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TOPICAL REVIEW — Strong field atomic and molecular physics

Bohmian trajectory perspective on strong field atomic processes*

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The interaction of an atom with an intense laser field provides an important approach to explore the ultrafast electron dynamics and extract the information of the atomic and molecular structures with unprecedented attosecond temporal and angstrom spatial resolution. To well understand the strong field atomic processes, numerous theoretical methods have been developed, including solving the time-dependent Schrödinger equation (TDSE), classical and semiclassical trajectory method, quantum *S*-matrix theory within the strong-field approximation, *etc*. Recently, an alternative and complementary quantum approach, called Bohmian trajectory theory, has been successfully used in the strong-field atomic physics and an exciting progress has been achieved in the study of strong-field phenomena. In this paper, we provide an overview of the Bohmian trajectory method and its perspective on two strong field atomic processes, *i.e*., atomic and molecular ionization and high-order harmonic generation, respectively.

Keywords: Bohmian trajectory, strong-field ionization, high-order harmonic generation

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1. Introduction

When interacting with an intense laser field, atoms and molecules may absorb many more photons than required for ionization. This very highly nonlinear process is known as above-threshold ionization (ATI) and has attracted considerable attention since the early work of Agostini and coworkers.[1] If the released electron revisits the parent ion in the presence of the laser field,[2,3] various additional highly nonlinear phenomena, such as high-order ATI (HATI),[4] high-order harmonic generation (HHG),[5–7] and nonsequential double ionization (NSDI),[8–11] are found in experiment. The study of these nonlinear strong-field phenomena has achieved great progress. For example, the HHG has been taken as a tabletop coherent extreme ultraviolet source[12] and as a source of attosecond pulses,[13] both the HHG and ATI have been employed as an important technique to explore the atomic and molecular structures and the subfemtosecond dynamics,[14–16] and the NSDI has created the opportunity for the study of strong-field electron–electron correlation.[17–19]

To well understand the underlying physics in these highly nonlinear strong-field phenomena, numerous theoretical methods have been developed. The most commonly used methods are the solution of the time-dependent Schrödinger equation (TDSE),[20,21] classical and semiclassical trajectory method,[22–25] quantum *S*-matrix theory within the strong-field approximation (SFA),[26–31] *etc*. The numerical solution of TDSE can accurately reproduce the experimental results and thus, it has been taken as a benchmark to evaluate the data in experiments and the calculations of other theories and models.[32–36] The classical and semiclassical trajectory method describes the electron dynamics in terms of the classical trajectory and thus, can provide clear physical insight into the strong-field phenomena of interest.[37–40] The quantum *S*-matrix theory within SFA is derived from the transition amplitude in quantum mechanics with suitable approximation[26,27] and can provide a clear description of the quantum effects relevant to the strong-field atomic processes, *e.g*., the channel-closing effect.[41–45] However, the solution of TDSE does not have the concept of particle trajectory, leading to that it is relatively difficult to extract the needed physical information from the TDSE simulations, while the theoretical models include several approximations, *e.g*., the neglecting of the quantum coherence and wave-packet dispersion in the classical and semiclassical trajectory method and of the ionic Coulomb effect and the excited states in the quantum *S*-matrix theory within SFA, resulting in that in some cases the experimental observation cannot be well reproduced by these theoretical models, frustrating a comprehensive understanding of the underlying physics behind the strong-field atomic phenomena. Therefore, in the strong-field atomic physics, both the TDSE and theoretical models are usually used together to confirm and explain the experimental observation. Although considerable success has been achieved over the past three decades, due to the limitation of these methods, the underlying mechanism of many strong-field phenomena is still under debate, for example, the below-threshold harmonic generation,[46–48] the population of

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the Rydberg states,[49–53] and so on.

To partly overcome the limitation of these theoretical methods, an alternative approach, called Bohmian mechanics, has been successfully used in the strong-field atomic physics recently. Bohmian mechanics,[54,55] first proposed by Louis de Broglie and then further developed by David Bohm, is a complementary quantum approach, which describes quantum phenomena in terms of point-like particle. Different from the traditional theoretical methods, Bohmian mechanics cannot only offer a trajectory-based explanation of the quantum phenomena, but also accurately reproduce the experimental measurement. Since first developed in 1952, Bohmian mechanics has been successfully used in a broad range of fields (see, *e.g.*, the related books[56–59] and a recent review in Ref. [60]), including ultracold atom physics,[61] nonadiabatic molecular dynamics,[62] nanoelectronic physics,[63] beyond spinless nonrelativistic scenarios,[64] and the quantum cosmology.[65–67] Recently, Bohmian mechanics has been adopted in the strong-field atomic physics to study the highly nonlinear phenomena and explore the ultrafast ionization dynamics.[60] For example, it provides a clear physical picture for the complex tunneling ionization dynamics in the classically forbidden region and strongly supports the use of the classical trajectory to describe the motion of an ionized electron in the classical or semiclassical trajectory methods. Moreover, by using a single Bohmian trajectory, the HHG spectrum can be qualitatively well reproduced, facilitating to a great extent the understanding of the harmonic generation.

Below is the outline of this review paper. In Section 2, we briefly introduce Bohmian trajectory formalism. In Section 3, we discuss the Bohmian trajectory perspective on the ionization of atoms and molecules. In Section 4, we discuss the Bohmian trajectory perspective on HHG. Conclusions and perspectives are given in Section 5.

2. Quantum trajectory formalism in Bohmian mechanics

Bohmian mechanics[54–57] is exactly derived from a subtle transformation of the Schrödinger equation (atomic units are used throughout, unless stated otherwise)

$$\mathrm{i}\frac{\partial}{\partial t}\psi(\boldsymbol{r},t)=\left[-\frac{1}{2}\nabla^2+W(\boldsymbol{r},t)\right]\psi(\boldsymbol{r},t), \tag{1}$$

where $W(\boldsymbol{r},t)$ is the potential and $\psi(\boldsymbol{r},t)$ is the time-dependent electronic wavefunction. For the interaction of atoms and molecules with the strong laser field, the potential is given by

$$W(\boldsymbol{r},t)=V(\boldsymbol{r},t)-\boldsymbol{r}\cdot\boldsymbol{E}(t), \tag{2}$$

where $V(\boldsymbol{r},t)$ is the Coulomb potential and $\boldsymbol{E}(t)$ is the electric field of the laser field. First of all, the electronic wavefunction is written in the polar form

$$\psi(\boldsymbol{r},t)=R(\boldsymbol{r},t)\,\mathrm{e}^{\mathrm{i}S(\boldsymbol{r},t)}, \tag{3}$$

where R and S are real functions. Secondly, inserting Eq. (3) into Eq. (1) leads to two coupled differential equations

$$\frac{\partial S}{\partial t}+\frac{(\nabla S)^2}{2}+W(\boldsymbol{r},t)+Q(\boldsymbol{r},t)=0, \tag{4}$$

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho v)=0, \tag{5}$$

where $\rho(\boldsymbol{r},t)=R^2(\boldsymbol{r},t)=|\psi(\boldsymbol{r},t)|^2$ is the probability density and $v(\boldsymbol{r},t)=\nabla S(\boldsymbol{r},t)$ is the velocity of the electron. Equation (5) is well known as the continuity equation and equation (4) is named as the quantum Hamilton–Jacobi equation in Bohmian mechanics. In comparison with the classical Hamilton–Jacobi equation, there is an extra term in Eq. (4):

$$Q(\boldsymbol{r},t)=-\frac{1}{2}\frac{\nabla^2 R(\boldsymbol{r},t)}{R(\boldsymbol{r},t)}, \tag{6}$$

which is called as the quantum potential in Bohmian mechanics and determines the quantum behavior of the electron.

According to Eqs. (4) and (5), a Bohm–Newton equation of motion can be obtained

$$\mathrm{d}^2\boldsymbol{r}/\mathrm{d}t^2=-\nabla\left[W(\boldsymbol{r},t)+Q(\boldsymbol{r},t)\right]. \tag{7}$$

If the quantum potential $Q\to 0$, the Bohm–Newton equation will reduce to the standard Newton equation and hence, the corresponding dynamics of the electron can be described with the classical mechanics. In practice, it is not easy to solve such second order differential equation with the quantum potential. Instead, the Bohmian trajectory can be equivalently obtained by solving a much simpler equation of motion[54–57]

$$\mathrm{d}\boldsymbol{r}/\mathrm{d}t=\nabla S(\boldsymbol{r},t), \tag{8}$$

in which the time-dependent phase $S(\boldsymbol{r},t)$ of the wavefunction is usually obtained by numerically solving the TDSE. The initial position of the Bohmian trajectory is selected according to the distribution function $|\psi(\boldsymbol{r},0)|^2$ of the electronic probability density.

In comparison with the classical trajectory, there are some special features for the Bohmian trajectory:[54–57,60]

(i) The quantum potential $Q(x,t)$ is responsible for the quantum behavior of the Bohmian trajectory.

(ii) The quantum potential is nonlocal, resulting in that the different Bohmian trajectories influence each other.

(iii) Non-crossing rule: Two Bohmian trajectories can never pass through the same point on configuration space at the same time due to the single-valuedness of the momentum field [see Eq. (8)].

3. Bohmian trajectory perspective on the ionization of atoms and molecules

Ionization is a fundamental process for atoms and molecules in the strong laser field and is the basis to investigate the highly nonlinear strong-field phenomena, *e.g.*, ATI, HHG, and NSDI. Recently, based on the atomic and molecular ionization, a strong-field photoelectron holography (PH)[33,34,68–71] and a laser-induced electron diffraction (LIED)[14–16,72,73] approach have been established and extensively employed, *e.g.*, in extracting the information of the atomic and molecular structures and ultrafast dynamics. The ionization of atoms in the strong laser field can be qualitatively understood with the simple man's model:[2,3] the outmost electron in an atom is firstly released by tunneling through the barrier formed by the Coulomb potential and the electric field of the laser and then the motion of the ionized electron is driven by the laser field. In this section, we will show that Bohmian trajectory method can be used to provide a deep understanding of the ultrafast ionization dynamics of atoms and molecules in the strong laser field.

3.1. Study of the tunneling ionization dynamics

The tunneling is a quantum-mechanical effect, which cannot be studied with classical mechanics. In the semiclassical trajectory,[22–25] the tunneling ionization is described with an analytical Ammosov–Delone–Krainov (ADK) theory without considering the detailed tunneling ionization dynamics, while in the quantum S-matrix theory,[26,27] the tunneling is treated with a simple direct transition from the ground state to the continuum states also without the detailed information of the tunneling ionization dynamics. In contrast, Bohmian mechanics can describe the tunneling in terms of the Bohmian trajectory and thus provide a deep understanding of the electron dynamics during the tunneling ionization. In 2013, Wei *et al.*[74] simulated the atomic ionization process in high-frequency laser pulses with the Bohmian-trajectory scheme. By analyzing the quantum potential, they found that the quantum force that acts on the Bohmian trajectory plays a crucial role in the ionization of atom. Their results show that the competition between the quantum force and the classical force leads to the ionization stabilization of the atom in a high-frequency intense laser pulse. Later, in 2015, Jooya *et al.*[75] studied the tunneling ionization process on a subfemtosecond time scale for the hydrogen atom subject to intense near-infrared laser fields in the Bohmian framework. They found that within each optical cycle of the external laser field, some portion of the ionized wave packets, represented by various groups of Bohmian trajectories (see Fig. 1(a)), return to the parent ion when the laser field changes sign, which causes the transitions to the excited bound and continuum states of the unperturbed atom. As a result, there are multiple peaks structure of the time-dependent ionization rate within a half optical cycle (see Fig. 1(b)). Recently, Douguet and Bartschat[76] showed that the ionized electron does not tunnel through the entire barrier, but starts already from the classically forbidden region (see the yellow region in Fig. 2), *i.e.*, from the tail of the initial ground-state wave function. Moreover, they constructed the correspondence between the probability of locating the electron at a particular initial position in the forbidden region and its asymptotic momentum. Their results provide new routes to understanding the ultrafast electron dynamics, *e.g.*, the mean tunneling time and the exit position of tunneling. Similar study of the tunneling exit during the strong field ionization using the Bohmian approach can be also found in Ref. [77].

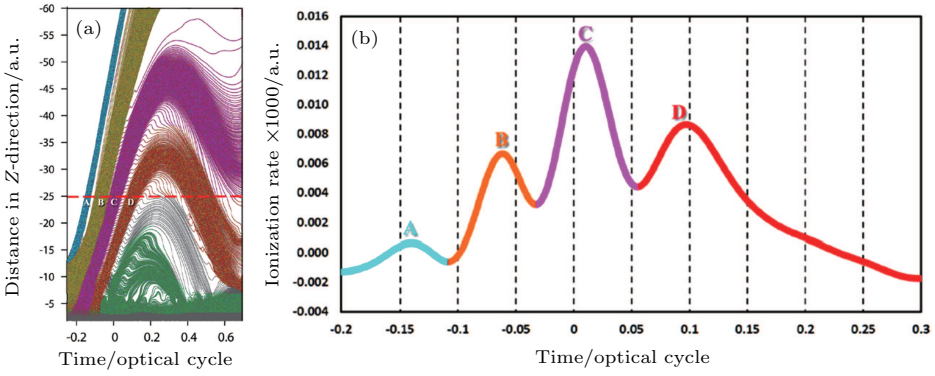


Fig. 1. (a) Four distinct groups of the Bohmian trajectories (labeled A–D), within a half optical cycle, shaded with different colors. (b) The corresponding peaks in the time-dependent ionization rate of the hydrogen atom within a half optical cycle.[75]

Recently, the study of the strong-field tunneling ionization with the Bohmian mechanics has been also extended from atoms to molecules.[78–80] For example, Takemoto and Becker[78] analyzed the attosecond electron dynamics in hydrogen molecular ion driven by an external intense laser field using the Bohmian trajectories. The Bohmian trajectories clearly visualize the electron transfer between the two protons in the laser field and, in particular, confirm the attosecond transient localization of the electron at one of the protons and the related multiple bunches of the ionization current within a half cycle of the laser field. Further analysis based on the quantum trajectories shows that the electron dynamics in the molecular

ion can be understood via the phase difference accumulated between the Coulomb wells at the two protons. Later, Sawada *et al.*[79] studied the mechanism of enhanced ionization of a two-electron molecule around the critical internuclear distance by analyzing Bohmian trajectories. They found that there are two kinds of electron trajectories (see Fig. 3(a)). The theoretical simulations show that around the critical internuclear distance, both the ejections from the up-field core and from the down-field core through the outer barrier are enhanced. Further analysis reveals that the trajectories of the two electrons are correlated with each other for the case of the ejection from the up-field core (see Fig. 3(b)), while the trajectory of the departing electron is not affected by that of the other electron for the case of the ejection from the down-field core (see Fig. 3(c)).

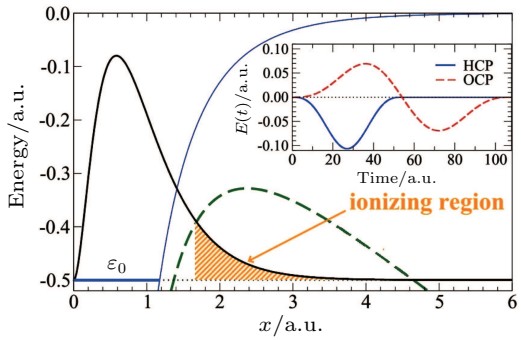


Fig. 2. The ionized electron starts from the classically forbidden region, *i.e.*, from the tail of the initial ground-state wave function. The ground-state probability (thick black line), the field-free Yukawa potential (thin blue line), and the classical potential at the maximum field strength (dashed green line). The inset shows the two kinds of the electric fields of the laser fields.[76]

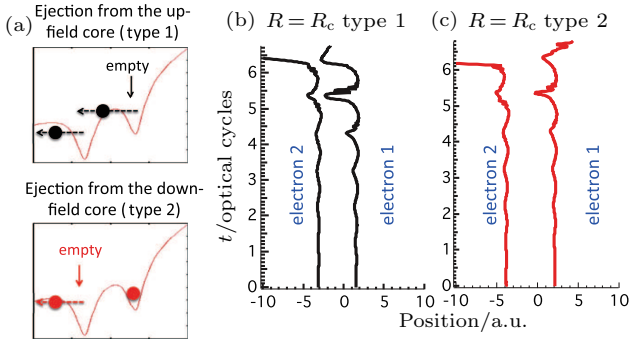


Fig. 3. (a) Schematic diagram of the ejection from the up-field core (type 1) and down-field core (type 2). (b) The trajectory at the critical distance for the case of the ejection from the up-field core (type 1). (c) The trajectory at the critical distance for the case of the ejection from the down-field core (type 2).[79]

3.2. Study of the electron dynamics after tunneling ionization

In the strong-field physics, the ionized electron in the presence of the laser field and the Coulomb potential is usually assumed to follow the Newton's equation of motion. However, whether such an assumption is accurate is still under debate. Bohmian trajectory theory may also provide an approach to accurately investigate the behavior of electron after the ionization. In 2009, Lai *et al.*[81] simulated the ionization of hydrogen atom in an intense laser pulse with the Bohmian mechanics. After solving a large number of the Bohmian trajectories, a photoelectron spectrum with clear ATI peaks is obtained, which is in a good agreement with the TDSE results. The consistency between the simulation with Bohmian trajectory and the TDSE calculation can be also found for, *e.g.*, the time-dependent ionization probabilities[82] and the time evolution of the electron wavepacket.[83,84] Furthermore, to compare with the classical trajectory, Lai *et al.*[85] studied the specific quantum potential along the Bohmian trajectories of the ionized electron. Figure 4 shows that the quantum potential is significant before ionization, but becomes negligible after the electron is ionized with the total energy larger than zero. According to Eq. (7), when the quantum potential Q tends to be negligible, the Bohm–Newton equation will reduce to the Newton's equation of motion and hence, the corresponding dynamics of the electron can be described with the classical mechanics. This result strongly supports the use of the classical trajectory in the semiclassical trajectory method to describe the motion of an ionized electron.[22–25,86] Very recently, in the Bohmian scenario, Jooya *et al.*[87] and Li *et al.*[88] found the laser-driven electron-multirescattering trajectories, which is consistent with the commonly accepted multi-return trajectories of the ionized electron calculated by the semiclassical trajectory method.

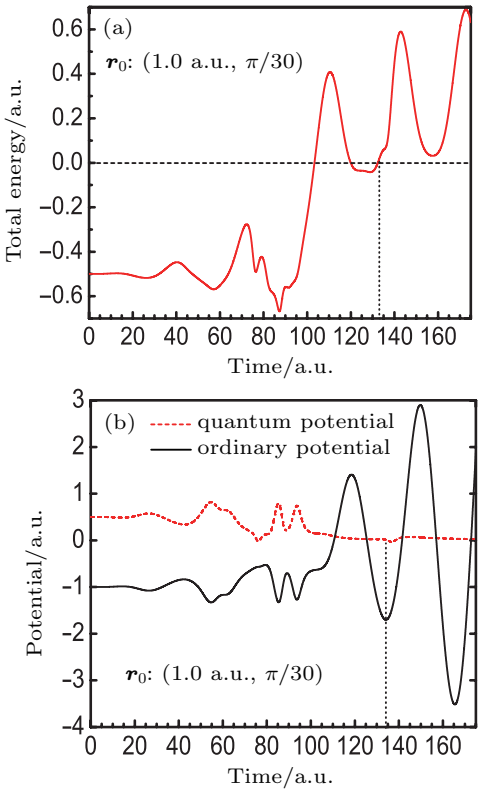


Fig. 4. (a) The time-dependent total energy (including kinetic energy, Coulomb potential, and quantum potential) of one Bohmian trajectory with the initial position of r_0. The electron is ionized when the total energy is larger than zero. (b) The corresponding the quantum potential Q and the ordinary potential W of the Bohmian trajectory.[85]

Furthermore, Bohmian trajectory method has also been employed in the study of the multiple ionization of atoms in the strong laser fields. After the tunneling ionization, the ionized electron can significantly modify the motion of the rest of the electrons in their orbits. Such electron–electron correlation effects can influence the ionization rate of the rest electrons, for example, the famous NSDI[8–11] in the strong-field physics. However, it is time consuming to exactly solve the TDSE to obtain the multi-electron time-dependent wave function, because the computational effort increases exponentially with the system dimensionality. To obtain the quantum trajectories for multidimenstional dynamics, several quantum trajectory methods based on the Bohmian scenario have been developed.[59,83,84,89] For example, Christov[89] introduced a quantum trajectory method based on the Bohmian trajectory method, by transforming the multple-electron TDSE into a set of equations for the single-electron orbitals to reduce significantly the computational time. With such a trajectory method, the NSDI yield as a function of the laser intensity qualitatively agrees well with the exact results[89] and further analysis shows that the NSDI output is predetermined by the early stage of the correlated electron motion during the rising front of the laser pulse.[90]

4. Bohmian trajectory perspective on the HHG

HHG[5–7] has attracted a great deal of attention due to its broad applications, such as the coherent extreme-ultraviolet sources,[12] attosecond pulses,[91] and attosecond imaging of dynamic processes.[92] The harmonic generation can be understood with a semiclassical three-step model:[2,3] i) the outmost electron in an atom is firstly liberated by the strong laser field; ii) secondly, the ionized electron is accelerated in the laser field before being turned around by the laser field and returning to the parent ion, and iii) finally, the returning electron can recombine with the parent ion, releasing its kinetic energy as a high-energy photon. The corresponding HHG spectrum exhibits a plateau with a cutoff energy of $3.17U_{\rm p}+I_{\rm p}$, where $U_{\rm p}$ is the ponderomotive potential of the laser field and $I_{\rm p}$ is the ionization potential of the atomic target under investigation. Similar to the ionization of atoms and molecules, Bohm mechanics provides an alternative explanation of the HHG processes.

4.1. The theoretical method of HHG from Bohmian mechanics[93–97]

After obtaining the Bohmian electron trajectories according to Eq. (8), the expectation value of the coordinate x is given by the following equation:

$$\bar{x}(t)=\frac{1}{N}\sum_{i=1}^{N}x_i(t), \tag{9}$$

and the expectation value of the dipole acceleration is written as

$$\bar{a}(t)=\frac{1}{N}\sum_{i=1}^{N}\frac{{\rm d}W(x,t)}{{\rm d}x}\Big|_{x=x_i}, \tag{10}$$

where N is the number of the Bohmian trajectories selected according to the initial distribution of $|\psi(\boldsymbol{r},0)|^2$. The corresponding high-order-harmonic spectra can be obtained by Fourier transformation of the expectation values:

$$P_l(\omega)=\left|\frac{1}{T}\int_0^T\bar{x}(t)\,{\rm e}^{-{\rm i}\omega t}\,{\rm d}t\right|^2, \tag{11}$$

or

$$P_a(\omega)=\left|\frac{1}{T}\frac{1}{\omega^2}\int_0^T\bar{a}(t)\,{\rm e}^{-{\rm i}\omega t}\,{\rm d}t\right|^2, \tag{12}$$

where T is the length of the laser pulse. The power spectra $P_l(\omega)$ and $P_a(\omega)$ should be the same if the Bohmian trajectory is solved accurately.

4.2. Simulation of the HHG spectrum with a single Bohmian trajectory

In the quantum S-matrix theory within SFA, the HHG spectrum can be simulated with the coherent superposition of a large number of rescattered electron trajectories with different ionization times and return times. Recently, it is found that by using a single Bohmian trajectory, the HHG spectrum can be qualitatively well reproduced. This simplicity facilitates to a great extent the understanding of the harmonic generation.

In 2010, Lai *et al.*[93] studied the HHG spectrum with Bohmian trajectory method. The simulated HHG spectrum with a large number of Bohmian trajectories is well consistent with the exact TDSE calculation. Similar results can be also found in Refs. [94–97], in which the HHG spectrum simulated with the Bohmian trajectory theory gets closer and closer to the TDSE result with the increase of the number of Bohmian trajectory (see Fig. 5).

Interestingly, it is found that the simulated HHG spectrum with two Bohmian trajectories with symmetric initial positions can qualitatively well reproduce the main feature of the TDSE result,[93] including the almost same plateau and the cutoff position. Very interestingly, Song *et al.*[94] and Wu *et al.*[96,97] further found that the main contributions to the high-harmonic spectra can be qualitatively well simulated with a single Bohmian trajectory located in the nuclear zone (see Fig. 6). Recently, Bohmian trajectory method is also employed

to study the HHG in a two-color laser field to demonstrate the effect of laser pulse shape on the characteristic properties of HHG.[98,99] Similarly, the HHG spectrum in the two-color laser field can be also well reproduced with a single Bohmian trajectory, but the initial position of the single trajectory is dependent on the symmetry of the electric field of the laser pulse.[98]

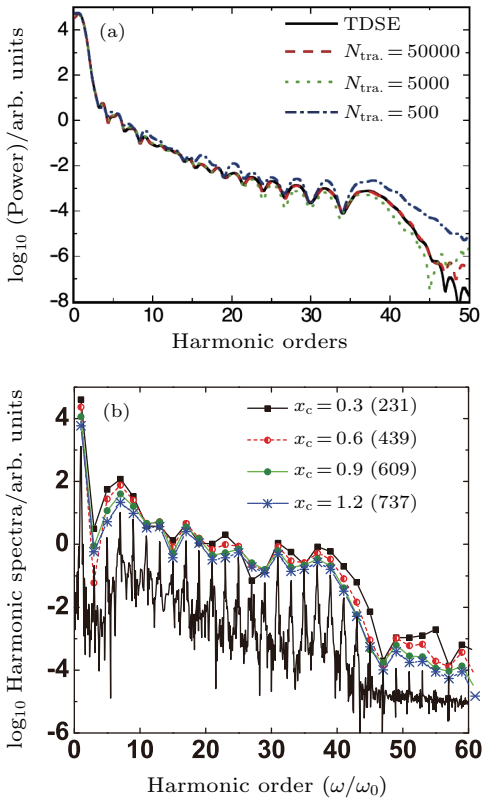


Fig. 5. The harmonic spectra calculated with TDSE and Bohmian mechanics with the different numbers of trajectories. Panel (a) is from Ref. [94] and panel (b) is from Ref. [97].

Furthermore, Wu *et al.*[96,97] discussed the underlying physics of the HHG from the central Bohmian trajectory. In the Bohmian scenario, this physical picture builds up nonlocally near the core via the quantum potential or the quantum mechanical phase of the wave function. A Bohmian trajectory evolves under the action of the wave function, which encompasses not only local information about the space variations of the potential function but also the information about global changes of the quantum phase. This implies that a Bohmian trajectory may be localized in the innermost part of the core and still contain bound and continuum dynamics. Any change in the wave function, be it far or close to the core region, will be transmitted nonlocally to the central trajectory via its phase.

It is worth noting that the simulation of the HHG spectrum with few Bohmian trajectories can be also applied for the molecular targets. For example, Wang *et al.*[100] investigated the HHG of diatomic molecular ions using Bohmian trajectory method. It is demonstrated that the main characteristics of the molecular harmonic spectrum can be also well reproduced by only two Bohmian trajectories which are located at the two ions.

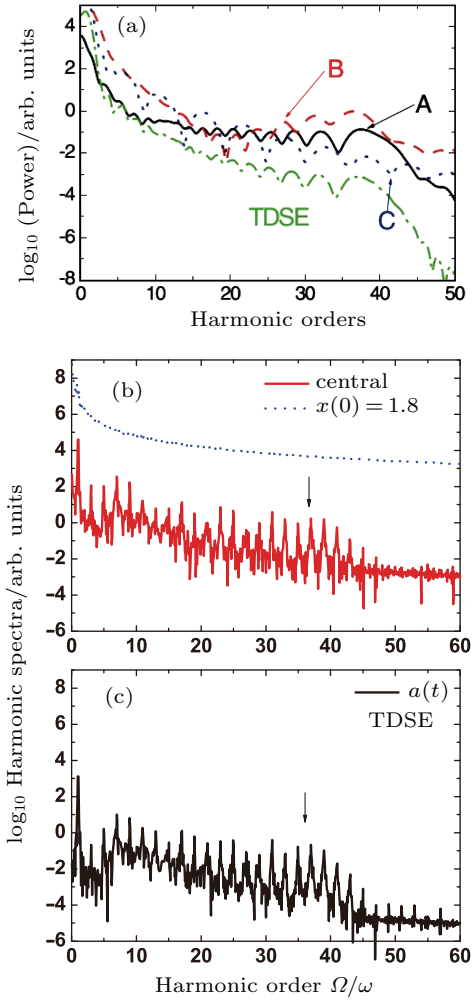


Fig. 6. (a) Harmonic spectra of Bohmian particles A, B, and C and those (dash-dotted green curve) obtained by numerically solving the TDSE. The initial position of particle A is located in the nuclear zone.[94] (b) High-order-harmonic spectra from the central Bohmian trajectory [$x(0) = 0$] (solid red line) and (c) the power spectra from the dipole acceleration computed from the TDSE (solid black line).[96]

5. Conclusion and perspectives

In summary, we briefly overview Bohmian trajectory perspective on strong field atomic processes. Bohmian mechanics can not only accurately reproduce the rich strong-field phenomena, but also provide a trajectory-based explanation of the ultrafast ionization dynamics of atom and molecule in the strong laser field. The use of the Bohmian trajectory method has achieved great progress in the study of strong field atomic processes. For example, in the Bohmian scenario, it provides a clear physical picture for the tunneling ionization dynamics in the classically forbidden region and strongly supports the use of the classical trajectory to describe the motion of an ionized electron in the conventional models. Moreover, the quantum trajectory method based on the Bohmian scenario has been

used to study the multiple ionization of atoms in the strong laser fields. In addition, it is found that the HHG spectrum can be qualitatively well reproduced with a single Bohmian trajectory, facilitating to a great extent the understanding of the harmonic generation. With this advantage, we expect that the Bohmian trajectory method may also provide a more comprehensive understanding of other strong field atomic phenomena, *e.g.*, the below-threshold harmonic generation[46–48] and the frustrated tunneling ionization (FTI),[49–53] to shed more light on the extensively debated excitation effects in strong-field atomic physics. Moreover, the quantum trajectory method based on the Bohmian scenario can be taken as a convenient computational tool to study the ionization dynamics of the more complex targets with multiple electrons, *e.g.*, the interatomic Coulombic decay (ICD) process in the dimers.[101]

References

[1] Agostini P, Fabre F, Mainfray G, Petite G and Rahman N K 1979 *Phys. Rev. Lett.* **42** 1127
[2] Schafer K J, Yang B R, DiMauro L F and Kulander K C 1993 *Phys. Rev. Lett.* **70** 1599
[3] Corkum P B 1993 *Phys. Rev. Lett.* **71** 1994
[4] Paulus G G, Nicklich W, Xu H, Lambropoulos P and Walther H 1994 *Phys. Rev. Lett.* **72** 2851
[5] Ferray M, L'Huillier A, Li X F, Lompré L A, Mainfray G and Manus C 1988 *J. Phys. B* **21** L31
[6] Krause J L, Schafer K J and Kulander K C 1992 *Phys. Rev. Lett.* **68** 3535
[7] Agostini P and DiMauro L F 2004 *Rep. Prog. Phys.* **67** 813
[8] Walker B, Sheehy B, DiMauro L F, Agostini P, Schafer K J and Kulander K C 1994 *Phys. Rev. Lett.* **73** 1227
[9] Becker A, Dörner R and Moshammer R 2005 *J. Phys. B* **38** S753
[10] Becker W, Liu X, Ho P J and Eberly J H 2012 *Rev. Mod. Phys.* **84** 1011
[11] de Morisson Faria C F and Liu X 2011 *J. Mod. Opt.* **58** 1076
[12] Spielmann Ch, Burnett N H, Sartania S, Kopptisch R, Schnürrer M, Kan C, Lenzner M, Wobrauschek P and Krausz F 1997 *Science* **278** 661
[13] Krausz F and Ivanov M 2009 *Rev. Mod. Phys.* **81** 163
[14] Blaga C I, Xu J L, DiChiara A D, Sistrunk E, Zhang K K, Agostini P, Miller T A, DiMauro L F and Lin C D 2012 *Nature* **483** 194
[15] Meckel M, Comtois D, Zeidler D, Staudte A, Pavičić D, Bandulet H C, Pépin H, Kieffer J C, Dörner R, Villeneuve D M and Corkum P B 2008 *Science* **320** 1478
[16] Sun R P, Lai X Y, Yu S G, Wang Y L, Xu S P, Quan W and Liu X J 2019 *Phys. Rev. Lett.* **122** 193202
[17] Quan W, Hao X L, Hu X Q, Sun R P, Wang Y L, Chen Y J, Yu S G, Xu S P, Xiao Z L, Lai X Y, Li X Y, Becker W, Wu Y, Wang J G, Liu X J and Chen J 2017 *Phys. Rev. Lett.* **119** 243203
[18] Hao X L, Chen J, Li W D, Wang B B, Wang X D and Becker W 2014 *Phys. Rev. Lett.* **112** 073002
[19] Maxwell A S and Figueira de Morisson Faria C 2016 *Phys. Rev. Lett.* **116** 143001
[20] Javanainen J, Eberly J H and Su Q C 1988 *Phys. Rev. A* **38** 3430
[21] Grobe R and Eberly J H 1993 *Phys. Rev. A* **48** 4664
[22] Hu B, Liu J and Chen S G 1997 *Phys. Lett. A* **236** 533
[23] Chen J, Liu J and Chen S G 2000 *Phys. Rev. A* **61** 033402
[24] Chen J, Liu J and Zheng W M 2002 *Phys. Rev. A* **66** 043410
[25] Ye D F, Liu X and Liu J 2008 *Phys. Rev. Lett.* **101** 233003
[26] Becker W, Grasbon F, Kopold R, Milošević D B, Paulus G G and Walther H 2002 *Adv. At. Mol. Opt. Phys.* **48** 35
[27] Milošević D B, Paulus G G, Bauer D and Becker W 2006 *J. Phys. B* **39** R203
[28] Figueira de Morisson Faria C, Schomerus H and Becker W 2002 *Phys. Rev. A* **66** 043413
[29] Keldysh L V 1964 *Zh. Eksp. Teor. Fiz.* **47** 1945 [1965 *Sov. Phys. JETP* **20** 1307]
[30] Faisal F H M 1973 *J. Phys. B* **6** L89
[31] Reiss H R 1980 *Phys. Rev. A* **22** 1786
[32] Blaga C I, Catoire F, Colosimo P, Paulus G G, Muller H G, Agostini P and Dimauro L F 2009 *Nat. Phys.* **5** 335
[33] Huismans Y, *et al.* 2011 *Science* **331** 61
[34] Huismans Y, *et al.* 2012 *Phys. Rev. Lett.* **109** 013002
[35] Lai X Y, Yu S G, Huang Y Y, Hua L Q, Gong C, Quan W, de Morisson Faria C F and Liu X J 2017 *Phys. Rev. A* **96** 013414
[36] Maxwell A S, Al-Jawahiry A, Das T and de Morisson Faria C F 2017 *Phys. Rev. A* **96** 023420
[37] Quan W, Lin Z, Wu M, Kang H, Liu H, Liu X, Chen J, Liu J, He X T, Chen S G, Xiong H, Guo L, Xu H, Fu Y, Cheng Y and Xu Z Z 2009 *Phys. Rev. Lett.* **103** 093001
[38] Wang Y L, Yu S G, Lai X Y, Kang H P, Xu S P, Sun R P, Quan W and Liu X J 2018 *Phys. Rev. A* **98** 043422
[39] Wu C Y, Yang Y D, Liu Y Q, Gong Q H, Wu M Y, Liu X, Hao X L, Li W D, He X T and Chen J 2012 *Phys. Rev. Lett.* **109** 043001
[40] Li M, Geng J W, Liu H, Deng Y, Wu C, Peng L Y, Gong Q and Liu Y 2014 *Phys. Rev. Lett.* **112** 113002
[41] Paulus G G, Grasbon F, Walther H, Kopold R and Becker W 2001 *Phys. Rev. A* **64** 021401
[42] Cornaggia C 2010 *Phys. Rev. A* **82** 053410
[43] Quan W, Lai X Y, Chen Y J, *et al.* 2013 *Phys. Rev. A* **88** 021401
[44] Quan W, Lai X Y, Chen Y J, *et al.* 2014 *Chin. J. Phys.* **52** 389
[45] Wang C, Tian Y, Luo S, *et al.* 2014 *Phys. Rev. A* **90** 023405
[46] L'Huillier A, Schafer K J and Kulander K C 1991 *J. Phys. B* **24** 3315
[47] Xiong W H, Geng J W, Tang J Y, Peng L Y and Gong Q H 2014 *Phys. Rev. Lett.* **112** 233001
[48] Li P C, Sheu Y L, Laughlin C and Chu S I 2014 *Phys. Rev. A* **90** 041401
[49] Nubbemeyer T, Gorling K, Saenz A, Eichmann U and Sandner W 2008 *Phys. Rev. Lett.* **101** 233001
[50] Manschwetus B, Nubbemeyer T, Gorling K, Steinmeyer G, Eichmann U, Rottke H and Sandner W 2009 *Phys. Rev. Lett.* **102** 113002
[51] Eichmann U, Saenz A, Eilzer S, Nubbemeyer T and Sandner W 2013 *Phys. Rev. Lett.* **110** 203002
[52] Zimmermann H, Buller J, Eilzer S and Eichmann U 2015 *Phys. Rev. Lett.* **114** 123003
[53] Zimmermann H, Patchkovskii S, Ivanov M and Eichmann U 2017 *Phys. Rev. Lett.* **118** 013003
[54] Bohm D 1952 *Phys. Rev.* **85** 16
[55] Bohm D 1952 *Phys. Rev.* **85** 180
[56] Holland P R 1993 *The Quantum Theory of Motion* (Cambridge: Cambridge University Press)
[57] Sanz Ángel S and Miret-Artés S 2013 *A Trajectory Description of Quantum Processes. I. Fundamentals* (New York: Springer), Vol. 850; *A Trajectory Description of Quantum Processes. II. Applications* (New York: Springer), Vol. 831
[58] Oriols X and Mompart J 2011 *Applied Bohmian Mechanics: From Nanoscale Systems to Cosmology* (Singapore: Pan Stanford Publishing)
[59] Wyatt R E 2005 *Quantum Dynamics with Trajectories: Introduction to Quantum Hydrodynamics* (New York and Heidelberg, Interdisciplinary Applied Mathematics, Springer)
[60] Benseny A, Albareda G, Sanz A S, Mompart J and Oriols X 2014 *Eur. Phys. J. D* **68** 286
[61] Benseny A, Bagudà J, Oriols X and Mompart J 2012 *Phys. Rev. A* **85** 053619
[62] Lopreore C L and Wyatt R E 1999 *Phys. Rev. Lett.* **82** 5190
[63] Albareda G, Marian D, Benali A, Yaro S, Zanghì N and Oriols X 2013 *J. Comput. Electron.* **12** 405
[64] Dürr D, Goldstein S, Tumulka R and Zanghì N 2004 *Phys. Rev. Lett.* **93** 090402
[65] Pinto-Neto N 2005 *Found. Phys.* **35** 577
[66] He D S, Gao D F and Cai Q Y 2014 *Phys. Rev. D* **89** 083510
[67] Wang B B 2008 *Chin. Phys. B* **17** 2817
[68] Bian X B and Bandrauk A D 2012 *Phys. Rev. Lett.* **108** 263003

[69] Zhou Y M, Tolstikhin O I and Morishita T 2016 *Phys. Rev. Lett.* **116** 173001
[70] Walt S G, Ram N B, Atala M, Shvetsov-Shilovski N I, Von Conta A, Baykusheva D, Lein M and Wörner H J 2017 *Nat. Commun.* **8** 15651
[71] He M R, Li Y, Zhou Y M, Li M, Cao W and Lu P X 2018 *Phys. Rev. Lett.* **120** 133204
[72] Meckel M, Comtois D, Zeidler D, Staudte A, Pavičić D, Bandulet H C, Pépin H, Kieffer J C, Dörner R, Villeneuve1 D M and Corkum P B 2008 *Science* **320** 1478
[73] Xu J, Blaga C I, Zhang K, Lai Y H, Lin C D, Miller T A, Agostini P and DiMauro L F 2014 *Nat. Commun.* **5** 4635
[74] Wei S S, Li S Y, Guo F M, Yang Y J and Wang B B 2013 *Phys. Rev. A* **87** 063418
[75] Jooya H Z, Telnov D A, Li P C and Chu S I 2015 *Phys. Rev. A* **91** 063412
[76] Douguet N and Bartschat K 2018 *Phys. Rev. A* **97** 013402
[77] Ivanov I A, Nam C H and Kim K T 2017 *Sci. Rep.* **7** 39919
[78] Takemotoa N and Becker A 2011 *J. Chem. Phys.* **134** 074309
[79] Sawada R, Sato T and Ishikawa K L 2014 *Phys. Rev. A* **90** 023404
[80] Xu L and He F 2019 *J. Opt. Soc. Am. B* **36** 840
[81] Lai X Y, Cai Q Y and Zhan M S 2009 *Eur. Phys. J. D* **53** 393
[82] Song Y, Yang Y J, Guo F M and Li S Y 2017 *J. Phys. B* **50** 095003
[83] Botheron P and Pons B 2010 *Phys. Rev. A* **82** 021404
[84] Cruz-Rodriguez L, Uranga-Pina L, Martinez-Mesa A and Meier C 2019 *Chem. Phys. Lett.* **715** 211
[85] Lai X Y, Cai Q Y and Zhan M S 2009 *New J. Phys.* **11** 113035
[86] Ni H C, Saalmann U and Rost J M 2018 *Phys. Rev. A* **97** 013426
[87] Jooya H Z, Telnov D A and Chu S I 2015 *Phys. Rev. A* **93** 063405
[88] Li P C, Sheu Y L, Jooya H Z, Zhou X X and Chu S I 2016 *Sci. Rep.* **6** 32763
[89] Christov I P 2006 *Opt. Express* **14** 6906
[90] Christov I P 2019 *Appl. Phys. B* **125** 209
[91] Hentschel M, Kienberger R, Spielmann C, Reider G A, Milosevic N, Brabec T, Corkum P B, Heinzmann U, Drescher M and Krausz F 2001 *Nature* **414** 509
[92] Itatani J, Levesque J, Zeidler D, Niikura H, Pépin H, Keiffer J C, Corkum P B and Villeneuve D M 2004 *Nature* **432** 867
[93] Lai X Y, Cai Q Y and Zhan M S 2010 *Chin. Phys. B* **19** 020302
[94] Song Y, Guo F M, Li S Y, Chen J G, Zeng S L and Yang Y J 2012 *Phys. Rev. A* **86** 033424
[95] Song Y, Li S Y, Liu X S, Guo F M and Yang Y J 2013 *Phys. Rev. A* **88** 053419
[96] Wu J, Augstein B B and Figueira de Morisson Faria C 2013 *Phys. Rev. A* **88** 023415
[97] Wu J, Augstein B B and Figueira de Morisson Faria C 2013 *Phys. Rev. A* **88** 063416
[98] Huang Y Y, Lai X Y and Liu X J 2018 *Chin. Phys. B* **27** 073204
[99] Jooya H Z, Telnov D A, Li P C and Chu S I 2015 *J. Phys. B* **48** 195401
[100] Wang J, Wang B B, Guo F M, Li S Y, Ding D J, Chen J G, Zeng S L and Yang Y J 2014 *Chin. Phys. B* **23** 053201
[101] Morishita Y, Liu X J, Saito N, Lischke T, Kato M, Prümper G, Oura M, Yamaoka H, Tamenori Y, Suzuki I H and Ueda K 2006 *Phys. Rev. Lett.* **96** 243402