|^{r\theta_{ji}^*} L_i^{r\varepsilon_{ji}^*}\}$, $r = \sum_{k=1}^{N_1}\omega_k + 1 = \sum_{k=1}^{N_2}\gamma_k + 1$, $N_1\xi_{ij} + \xi_{ij}^* = 1$, $N_1\delta_{ij} + \delta_{ij}^* = 1$, $N_2\theta_{ij} + \theta_{ij}^* = 1$, $N_2\varepsilon_{ij} + \varepsilon_{ij}^* = 1$, and $i,j = 1,2,\ldots,n$.

**Proof** According to Lemma 6 and Lemma 7, there exist $\prod_{i=1}^n (K_i+1)$ equilibrium points in $\Theta^{(1)}$. Let $\phi$ is an arbitrary subregion located in $\Theta^{(1)}$. Then $\phi$ is positively invariant. Suppose $x^*$ is the equilibrium point located in $\phi$, and $x^* = (x_1^*,x_2^*,\ldots,x_n^*)^{\mathrm{T}}$. Denote $e_i(t) = x_i(t) - x_i^*$. Denote the following function:

$$V(t) = \frac{1}{r}\sum_{i=1}^{n} |e_i(t)|^r.$$

Then

$$
\begin{aligned}
&D^{\alpha}V(t)\\
&=\sum_{i=1}^{n}|e_i(t)|^{r-1}sign(e_i(t))\{-a_i(x_i(t))[b_i(x_i(t))\\
&\quad-\sum_{j=1}^{n}c_{ij}f_j(x_j(t))-\sum_{j=1}^{n}d_{ij}f_j(x_j(t-\tau))-I_i]\}\\
&=\sum_{i=1}^{n}|e_i(t)|^{r-1}sign(e_i(t))\{-a_i(x_i(t))[(b_i(x_i(t))-b_i(x_i^*))\\
&\quad-\sum_{j=1}^{n}c_{ij}(f_j(x_j(t))-f_j(x_j^*)\\
&\quad-\sum_{j=1}^{n}d_{ij}(f_j(x_j(t-\tau))-f_j(x_j^*))]\}\\
&\le-\sum_{i=1}^{n}\underline{a}_i l_i|e_i(t)|^r+\sum_{i=1}^{n}\overline{a}_i\sum_{j=1}^{n}|c_{ij}|L_j|e_i(t)|^{r-1}|e_j(t)|\\
&\quad+\sum_{i=1}^{n}\overline{a}_i\sum_{j=1}^{n}|d_{ij}|L_j|e_i(t)|^{r-1}|e_j(t-\tau)|\\
&=\sum_{i=1}^{n}[-\underline{a}_i l_i|e_i(t)|^r\\
&\quad+\overline{a}_i\sum_{j=1}^{n}\prod_{k=1}^{N_1}(|c_{ij}|^{\xi_{ij}/\omega_k}L_j^{\delta_{ij}/\omega_k}|e_i(t)|)^{\omega_k}|c_{ij}|^{\xi^*_{ij}}L_j^{\delta^*_{ij}}|e_j(t)|\\
&\quad+\overline{a}_i\sum_{j=1}^{n}\prod_{k=1}^{N_2}(|d_{ij}|^{\theta_{ij}/\gamma_k}L_j^{\varepsilon_{ij}/\gamma_k}|e_i(t)|)^{\gamma_k}|d_{ij}|^{\theta^*_{ij}}L_i^{\varepsilon^*_{ij}}|e_j(t-\tau)|].
\end{aligned}
$$

According to Lemma 2,

$$
\begin{aligned}
&D^{\alpha}V(t)\\
&\le\sum_{i=1}^{n}\Bigg[-\underline{a}_i l_i|e_i(t)|^r+\overline{a}_i\sum_{j=1}^{n}\Bigg(\frac{1}{r}\sum_{k=1}^{N_1}\omega_k|c_{ij}|^{r\xi_{ij}/\omega_k}L_j^{r\delta_{ij}/\omega_k}|e_i(t)|^r\\
&\quad+\frac{1}{r}|c_{ij}|^{r\xi^*_{ij}}L_j^{r\delta^*_{ij}}|e_j(t)|^r\Bigg)\\
&\quad+\overline{a}_i\sum_{j=1}^{n}\Bigg(\frac{1}{r}\sum_{k=1}^{N_2}\gamma_k|d_{ij}|^{r\theta_{ij}/\gamma_k}L_j^{r\varepsilon_{ij}/\gamma_k}|e_i(t)|^r\\
&\quad+\frac{1}{r}|d_{ij}|^{r\theta^*_{ij}}L_j^{r\varepsilon^*_{ij}}|e_j(t-\tau)|^r\Bigg)\Bigg]\\
&=-\frac{1}{r}\sum_{i=1}^{n}[r\underline{a}_i l_i-\overline{a}_i\sum_{j=1}^{n}\sum_{k=1}^{N_1}\omega_k|c_{ij}|^{r\xi_{ij}/\omega_k}L_j^{r\delta_{ij}/\omega_k}\\
&\quad-\sum_{j=1}^{n}\overline{a}_j|c_{ji}|^{r\xi^*_{ji}}L_i^{r\delta^*_{ji}}-\overline{a}_i\sum_{j=1}^{n}\sum_{k=1}^{N_2}\gamma_k|d_{ij}|^{r\theta_{ij}/\gamma_k}L_j^{r\varepsilon_{ij}/\gamma_k}]|e_i(t)|^r\\
&\quad+\frac{1}{r}\sum_{i=1}^{n}(\sum_{j=1}^{n}\overline{a}_j|d_{ji}|^{r\theta^*_{ji}}L_i^{r\varepsilon^*_{ji}}|e_i(t-\tau)|^r)\\
&\le-\frac{1}{r}\sum_{i=1}^{n}\sigma_1|e_i(t)|^r+\frac{1}{r}\sum_{i=1}^{n}\sigma_2|e_i(t-\tau)|^r\\
&=-\sigma_1V(t)+\sigma_2V(t-\tau).
\end{aligned}
$$

From Lemma 3, we can obtain that

$$V(t)\le\frac{-\sigma_1}{-\sigma_1+\sigma_2}V(0)+M_0,$$

where $M_0=\sup_{-\tau\le\theta\le0}|\psi(\theta)|$, $|\psi(\theta)|=(1/r)\sum_{i=1}^{n}|x_i(\theta)-x_i^*|^r$, $\theta\in[-\tau,0]$. According to Definition 1, $x^*$ is a locally Lagrange-stable equilibrium point in $\phi$. Because of the arbitrariness of $\phi$, there exist $\prod_{i=1}^{n}(K_i+1)$ locally Lagrange-stable equilibrium points of FOCGNN (1) with time delays.

According to Theorem 1, every subregion of $\Theta^{(1)}$ (such as $\Lambda_1$, $\Lambda_2$, $\Lambda_3$, and $\Lambda_4$ in Fig. 1) has a locally Lagrange-stable equilibrium point. From the definition of $\Theta^{(1)}$ and Lemma 7, subregions of $\Theta^{(1)}$ are positively invariant and are separated from each other. The boundaries of these subregions can be seen as Lagrange boundaries between different stable equilibrium points, which are separated from each other. The neuron states in these subregions will converge to the Lagrange stable equilibrium points in the corresponding regions. In fact, there are some blank regions (such as $\Theta_1$, $\Theta_2$, $\Theta_3$, $\Theta_4$ and $\Xi$ in Fig. 1) between different invariant regions. Because of the high dimension and the existence of time delays, it is difficult to discuss the neural state trajectories in these blank regions. In the future work, we will further study the neural state trajectories in the blank regions and further determine the Lagrange boundaries between different stable equilibrium points.

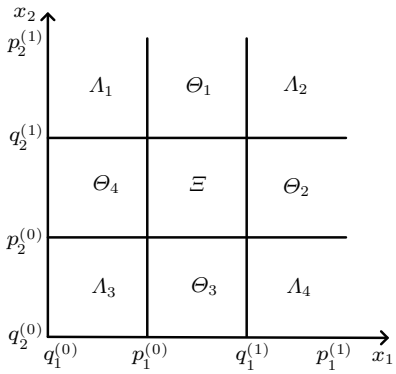



**Fig. 1.** Disassemble 2-dimensional space.

**Remark 1** In this paper, the activation functions are generally nonlinear, nonmonotonic, and have multiple corner points, which make FOCGNNs could have a large number of stable equilibrium points. In pattern recognition and associative memory, multistability of recurrent neural networks is an important problem. Each stable equilibrium point can be designed as one desired pattern. So a large number of stable equilibrium points correspond to high storage capacity. Obtaining large storage capacity is an important issue. Hence, our work is important and meaningful.

**Remark 2** The authors have concerned local Lagrange stability for delayed recurrent neural networks in Ref. [32], in which the activation functions were restricted bo be piecewise and linear. The activation functions used herein are more general and less conservative, which are nonmonotonic and nonlinear in any open interval. In fact, piecewise linear activation functions can be seen as special cases of general nonlinear activation functions in this paper.

**Theorem 2** Assume Assumption 1–Assumption 4 and expressions (3) hold. There exist $\prod_{i=1}^{n}(K_i+1)$ locally Lyapunov asymptotically stable equilibrium points for FOCGNN (1) if there exist constants $\omega_k > 0$ $(k = 1,2,\ldots,N_1)$, $\gamma_k > 0$ $(k = 1,2,\ldots,N_2)$, $\xi_{ij}$, $\xi_{ij}^*$, $\delta_{ij}$, $\delta_{ij}^*$, $\theta_{ij}$, $\theta_{ij}^*$, $\varepsilon_{ij}$, $\varepsilon_{ij}^* \in \mathcal{R}$, such that

$$\sigma_1 \sin\left(\frac{\alpha\pi}{2}\right) > \sigma_2 \tag{6}$$

holds, where

$$\begin{aligned}\sigma_1 = \min_{1\le i\le n}\Bigg\{ & r\underline{a}_i l_i - \overline{a}_i \sum_{j=1}^{n}\sum_{k=1}^{N_1} \omega_k |c_{ij}|^{r\xi_{ij}/\omega_k} L_j^{r\delta_{ij}/\omega_k} \\ & - \sum_{j=1}^{n} \overline{a}_j |c_{ji}|^{r\xi_{ji}^*} L_i^{r\delta_{ji}^*} \\ & - \overline{a}_i \sum_{j=1}^{n}\sum_{k=1}^{N_2} \gamma_k |d_{ij}|^{r\theta_{ij}/\gamma_k} L_j^{r\varepsilon_{ij}/\gamma_k} \Bigg\},\end{aligned}$$

$\sigma_2 = \max_{1\le i\le n}\{\sum_{j=1}^{n} \overline{a}_j |d_{ji}|^{r\theta_{ji}^*} L_i^{r\varepsilon_{ji}^*}\}$, $r = \sum_{k=1}^{N_1}\omega_k + 1 = \sum_{k=1}^{N_2}\gamma_k + 1$, $N_1\xi_{ij} + \xi_{ij}^* = 1$, $N_1\delta_{ij} + \delta_{ij}^* = 1$, $N_2\theta_{ij} + \theta_{ij}^* = 1$, $N_2\varepsilon_{ij} + \varepsilon_{ij}^* = 1$, and $i, j = 1,2,\ldots,n$.

**Proof** By the same definition and proof as the first half of Theorem 1, we can get

$$D^\alpha V(t) \le -\sigma_1 V(t) + \sigma_2 V(t-\tau).$$

Consider delayed fractional-order differential systems as follows:

$$D^\alpha W(t) = -\sigma_1 W(t) + \sigma_2 W(t-\tau), \tag{7}$$

where $W(t) \ge 0$. The initial conditions of $W(t)$ are the same with $V(t)$. We can get Laplace transformation of Eq. (7) that

$$\begin{aligned} & s^\alpha W(s) - s^{\alpha-1}W(0) \\ = & -\sigma_1 W(s) + \sigma_2 \mathrm{e}^{-s\tau} W(s) + \sigma_2 \mathrm{e}^{-s\tau}\int_{-\tau}^{0} \mathrm{e}^{-st} W(t)\,\mathrm{d}t, \\ & (s^\alpha + \sigma_1 - \sigma_2 \mathrm{e}^{-s\tau}) W(s) \\ = & s^{\alpha-1}W(0) + \sigma_2 \mathrm{e}^{-s\tau}\int_{-\tau}^{0} \mathrm{e}^{-st} W(t)\,\mathrm{d}t. \end{aligned}$$

Let $\Delta(s) = s^\alpha + \sigma_1 - \sigma_2 \mathrm{e}^{-s\tau}$, $\Gamma(s) = s^{\alpha-1}W(0) + \sigma_2 \mathrm{e}^{-s\tau}\int_{-\tau}^{0} \mathrm{e}^{-st} W(t)dt$. Then $\Delta(s)W(s) = \Gamma(s)$. We claim that the characteristic equation $\det(\Delta(s)) = \det(s^\alpha + \sigma_1 - \sigma_2 \mathrm{e}^{-s\tau}) = 0$ $(\tau > 0)$ has no pure imaginary roots. If this is not true, there exists one pure imaginary root $s = \mathrm{i}\zeta = |\zeta|[\cos(\pi/2) + \mathrm{i}\sin(\pm\pi/2)]$, where i is the imaginary unit. Substituting $s = \mathrm{i}\zeta$ into $\det(\Delta(s)) = 0$ gives

$$\begin{aligned} & |\zeta|^\alpha \left[\cos\left(\frac{\alpha\pi}{2}\right) + \mathrm{i}\sin\left(\pm\frac{\alpha\pi}{2}\right)\right] \\ & + \sigma_1 - \sigma_2[\cos(\zeta\tau) - \mathrm{i}\sin(\zeta\tau)] = 0. \end{aligned} \tag{8}$$

Therefore, we have

$$|\zeta|^\alpha \cos\left(\frac{\alpha\pi}{2}\right) + \sigma_1 = \sigma_2 \cos(\zeta\tau), \tag{9}$$

$$|\zeta|^\alpha \sin\left(\pm\frac{\alpha\pi}{2}\right) = -\sigma_2 \sin(\zeta\tau). \tag{10}$$

We can get from Eqs. (9) and (10) that

$$|\zeta|^{2\alpha} + 2\sigma_1|\zeta|^\alpha \cos\left(\frac{\alpha\pi}{2}\right) + \sigma_1^2 - \sigma_2^2 = 0. \tag{11}$$

The discriminant of Eq. (6) is as follows:

$$\begin{aligned}\Delta &= \left[2\sigma_1 \cos\left(\frac{\alpha\pi}{2}\right)\right]^2 - 4(\sigma_1^2 - \sigma_2^2) \\ &= 4\left[\sigma_2^2 - \sigma_1^2 \sin^2\left(\pm\frac{\alpha\pi}{2}\right)\right].\end{aligned}$$

According to condition (11), we have $\Delta < 0$. That is, $\det(\Delta(s)) = 0$ has no pure imaginary roots. Next we claim $|\arg(\overline{\lambda})| > \pi/2$ for all the eigenvalues $\overline{\lambda}$ of $\sigma_2 - \sigma_1$. In fact,

$$\sigma_2 < \sigma_1 \sin(\frac{\alpha\pi}{2}) < \sigma_1.$$

Therefore, the eigenvalues of $\sigma_2 - \sigma_1$ are negative. According to Lemma 4, the zero solution of system (7) is locally Lyapunov asymptotically stable. According to Lemma 5,

$$0 < V(t) \le W(t), \forall t \in [0, +\infty).$$

So $V(t)$ is also locally Lyapunov asymptotically stable. Therefore, $x^*$ is a locally Lyapunov asymptotically stable equilibrium point in $\phi$. Because of arbitrariness of $\phi$, there exist $\prod_{i=1}^{n}(K_i+1)$ locally Lyapunov asymptotically stable equilibrium points.

**Remark 3** In the proofs of Theorem 1 and Theorem 2, we have defined some local Lyapunov functions to deal with the stability of different equilibrium points of neural networks. These Lyapunov functions correspond to different invariant regions. We can synthesize a piecewise continuous Lyapunov function from these local Lyapunov functions as follows:

$$V(t) = \frac{1}{r}\sum_{i=1}^{n} |e_i(t)|^r,$$

where the definitions of parameters are the same as those in Theorem 1. Because we do not study the global stability of some one equilibrium point in global space, we don't construct a Lyapunov function for the whole space. Because of the existences of some not discussed regions between the invariant regions, high dimensions and time delays of neural networks, it is not easy to synthesize a continuous Lyapunov function for the whole space to investigate local stability of equilibrium points. In the future, we will study how to construct a Lyapunov function for the whole space to discuss the local stability of neural networks.

**Remark 4** Multiple Mittag–Leffler stability of FONNs without time delays has been investigated in Ref. [30]. The FONN models could be seen as special cases of delayed

FONN models in this paper. Meanwhile, the authors have pointed out that it was necessary to consider the influence of time delays on FONNs. And it was a problem to be solved to build or search for effective methods to investigate multistability of FONNs with time delays.[30] In this paper, the time delays have been considered, and some relevant methods have been introduced to discuss the multistability of FOCGNNs with time delays.

**Remark 5** Multiple $\psi$-type stability of delayed CGNNs has been investigated in Ref. [33], where the neural network models were integer-order. In fact, the existing methods for studying the multistability of integer-order CGNNs are not suitable for the discussion of the multistability of FOCGNNs. It is necessary to introduce new methods to discuss the stability of FOCGNNs. So Lemmas 1–5 are necessary to discuss the stability of delayed FOCGNNs. In Refs. [48,52], the authors have discussed the existence of equilibrium points of recurrent neural networks. The neural network models were still integer-order. Noting that fractional-order systems provide better hereditary and memory properties than integer-order ones, so the results herein are more superior.

**Remark 6** In Theorem 1 and Theorem 2, it has been proved that there exist $\prod_{i=1}^{n}(K_i+1)$ locally Lagrange and Lyapunov asymptotically stable equilibrium points for FOCGNN (1). In previous results of Refs. [21,27], there were $2^n$ locally stable equilibrium points for recurrent neural networks. It can be seen that this paper has more number of locally stable equilibrium points when $K_i \geq 2$. It indicates that our work is also effective to improve the storage capacity, which is an improvement of the results in Refs. [21,27].

It can be noted that the above results can be generated in the following integral-order CGNNs:

$$\begin{aligned}\dot{x}_i(t) = &-a_i(x_i(t))[b_i(x_i(t)) - \sum_{j=1}^{n} c_{ij} f_j(x_j(t)) \\ &- \sum_{j=1}^{n} d_{ij} f_j(x_j(t-\tau)) - I_i],\end{aligned} \tag{12}$$

where $i = 1,2,\ldots,n$. According to the similar discussion of the above results, we can obtain some corollaries.

**Corollary 1** Assume that Assumption 1–Assumption 4 and expressions (3) hold. There exist $\prod_{i=1}^{n}(K_i+1)$ locally Lagrange-stable equilibrium points for CGNN (12), if there exist constants $\omega_k > 0$ $(k = 1,2,\ldots,N_1)$, $\gamma_k > 0$ $(k = 1,2,\ldots,N_2)$, $\xi_{ij}, \xi_{ij}^*, \delta_{ij}, \delta_{ij}^*, \theta_{ij}, \theta_{ij}^*, \varepsilon_{ij}, \varepsilon_{ij}^* \in \mathcal{R}$ such that

$$\sigma_1 > \sigma_2 \tag{13}$$

holds, where

$$\begin{aligned}\sigma_1 = \min_{1\leq i\leq n}\Bigg\{ & r\underline{a}_i l_i - \overline{a}_i \sum_{j=1}^{n}\sum_{k=1}^{N_1} \omega_k |c_{ij}|^{r\xi_{ij}/\omega_k} L_j^{r\delta_{ij}/\omega_k} \\ & - \sum_{j=1}^{n} \overline{a}_j |c_{ji}|^{r\xi_{ji}^*} L_i^{r\delta_{ji}^*} \\ & - \overline{a}_i \sum_{j=1}^{n}\sum_{k=1}^{N_2} \gamma_k |d_{ij}|^{r\theta_{ij}/\gamma_k} L_j^{r\varepsilon_{ij}/\gamma_k} \Bigg\},\end{aligned}$$

$\sigma_2 = \max_{1\leq i\leq n}\{\sum_{j=1}^{n} \overline{a}_j |d_{ji}|^{r\theta_{ji}^*} L_i^{r\varepsilon_{ji}^*}\}$, $r = \sum_{k=1}^{N_1}\omega_k + 1 = \sum_{k=1}^{N_2}\gamma_k + 1$, $N_1\xi_{ij} + \xi_{ij}^* = 1$, $N_1\delta_{ij} + \delta_{ij}^* = 1$, $N_2\theta_{ij} + \theta_{ij}^* = 1$, $N_2\varepsilon_{ij} + \varepsilon_{ij}^* = 1$, and $i, j = 1,2,\ldots,n$.

**Corollary 2** Assume Assumption 1–Assumption 4 and expressions (3) hold. There exist $\prod_{i=1}^{n}(K_i+1)$ locally Lyapunov asymptotically stable equilibrium points for FOCGNN (12) if there exist constants $\omega_k > 0$ $(k = 1,2,\ldots,N_1)$, $\gamma_k > 0$ $(k = 1,2,\ldots,N_2)$, $\xi_{ij}$, $\xi_{ij}^*$, $\delta_{ij}$, $\delta_{ij}^*$, $\theta_{ij}$, $\theta_{ij}^*$, $\varepsilon_{ij}$, $\varepsilon_{ij}^* \in \mathcal{R}$ such that

$$\sigma_1 \sin\left(\frac{\alpha\pi}{2}\right) > \sigma_2 \tag{14}$$

holds, where

$$\begin{aligned}\sigma_1 = \min_{1\leq i\leq n}\Bigg\{ & r\underline{a}_i l_i - \overline{a}_i \sum_{j=1}^{n}\sum_{k=1}^{N_1} \omega_k |c_{ij}|^{r\xi_{ij}/\omega_k} L_j^{r\delta_{ij}/\omega_k} \\ & - \sum_{j=1}^{n} \overline{a}_j |c_{ji}|^{r\xi_{ji}^*} L_i^{r\delta_{ji}^*} \\ & - \overline{a}_i \sum_{j=1}^{n}\sum_{k=1}^{N_2} \gamma_k |d_{ij}|^{r\theta_{ij}/\gamma_k} L_j^{r\varepsilon_{ij}/\gamma_k} \Bigg\},\end{aligned}$$

$\sigma_2 = \max_{1\leq i\leq n}\{\sum_{j=1}^{n} \overline{a}_j |d_{ji}|^{r\theta_{ji}^*} L_i^{r\varepsilon_{ji}^*}\}$, $r = \sum_{k=1}^{N_1}\omega_k + 1 = \sum_{k=1}^{N_2}\gamma_k + 1$, $N_1\xi_{ij} + \xi_{ij}^* = 1$, $N_1\delta_{ij} + \delta_{ij}^* = 1$, $N_2\theta_{ij} + \theta_{ij}^* = 1$, $N_2\varepsilon_{ij} + \varepsilon_{ij}^* = 1$, and $i, j = 1,2,\ldots,n$.

**Remark 7** On the basis of studying the multistability of recurrent neural networks, the authros in Refs. [24] and [29] have also investigated the existence of unstable equilibrium points, analyzed the trajectory of neuron states in the subregions of these unstable equilibrium points, and established larger domains of attraction of stable equilibrium points than invariant subregions. Because of the existence of fractional order, it is complex to establish larger domains of attraction of stable equilibrium points of FOCGNNs (1). In the future, we will study this problem.

**Remark 8** In the application of pattern recognition, increase of the number of equilibrium points does not affect the accuracy of the individual pattern design. When the number of equilibrium points increases, the domains of attraction of stable equilibrium points may become smaller. But the initial states in the domains of attraction will converge to the corresponding stable equilibrium points. The result of recognition is accurate.

**Remark 9** Condition (3) is a very important condition to ensure the existence of multiple local stable equilibrium points for FOCGNNs (1). Condition (3) is some inequalities composed of connection weight, activation function, external input and other parameters of neural network model. The existence of inequality can guarantee the robustness of our method, which allows the parameters of neural networks to fluctuate in a certain range. As long as the condition (3) is satisfied, FOCGNNs (1) will have multiple local stable equilibrium points under some additional conditions.

## 4. Illustrative examples

Now, we give two numerical examples to verify the validity of the main results.

**Example 1** Discuss the following 2-neuronal FOCGNN:

$$\begin{aligned}
&D^{0.9}x_1(t)\\
=\ &-(2+\cos(x_1(t)))[1.2x_1(t)-2.7f_1(x_1(t))-0.1f_2(x_2(t))\\
&-2.3f_1(x_1(t-1))-0.1f_2(x_2(t-1))-1],\\
&D^{0.9}x_2(t)\\
=\ &-(2+\sin(x_2(t)))[2x_1(t)-0.2f_1(x_1(t))-3.2f_2(x_2(t))\\
&-0.2f_1(x_1(t-1))-2.8f_2(x_2(t-1))-1], \qquad (15)
\end{aligned}$$

where $f_1(\cdot)=f_2(\cdot)=f(\cdot)=\tanh(\cdot)$.

It is easy to verify that Assumptions Assumption 1–Assumption 4 hold. The parameters of Assumption 1–Assumption 4 are chosen as follows: $\bar{a}_i=3$, $\underline{a}_i=1$, $l_1=1.2$, $l_2=2$, $m_i=-1$, $M_i=1$ for $i=1,2$.

The bounding functions are

$$\begin{aligned}
F_1^-(x)&=-1.2x+5\tanh(x)+0.8,\\
F_1^+(x)&=-1.2x+5\tanh(x)+1.2,\\
F_2^-(x)&=-2x+6\tanh(x)+0.6,\\
F_2^+(x)&=-2x+6\tanh(x)+1.4.
\end{aligned}$$

Choose $p_i^{(0)}=-2$, $q_i^{(1)}=2$, $L_i=0.0707$ for $i=1,2$. Then condition (3) holds. Choose $N_1=N_2=1$, $\omega_1=0.5$, $\gamma_1=0.5$, $r=1.5$ and

$$\begin{array}{llll}
\xi_{11}=1, & \xi_{12}=0.5, & \xi_{21}=0.44, & \xi_{22}=1,\\
\xi_{11}^*=0, & \xi_{12}^*=0.5, & \xi_{21}^*=0.56, & \xi_{22}^*=0,\\
\delta_{11}=0.55, & \delta_{12}=0.6, & \delta_{21}=0.36, & \delta_{22}=0.5,\\
\delta_{11}^*=0.45, & \delta_{12}^*=0.4, & \delta_{21}^*=0.64, & \delta_{22}^*=0.5,\\
\theta_{11}=1, & \theta_{12}=0.36, & \theta_{21}=0.35, & \theta_{22}^*=1,\\
\theta_{11}^*=0, & \theta_{12}^*=0.64, & \theta_{21}^*=0.65, & \theta_{22}^*=0,\\
\varepsilon_{11}=0.48, & \varepsilon_{12}=0.48, & \varepsilon_{21}=0.35, & \varepsilon_{22}=0.45,\\
\varepsilon_{11}^*=0.52, & \varepsilon_{12}^*=0.52, & \varepsilon_{21}^*=0.65, & \varepsilon_{22}^*=0.55.
\end{array}$$

Then condition (5) holds. That is, conditions of Theorem 1 are satisfied. There exist 4 locally Lagrange-stable equilibrium points for FOCGNN (15). The dynamics of $(x_1,x_2)$ for FOCGNN (15) with 120 initial values are illustrated in Fig. 2. The dynamics of $x_1$ and $x_2$ with 40 initial values are illustrated in Figs. 3 and 4. The effectiveness of the theoretical results has been verified.

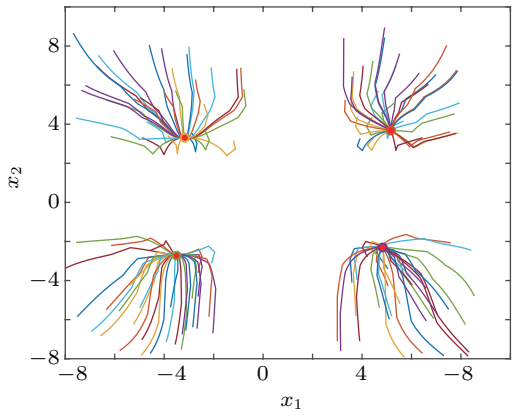



**Fig. 2.** Transient states of FOCGNN (15).

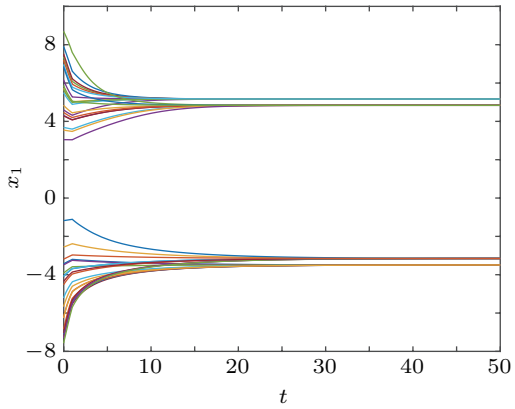



**Fig. 3.** Dynamics of $x_1(t)$ of FOCGNN (15).

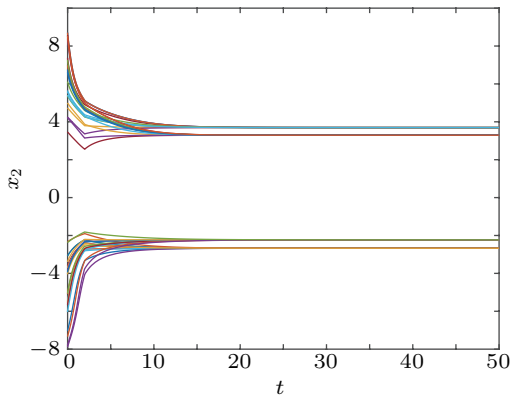



**Fig. 4.** Dynamics of $x_2(t)$ of FOCGNN (15).

**Example 2** Consider the following 2-neuronal FOCGNN:

$$\begin{aligned}
&D^{0.9}x_1(t)\\
=\ &-(2.4+\sin(x_1(t)))[2.1x_1(t)-2.1f(x_1(t))-0.1f(x_2(t))\\
&-2.9f(x_1(t-1))-0.3f(x_2(t-1))-0.05],\\
&D^{0.9}x_2(t)\\
=\ &-(2+\cos(x_2(t)))[2x_1(t)-0.1f(x_1(t))-2.2f(x_2(t))\\
&-0.2f(x_1(t-1))-2.8f(x_2(t-1))-0.4], \qquad (16)
\end{aligned}$$

where the activation function $f(x) = (1/2)(|x+1| - |x-1|) - (|x+4| - |x-4|) + (|x+7| - |x-7|)$.

It is easy to verify that Assumptions 1–4 hold. The parameters of Assumption 1–Assumption 4 are chosen as follows: $\overline{a}_1 = 3.4$, $\underline{a}_1 = 1.4$, $\overline{a}_2 = 3$, $\underline{a}_2 = 1$, $l_1 = 2.1$, $l_2 = 2$, $m_i = -7$, $M_i = 7$ for $i = 1,2$. Choose $q_i^{(0)} = -\infty$, $p_i^{(0)} = -7$, $q_i^{(1)} = -4$, $p_i^{(1)} = -1$, $q_i^{(2)} = 1$, $p_i^{(2)} = 4$, $q_i^{(3)} = 7$, $p_i^{(3)} = +\infty$, $L_i = 0$ for $i = 1,2$. Then condition (3) holds. Because $L_i = 0$, condition (6) holds. That is, conditions of Theorem 2 are satisfied. There exist 16 locally Lyapunov asymptotically equilibrium points for FOCGNN (16). The dynamics of $(x_1, x_2)$ for FOCGNN (16) with 160 initial values are illustrated in Fig. 5. The dynamics of $x_1$ and $x_2$ with 160 initial values are illustrated in Figs. 6–7. The effectiveness of the theoretical results has been verified.

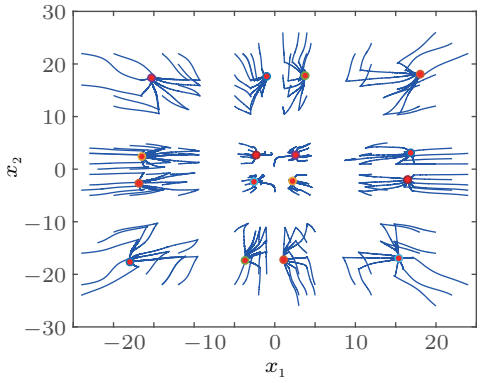



**Fig. 5.** Transient states of system (16).

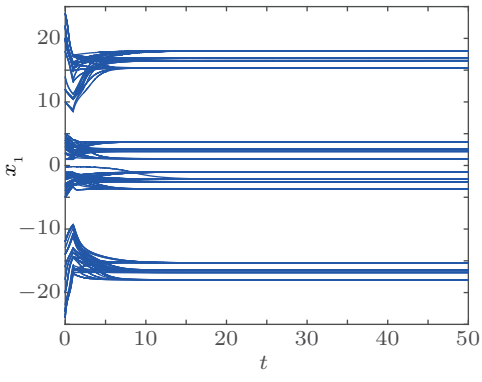



**Fig. 6.** Dynamics of $x_1(t)$ of FOCGNN (16).

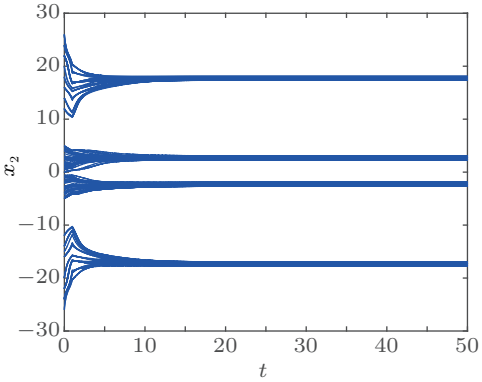



**Fig. 7.** Dynamics of $x_2(t)$ of FOCGNN (16).

## 5. Conclusion

In this paper, we discussed the multistability of FOCGNNs with time delays. By using decomposition of state space, Brouwer's fixed point theorem, Hardy inequality, fractional Halanay inequality, and Lyapunov theory, sufficient conditions have been derived to ensure the existence of $\prod_{i=1}^{n}(K_i+1)$ locally Lagrange-stable and locally Lyapunov asymptotically stable equilibrium points. The neural network models of this paper are general, which encompass some conventional models as special cases. The activation functions are generally nonlinear, nonmonotonic, and have multiple corner points, which make FOCGNNs could have a larger number of stable equilibrium points. Each stable equilibrium point can be designed as one desired pattern. So a large number of stable equilibrium points correspond to high storage capacity. Increasing storage capacity is a fundamental problem. Hence, our work is important and meaningful.

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