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Effect of system–reservoir correlations on temperature estimation*

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In many previous temperature estimation schemes, the temperature of a sample is directly read out from the final steady state of a quantum probe, which is coupled to the sample. However, in these studies, information of correlations between system (the probe) and reservoir (the sample) is usually eliminated, leading the steady state of the probe is a canonical equilibrium state with respect solely to system's Hamiltonian. To explore the influence of system–reservoir correlations on the estimation precision, we investigate the equilibration dynamics of a spin interacting with a finite temperature bosonic reservoir. By incorporating an intermediate harmonic oscillator or a collective coordinate into the spin, the system–reservoir correlations can be correspondingly encoded in a Gibbs state of an effective Hamilton, which is size consistent with the original bare spin. Extracting information of temperature from this corrected steady state, we find the effect of the system-reservoir correlations on the estimation precision is highly sensitive to the details of the spectral density function of the measured reservoir.

Keywords: quantum Fisher information, spin-boson model, system–reservoir correlations, temperature sensing

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1. Introduction

Temperature sensing lies at the heart of the thermodynamics of quantum devices, which has many potential applications in microelectronics,[1] biochemistry,[2] and material science.[3,4] How to achieve an accurate measurement of unknown temperature of a heat reservoir has attracted much attention over the last few years.[5–10] The most simple and straightforward strategy is to contact the heat reservoir with an individual quantum probe, which acts as a thermometry. Due to the probe–reservoir coupling, the message of reservoir's temperature is labeled in the quantum state of the probe and can be extracted out directly. Commonly, the steady state of the probe is utilized to draw out the information of temperature. The above scheme has been extensively studied in literature,[5–15] for example, quantum dot is used as the temperature probe to accurately estimate the temperature of both fermionic[12,13] and bosonic[14,15] reservoirs.

Physically speaking, the probe and the heat reservoir comprise a so-called quantum dissipative system,[16] which is a prominent physical model to simulate relaxation and decoherence in a noisy environment. In this sense, one can gain additional insight into the problem of quantum thermometry from a quantum dissipative system perspective. Unfortunately, the exact expression of the final steady-state of the system (probe), namely $\hat{\rho}_{\rm ss} = {\rm Tr}_{\rm reservoir}[\hat{\rho}_{\rm tot}(t\to\infty)]$, is generally difficult to obtain. To derive a tractable result, a series of assumptions, for example, the weak system–reservoir coupling condition and the Markovian approximation, are employed in many previous studies.[5,7,8] Under these restrictions, $\hat{\rho}_{\rm ss}$ can be approximated as a canonical equilibrium state with respect solely to the system's Hamiltonian, if the system–reservoir interaction time is sufficiently long. The above process, *i.e.*, thermalization from a given initial state to a canonical equilibrium state, can be alternatively achieved by using the hypothesis of ergodicity in statistical mechanics, suggesting ensemble and time averages are equivalent. Consequently, many of previous temperature estimating schemes neglect the influence of system–reservoir correlations, which shall modify the canonical equilibrium state. Several studies[17–21] have pointed out that the system–reservoir correlations play a significant role in strong system-reservoir coupling regimes, resulting in the occurrence of noncanonical distribution properties. Thus, a more precise temperature measuring scheme should take system–reservoir correlations into account.

On the other hand, in quantum metrology theory, the fundamental limitation of an arbitrary parameter estimation is bounded by the Cramér–Rao theorem: for a family of quantum state $\hat{\rho}(\theta)$, carrying a unknown parameters θ, the estimating precision of θ from the quantum state $\hat{\rho}(\theta)$ is theoretically determined by[22]

$$\mathrm{var}(\theta) \geq \frac{1}{\upsilon \mathcal{F}(\theta)}, \tag{1}$$

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where $\mathrm{var}(\theta)$ is the variance of parameter θ, parameter υ represents the number of repeated measurements, and $\mathcal{F}(\theta)$ denotes the quantum Fisher information (QFI) of θ with respect to the quantum state $\hat{\rho}(\theta)$. By diagonalizing the quantum state $\hat{\rho}(\theta)$ as $\hat{\rho}(\theta)=\sum_i \varepsilon_i|\psi_i\rangle\langle\psi_i|$, the QFI with arbitrary ranks can be computed as[23,24]

$$\mathcal{F}(\theta)=\sum_i \frac{(\partial_\theta \varepsilon_i)^2}{\varepsilon_i}+\sum_i 4\varepsilon_i\langle\partial_\theta\psi_i|\partial_\theta\psi_i\rangle -\sum_{i,j}\frac{8\varepsilon_i\varepsilon_j}{\varepsilon_i+\varepsilon_j}|\langle\psi_i|\partial_\theta\psi_j\rangle|^2, \tag{2}$$

where $\partial_\theta \equiv \partial/\partial\theta$ denotes partial derivative. From Eq. (1), one can find that, with a fixed number of measurements, the ultimate precision of a parameter sensing is completely decided by the QFI: more larger the value of QFI, more higher estimation precision is. How to improve the value of QFI is the crucial problem in quantum metrology.[24]

In many previous temperature sensing schemes,[5–10] the temperature is read out from the Gibbs state of the system (probe), namely $\hat{\rho}=\mathcal{Z}^{-1}\mathrm{e}^{-\beta\hat{H}}$, where $\mathcal{Z}\equiv\mathrm{Tr}(\mathrm{e}^{-\beta\hat{H}})$ denotes the partition function and $\beta\equiv T^{-1}$ denotes inverse temperature (we set the Boltzmann constant $k_\mathrm{B}=1$ through this paper). The QFI of temperature with respect to the above Gibbs state can be exactly evaluated, and the result is given by[5]

$$\mathcal{F}(T)=\frac{\mathrm{var}(\hat{H})}{T^4}, \tag{3}$$

where $\mathrm{var}(\hat{H})\equiv\langle\hat{H}^2\rangle-\langle\hat{H}\rangle^2$ with $\langle\hat{H}\rangle\equiv\mathrm{Tr}(\hat{\rho}\hat{H})$. Equation (3) indicates that the QFI is proportional to the thermal fluctuation of the system's energy. By introducing the definition of heat capacity as $C(T)\equiv\partial_T\langle\hat{H}\rangle$, one can find the QFI is related to the heat capacity $\mathcal{F}(T)=C(T)/T^2$. In this sense, the QFI is closely associated with an experimentally measurable quantity, suggesting a experimental scheme to retrieve QFI by detecting heat capacities in bulk systems.

In this paper, we investigate the effect of system–reservoir correlations on the precision of temperature estimation in a spin-boson (SB) model with arbitrary spin sizes, in which a spin acts as the probe to measure the temperature of a bosonic reservoir, by making use of the reaction coordinate mapping method.[20,21,25–27] The reaction coordinate mapping, which was originally proposed by Garg *et al.* in Ref. [25], can exactly convert the conventional SB model into an equivalent model without any approximations and has been widely applied in studies of electron transfer in biomolecules[25] as well as heat transport in a nonequilibrium environment.[28] As shown in Fig. 1, in this equivalent model, instead of interacting with the bosonic reservoir directly, the spin couples to an intermediate harmonic oscillator (IHO), which is embedded into a residual bosonic reservoir in turn. Compared with the conventional SB model, the "system's part" of the mapped spin-IHO-boson (SIB) model includes system–environment correlations, because parts of environmental information (reflected by the spin-IHO coupling) are incorporated into the mapped system's Hamiltonian. In Ref. [21], the authors have demonstrated that the robust system–reservoir correlations can be even persisted into equilibrium state of the mapped system. Therefore, the long-lived system–reservoir correlations lead to a clear department from the canonical equilibration dynamics of a quantum dissipative system.[20,21] By tracing out the degrees of freedom of the IHO, a corrected steady state of the spin can be obtained. We use this corrected state to compute the QFI and explore the influence of system-reservoir correlations on the estimation precision.

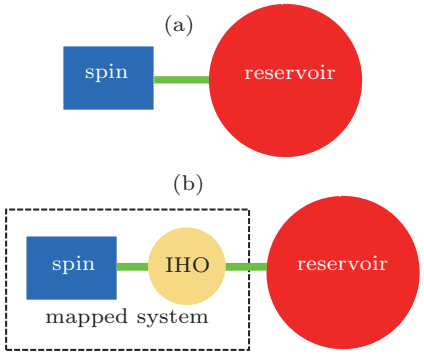


Fig. 1. (a) Original scheme: a spin directly coupled to a bosonic reservoir. (b) Our scheme: a spin interacting with an IHO, which is in turn embedded in a residual bosonic reservoir.

2. Reaction coordinate mapping

The full SB model with arbitrary spin sizes considered in our temperature estimation scheme is given by (we adopt the unit $\hbar=1$ throughout the paper)

$$\hat{H}_\mathrm{sb}=\varepsilon\hat{J}_z+\sum_k\omega_k\hat{b}_k^\dagger\hat{b}_k+\hat{J}_x\sum_k g_k(\hat{b}_k^\dagger+\hat{b}_k), \tag{4}$$

where $\hat{J}_{x,z}$ are components of an arbitrary spin vector $\boldsymbol{J}$ with size J, the parameter ε stands for the level splitting. Operator $\hat{b}_k^\dagger$ ($\hat{b}_k$) is the bosonic creation (annihilation) operator of the k-th environmental mode with frequency ω_k. Parameter g_k quantifies the coupling strength between the spin and the k-th environmental mode. Commonly, the frequency dependence of the system–reservoir interaction strengths are fully characterized by the spectral density function, which is introduced by $\mathcal{J}_\mathrm{sb}(\omega)\equiv\sum_k g_k^2\delta(\omega-\omega_k)$.

To take system–reservoir correlations into account, a transformation is applied to the original Hamiltonian $\hat{H}_\mathrm{sb}$ to formulate an exactly equivalent SIB model. Following the procedure outlined by Garg *et al.* in Ref. [25], one can obtain the equivalent system, in which, the spin interacts with a collective coordinate of reservoir, known as an IHO[26,27] or a

reaction coordinate,[20,21] which is in turn coupled to the residual bosonic reservoir. The Hamiltonian of this equivalent SIB model is given by[20,21,25–27]

$$\hat{H}_{\rm sib} = \hat{H}_{\rm sys} + \hat{H}_{\rm b} + \hat{H}_{\rm int}, \tag{5}$$

where

$$\hat{H}_{\rm sys} = \varepsilon \hat{J}_z + \Omega \hat{a}^\dagger \hat{a} + \lambda \hat{J}_x (\hat{a}^\dagger + \hat{a}), \tag{6}$$

$$\hat{H}_{\rm b} = \sum_k \tilde{\omega}_k \hat{c}_k^\dagger \hat{c}_k + (\hat{a}^\dagger + \hat{a})^2 \sum_k \frac{\tilde{g}_k^2}{\tilde{\omega}_k}, \tag{7}$$

$$\hat{H}_{\rm int} = (\hat{a}^\dagger + \hat{a}) \sum_k \tilde{g}_k (\hat{c}_k^\dagger + \hat{c}_k). \tag{8}$$

Here, operator $\hat{a}^\dagger$ ($\hat{a}$) is the creation (annihilation) operator of the IHO with frequency Ω, and λ accounts for the coupling strength between the spin and the IHO. The k-th residual environmental mode associated with frequency $\tilde{\omega}_k$, denoted by creation (annihilation) operator $\hat{c}_k^\dagger$ ($\hat{c}_k$), now only couples to the IHO with coupling strength $\tilde{g}_k$. The spectral density function of the residual bosonic reservoir is characterized by $\mathcal{J}_{\rm sib}(\omega) = \sum_k \tilde{g}_k^2 \delta(\omega - \tilde{\omega}_k)$. Note that there is a counter term in Eq. (7), which is related to the choice of the spectral density function. When deriving a quantum master equation of the SIB model, integral divergences caused by this counter term can be eliminated by using the principal value integral method.[21]

As pointed out in Refs. [20,21,26], the map from Eq. (4) to Eq. (5) does not involve the spin, *i.e.*, the spin is unaffected by the transformation. Thus, one can replace the spin with a classical coordinate, which moves in a potential. By comparing the Fourier transformed equations of motion for the classical coordinate before and after the reaction coordinate mapping, one can connect the spectral density function $\mathcal{J}_{\rm sib}(\omega)$ to the original spectrum $\mathcal{J}_{\rm sb}(\omega)$, see Appendix A for more details. If $\mathcal{J}_{\rm sib}(\omega)$ has an Ohmic form, *i.e.*, $\mathcal{J}_{\rm sib}(\omega) = \gamma\omega$, the spectral density function of its equivalent SB model is described by a Lorentzian structured spectrum, it reads[20,21]

$$\mathcal{J}_{\rm sb}(\omega) = \frac{2\alpha\omega\Omega^4}{(\Omega^2 - \omega^2)^2 + (\Gamma\omega)^2}. \tag{9}$$

The above spectral density function has a Lorentzian peak of width $\Gamma = 2\pi\gamma\Omega$ at the characteristic frequency $\omega = \Omega$, and it behaves linearly in low frequency regime with a dimensionless coupling strength $\alpha = 2\gamma\lambda^2/\Omega^2$. The reduced dynamics of the SB model with the structured Lorentzian spectral density function given in Eq. (9) exhibits a rich set of relaxation and decoherence behaviors, at both zero[29] and finite temperature.[26,27,30]

In Refs. [20,21], the authors have studied the reduced dynamics (tracing out the degrees of freedom of the residual bosonic reservoir) of the SIB model by a numerically exact hierarchical equations of motion[31,32] as well as a perturbative quantum master equation approach. Both methods showed that the final steady state can be faithfully described by the Gibbs state of the mapped spin-IHO Hamiltonian $\hat{H}_{\rm sys}$. Therefore, eliminating the degrees of freedom of the IHO, a corrected steady state of the spin can be extracted from the canonical equilibrium state of $\hat{H}_{\rm sys}$, namely,

$$\hat{\rho}_{\rm ss} = {\rm Tr}_{\rm i}\left(\frac{{\rm e}^{-\beta\hat{H}_{\rm sys}}}{\mathcal{Z}}\right), \tag{10}$$

where $\mathcal{Z} = {\rm Tr}_{\rm si}({\rm e}^{-\beta\hat{H}_{\rm sys}})$. We argue that the steady state given by Eq. (10) contains system–reservoir correlations, compared with the Gibbs state with respect to the bare spin's Hamiltonian.

3. Effective Hamiltonian

The corrected steady state in Eq. (10) can be numerically evaluated. However, to obtain a more clear physical picture, we prefer an analytical approach. Motivated by Refs. [33–35], we find $\hat{\rho}_{\rm ss}$ can be cast in a form of canonical equilibrium state with respect to an effective Hamiltonian, which preserves the same size with the original bare spin.

To this aim, we first separate $\hat{H}_{\rm sys}$ into two parts $\hat{H}_{\rm sys} = \hat{H}_0 + \hat{H}_1$, where $\hat{H}_0 = \Omega\hat{a}^\dagger\hat{a}$ and $\hat{H}_1 = \varepsilon\hat{J}_z + \lambda\hat{J}_x(\hat{a}^\dagger + \hat{a})$. The next key step is the employment of the following identity[33–35]

$${\rm e}^{-\beta(\hat{H}_0+\hat{H}_1)} = {\rm e}^{-\beta(\hat{L}_0+\hat{H}_1)}\,{\rm e}^{-\beta\hat{H}_0}, \tag{11}$$

where $\hat{L}_0$ is the Liouvillian operator associated with $\hat{H}_0$: for an arbitrary operator $\hat{O}$, the Liouvillian operator $\hat{L}_0$ is defined though the commutator $\hat{L}_0\hat{O} = \hat{H}_0\hat{O} - \hat{O}\hat{H}_0$. With the help of the identity in Eq. (11), the partition function $\mathcal{Z}$ can be rewritten as

$$\begin{aligned}\mathcal{Z} &= {\rm Tr}_{\rm si}\left[{\rm e}^{-\beta(\hat{L}_0+\hat{H}_1)}\,{\rm e}^{-\beta\hat{H}_0}\right] \\ &= \mathcal{Z}_0 {\rm Tr}_{\rm si}\left[{\rm e}^{-\beta(\hat{L}_0+\hat{H}_1)}\frac{{\rm e}^{-\beta\hat{H}_0}}{\mathcal{Z}_0}\right] \\ &= \mathcal{Z}_0 {\rm Tr}_{\rm s}\left[\langle {\rm e}^{-\beta(\hat{L}_0+\hat{H}_1)}\rangle_0\right] \\ &= \mathcal{Z}_0\mathcal{Z}_{\rm eff},\end{aligned} \tag{12}$$

where $\mathcal{Z}_0 \equiv {\rm Tr}_{\rm i}({\rm e}^{-\beta\hat{H}_0})$, and the symbol $\langle\hat{O}\rangle_0$ is defined as the ensemble averaging with respect to the Gibbs state of $\hat{H}_0$, namely,

$$\langle\hat{O}\rangle_0 \equiv \frac{1}{\mathcal{Z}_0}{\rm Tr}_{\rm i}\left(\hat{O}e^{-\beta\hat{H}_0}\right).$$

Thus, we find the corrected steady state in Eq. (10) can be re-expressed as

$$\begin{aligned}\hat{\rho}_{\rm ss} &= {\rm Tr}_{\rm i}\left[\frac{{\rm e}^{-\beta(\hat{L}_0+\hat{H}_1)}\,{\rm e}^{-\beta\hat{H}_0}}{\mathcal{Z}}\right] \\ &= \frac{1}{\mathcal{Z}_{\rm eff}}{\rm Tr}_{\rm i}\left[{\rm e}^{-\beta(\hat{L}_0+\hat{H}_1)}\frac{{\rm e}^{-\beta\hat{H}_0}}{\mathcal{Z}_0}\right]\end{aligned}$$

$$= \frac{1}{\mathcal{Z}_{\rm eff}} \left\langle {\rm e}^{-\beta(\hat{L}_0+\hat{H}_1)} \right\rangle_0$$
$$= \frac{1}{\mathcal{Z}_{\rm eff}} {\rm e}^{-\beta\hat{H}_{\rm eff}}, \tag{13}$$

here

$$\hat{H}_{\rm eff} \equiv -\frac{1}{\beta} \ln \left\langle {\rm e}^{-\beta(\hat{L}_0+\hat{H}_1)} \right\rangle_0 \tag{14}$$

is introduced as an effective Hamilton of $\hat{H}_{\rm sys}$, which is size consistent with the original bare spin.

The above effective Hamiltonian $\hat{H}_{\rm eff}$ can be perturbatively computed as follows:[34,35]

$$\begin{aligned}\hat{H}_{\rm eff} &= -\frac{1}{\beta} \ln \left\langle {\rm e}^{-\beta(\hat{L}_0+\hat{H}_1)} \right\rangle_0 \\ &= -\frac{1}{\beta} \left\langle {\rm e}^{-\beta(\hat{L}_0+\hat{H}_1)} - 1 \right\rangle_0^{\mathbb{C}} \\ &= -\frac{1}{\beta} \sum_{\ell=1}^{\infty} \frac{(-\beta)^\ell}{\ell!} \left\langle (\hat{L}_0+\hat{H}_1)^\ell \right\rangle_0^{\mathbb{C}},\end{aligned} \tag{15}$$

here the upper index $\mathbb{C}$ denotes the cumulant averaging, which has been systematically discussed by Kubo in Ref. [36]. In high temperature regimes, one can only retain the first and the second cumulants in the series, then the effective Hamiltonian is approximated as

$$\begin{aligned}\hat{H}_{\rm eff} &\simeq \langle \hat{L}_0+\hat{H}_1 \rangle_0^{\mathbb{C}} - \frac{\beta}{2} \langle (\hat{L}_0+\hat{H}_1)^2 \rangle_0^{\mathbb{C}} \\ &= \langle \hat{H}_1 \rangle_0 - \frac{\beta}{2} \left(\langle \hat{H}_1^2 \rangle_0 - \langle \hat{H}_1 \rangle_0^2 \right) \\ &= \varepsilon \hat{J}_z - \frac{\beta\lambda^2}{2} \coth\left(\frac{\beta\Omega}{2}\right) \hat{J}_x^2.\end{aligned} \tag{16}$$

A similar result is also reported in Refs. [35,37]. We find the first term in Eq. (16) is just the original bare spin's Hamiltonian, and the second term can be viewed as modifications induced by the coupling between the spin and the IHO, which reflects the influence of system-reservoir correlations on the final steady state. We also notice that, for the spin-$1/2$ case, $\hat{J}_x^2$ reduces to $(1/4)\hat{\mathbf{1}}_2$, which means ${\rm var}(\hat{H}_{\rm eff}) = {\rm var}(\varepsilon\hat{J}_z)$, indicating the system–reservoir correlations have no influence on the QFI up to the second order cumulants in this spin-$1/2$ case.

4. Result

The QFI from the corrected steady state given in Eq. (13) can be analytically computed, and we express the result as

$$\mathcal{F} = \mathcal{F}_{\rm s} + \delta\mathcal{F},$$

where the first term $\mathcal{F}_{\rm s}$ is the QFI contributed by the Gibbs state with respect solely to the bare spin, and the second term $\delta\mathcal{F}$ can be regarded as a correction due to the system–reservoir correlations. If $\delta\mathcal{F} > 0$, the system–reservoir correlations improve the estimation precision; on the contrary, $\delta\mathcal{F} < 0$ means system–reservoir correlations play a negative role in our scheme. As illustrative examples, we consider two different spin sizes: $J = 1$ and $J = 3/2$ in this paper. In the above two cases, $\mathcal{F}_{\rm s}$ and $\delta\mathcal{F}$ can be exactly computed, but we only list the expressions of $\mathcal{F}_{\rm s}$ here, because the explicit expressions of $\delta\mathcal{F}$ are rather tedious. We find

$$\mathcal{F}_{\rm s}^{(1)} = \frac{2\varepsilon^2}{T^4} \frac{2+\cosh(2\zeta)}{[1+2\cosh(2\zeta)]^2}, \tag{17}$$

$$\mathcal{F}_{\rm s}^{(3/2)} = \frac{\varepsilon^2}{4T^4} \left[{\rm sech}^2(\zeta) + 4{\rm sech}^2(2\zeta) \right], \tag{18}$$

where $\zeta \equiv \varepsilon/2T$.

In Figs. 2(a) and 2(b), we plot $\mathcal{F}$ *versus* temperature T for spin-1 and spin-$3/2$ systems. In both cases, we find the QFI vanishes as $T \to 0$ and decays exponentially at high temperatures. This result suggests there exists an optimal temperature that maximizes the value of QFI. In Figs. 2(c) and 2(d), we display $\delta\mathcal{F}$ as the function of Ω with different spin sizes. We find $\delta\mathcal{F} < 0$ when Ω is small, which suggests the system–reservoir correlations destroy the estimation precision; as Ω increases, $\delta\mathcal{F}$ becomes positive, indicating the system-reservoir correlations may improve the value of QFI. Our result implies that the system–reservoir correlations play a complicated role in the estimation scheme, it effect (improving or decreasing the estimation precision) depends strongly on the details of the reservoir spectral density function. Due to the fact that $\mathcal{F}(T) \propto C(T)$, our result is consistent with the result reported in Ref. [38], in which the authors found the ratio of $C_{\rm cor}/C_{\rm bare}$, $C_{\rm cor}$ is the heat capacity of a harmonic oscillator strongly correlated with its surrounding environment, while $C_{\rm bare}$ is the heat capacity of a bare harmonic oscillator without system-reservoir coupling, is sensitive to parameters in the spectral density: by adjusting the coupling strength, the ratio of $C_{\rm cor}/C_{\rm bare}$ can be larger or smaller than 1 at different temperatures. Moreover, as demonstrated in Ref. [39], parameters in the spectral density function are controllable by tuning experimental conditions, when simulating the dephasing process of trapped ultracold atoms. Thus, our result suggests that one can improve the estimation precision by employing the spectrum engineering technique.

At the end of this section, we would like to briefly discuss the experimental feasibility of measuring temperature of a reservoir by making use of spins. Many previous studies[40] have showed that a superconducting circuit based on the Cooper-pair box behaves as a spin-$1/2$ system. Nuclear spins and molecular nano-magnets,[41] which are small clusters of a few atoms embedded into a crystal, can be used to simulate spins greater than one half in experiments. As pointed out in Ref. [13], the level populations of a spin can be measured by recording resonance fluorescence signals, then via fitting experimental data into a Fermi–Dirac distribution or a Bose–Einstein distribution, which are the thermal distributions of spins in steady state, one can extract the temperature of the back contact (reservoir).

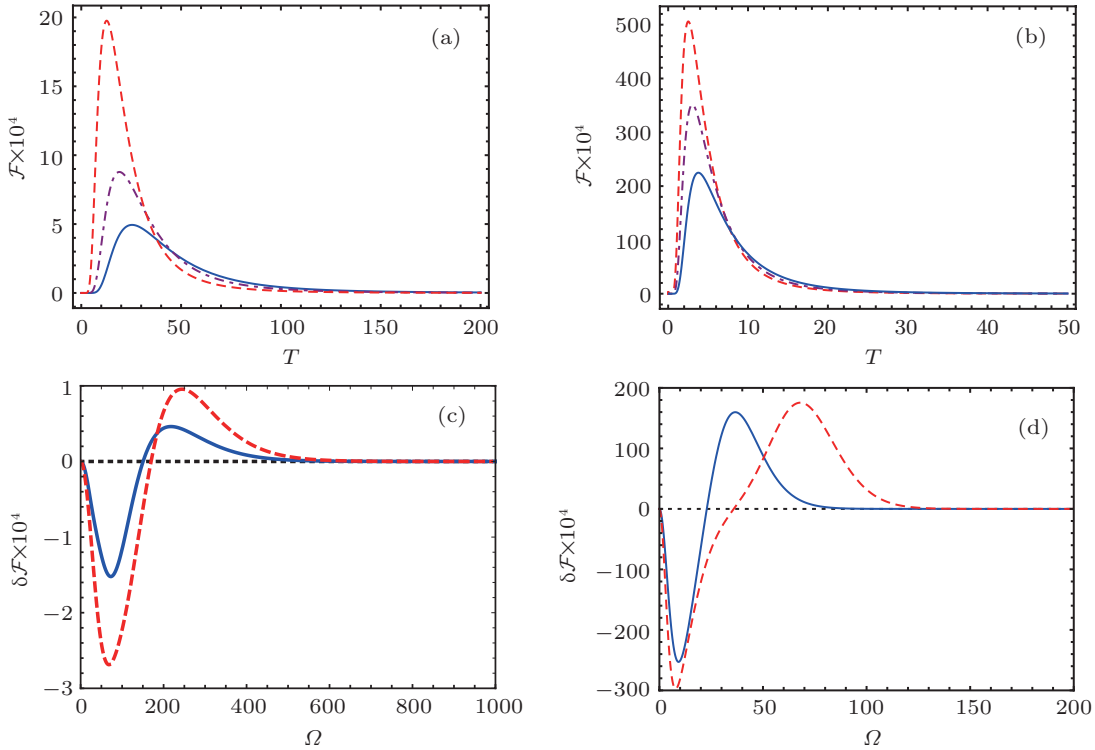


Fig. 2. (a) The total QFI $\mathcal{F}$ with $J=1$ is plotted as the function of temperature T: $\Omega=50$ (red dashed line), $\Omega=75$ (purple dotdashed line), and $\Omega=100$ (blue solid line). Other parameters are chosen as $\lambda=0.1\Omega$ and $\varepsilon=\Omega$. (b) The total QFI $\mathcal{F}$ with $J=3/2$ is plotted as the function of temperature T: $\Omega=10$ (red dashed line), $\Omega=12$ (purple dotdashed line), and $\Omega=15$ (blue solid line). Other parameters are chosen as $\lambda=0.1\Omega$ and $\varepsilon=\Omega$. (c) The corrected QFI $\delta\mathcal{F}$ at $T=50$ is plotted as the function of Ω: $J=1$ (blue solid line) and $J=3/2$ (red dashed line). Other parameters are chosen as $\lambda=100$ and $\varepsilon=\Omega$. (d) The corrected QFI $\delta\mathcal{F}$ at $T=5$ is plotted as the function of Ω: $J=1$ (blue solid line) and $J=3/2$ (red dashed line). Other parameters are chosen as $\lambda=30$ and $\varepsilon=\Omega$. In our calculation, temperature, the frequency of IHO, and QFI are expressed in arbitrary units.

5. Conclusion

In summary, we investigate the reduced equilibration dynamics of an SB model, which consists of an arbitrary size spin interacting with a finite temperature bosonic reservoir. By making use of the reaction coordinate mapping approach, an exact equivalent SIB model is obtained, in which an IHO (a collective coordinate) is partitioned into the mapped system's Hamiltonian, reflecting the influence of system–reservoir correlations on the reduced equilibration dynamics. We find the steady state of the spin can be expressed in a form of Gibbs state with respect to an effective Hamilton, which is size consistent with the original bare spin. Employing this corrected steady state, we explore how the system–reservoir correlations affect the estimation precision of reservoir's temperature. It is revealed that the effect of the system–reservoir correlations on the estimation precision is highly sensitive to the details of the spectral density function of the reservoir. Varying the parameters of the spectrum, the system–reservoir correlations play opposite roles in different parameter spaces. Technically speaking, our treatment is acceptable only in high temperature regimes, due to the the omission of the higher order cumulant averaging terms. It would be very interesting to further extend this study to the entire range of temperatures. Recently, in Ref. [42], the authors have generalized the reaction coordinate mapping to impurity systems coupled to fermionic reservoirs, which suggests our method can be also exploited to measure the temperature of a reservoir of free fermions. Finally, due to the generality of the SB model, we expect our result to be of interest for a wide range in quantum metrology tasks.

Appendix A

In this appendix, we briefly outline how to relate the original spectral density function $\mathcal{J}_{\rm sb}(\omega)$ to the mapped spectrum $\mathcal{J}_{\rm sib}(\omega)$ following the detailed exposition in Refs. [21,25–27,43]. Due to the fact that both spectrums, *i.e.*, $\mathcal{J}_{\rm sb}(\omega)$ and $\mathcal{J}_{\rm sib}(\omega)$, do not contain any information about the spin itself, but rather just the coupling between the spin and reservoirs, one can transform the spin into a continuous classical coordinate q moving in a potential $U(q)$. Then, the original Hamiltonian $\hat{H}_{\rm sb}$ becomes

$$\hat{H}_q=\frac{p^2}{2}+U(q)+\sum_k\left(\frac{\hat{P}_k^2}{2}+\frac{1}{2}\omega_k^2\hat{X}_k^2\right)+\sum_k\sqrt{2\omega_k}g_kq\hat{X}_k+q^2\sum_k\frac{g_k^2}{\omega_k},\tag{A1}$$

where $\hat{X}_k$ and $\hat{P}_k$ are defined as

$$\hat{X}_k=\sqrt{\frac{1}{2\omega_k}}(\hat{b}_k^\dagger+\hat{b}_k);\quad \hat{P}_k=\mathrm{i}\sqrt{\frac{\omega_k}{2}}(\hat{b}_k^\dagger-\hat{b}_k).$$

From the above Hamiltonian, the equation of motion of classical position $q(t)$ can be exactly derived.[21,25–27] In the Fourier

space, it reads

$$K(\varpi)\check{q}(\varpi) = -\check{U}'(\varpi), \tag{A2}$$

where

$$K(\varpi) = -\varpi^2\left[1+\int_0^\infty \mathrm{d}\omega \frac{\mathcal{J}_{\rm sb}(\omega)}{\omega(\omega^2-\varpi^2)}\right], \tag{A3}$$

and $\check{f}(\varpi)$ is defined as the Fourier transform of an arbitrary time-dependent function $f(t)$, *i.e.*,

$$\check{f}(\varpi) \equiv \int_{-\infty}^{+\infty} \mathrm{d}t f(t)\mathrm{e}^{-\mathrm{i}\varpi t}.$$

From Eq. (A3), the spectral density function $\mathcal{J}_{\rm sb}(\omega)$ can be obtained by making use of residue theorem

$$\mathcal{J}_{\rm sb}(\omega) = \frac{1}{\pi}\lim_{\varepsilon\to 0^+}\mathrm{Im}[K(\omega-\mathrm{i}\varepsilon)]. \tag{A4}$$

Similarity, by swapping the spin as a classical position q, the Hamiltonian of SIB model $\hat{H}_{\rm sib}$ is then transformed to

$$\begin{aligned}\hat{\mathcal{H}}_q =& \frac{p^2}{2}+U(q)+\frac{\hat{p}_0^2}{2}+\frac{1}{2}\Omega^2\hat{x}_0^2+\sqrt{2\Omega}\lambda q\hat{x}_0\\ &+\sum_k\left(\frac{\hat{\mathcal{P}}_k^2}{2}+\frac{1}{2}\tilde{\omega}_k^2\hat{\mathcal{X}}_k^2\right)+\sum_k\sqrt{2\omega_k}\tilde{g}_k\hat{x}_0\hat{\mathcal{X}}_k\\ &+\frac{\lambda^2}{\Omega}q^2+\hat{x}_0^2\sum_k\frac{\tilde{g}_k^2}{\tilde{\omega}_k},\end{aligned} \tag{A5}$$

where $\{\hat{x}_0,\ \hat{p}_0\}$ and $\{\hat{\mathcal{X}}_k,\ \hat{\mathcal{P}}_k\}$ are position–momentum operators of the IHO and the residual bosonic reservoir, respectively. From Eq. (A5), one can also attain the equation of motion for $q(t)$, moving into the Fourier space, it reads $\mathcal{K}(\varpi)\check{q}(\varpi) = -\check{U}'(\varpi)$, which has the same form with Eq. (A2). The expression of $\mathcal{K}(\varpi)$ in this case is given by

$$\mathcal{K}(\varpi) = -\varpi^2 - \frac{2\lambda^2}{\Omega}\frac{\mathcal{L}(\varpi)}{\Omega^2+\mathcal{L}(\varpi)}, \tag{A6}$$

where

$$\mathcal{L}(\varpi) \equiv -\varpi^2\left[1+4\Omega\int_0^\infty \mathrm{d}\omega\frac{\mathcal{J}_{\rm sib}(\omega)}{\omega(\omega^2-\varpi^2)}\right].$$

If one choose $\mathcal{J}_{\rm sib}(\omega) = \gamma\omega$, then $\mathcal{L}(\varpi)$ reduces to $\mathcal{L}(\varpi) = -\varpi^2 + 2\mathrm{i}\pi\Omega\gamma\varpi$. Plugging this expression into Eq. (A6), one can obtain $\mathcal{K}(\varpi)$.

Finally, using the dynamical consistency condition, namely, the equation of motion for $q(t)$ derived from $\hat{H}_q$ should be equivalent to that from $\hat{\mathcal{H}}_q$, we have

$$\mathcal{K}(\varpi) = K(\varpi).$$

Then, with the help of Eq. (A4), one can find

$$\begin{aligned}\mathcal{J}_{\rm sb}(\omega) &= \frac{1}{\pi}\lim_{\varepsilon\to 0^+}\mathrm{Im}[\mathcal{K}(\omega-\mathrm{i}\varepsilon)]\\ &= \frac{2\gamma\lambda^2\omega\Omega^2}{(\Omega^2-\omega^2)^2+(2\pi\gamma\Omega\omega)^2},\end{aligned} \tag{A7}$$

which recovers Eq. (9) in the main text.

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