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Heralded entanglement purification protocol using high-fidelity parity-check gate based on nitrogen-vacancy center in optical cavity*

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The decoherence of entangled states caused by the noisy channel is a salient problem for reducing the fidelity of quantum communication. Here we present a heralded two-photon entanglement purification protocol (EPP) using heralded high-fidelity parity-check gate (HH-PCG), which can increase the entanglement of nonlocal two-photon polarization mixed state. The HH-PCG is constructed by the input-output process of nitrogen-vacancy (NV) center in diamond embedded in a single-sided optical cavity, where the errors caused by the imperfect interaction between the NV center-cavity system and the photon can be heralded by the photon detector. As the unwanted components can be filtrated due to the heralded function, the fidelity of the EPP scheme can be enhanced considerably, which will increase the fidelity of quantum communication processing.

Keywords: quantum communication, heralded entanglement purification, heralded parity-check gate

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1. Introduction

Quantum entanglement is one of the marvellous phenomena of quantum mechanics, and it is the core resource for quantum communication such as quantum teleportation,[1,2] quantum dense coding,[3,4] quantum key distribution,[5–10] quantum secure direct communication,[11–16] and quantum secret sharing.[17,18] In quantum communication, photon systems are used as information carriers, and entangled photon systems are always prepared locally and distributed to remote communication nodes. During long-distance transmission, the photons will inevitably suffer from decoherence due to the environment and channel noises, which may change the maximally entangled state to partially entangled state (even to mixed state) and decrease the fidelity of quantum communication.

In order to ensure the quality of fiber-based long-distance quantum communication, the concept of quantum repeater was introduced.[19] A standard quantum repeater consists of three basic technologies:[20–22] entanglement swapping, entanglement purification, and quantum memory.[23–25] Recently, several quantum repeater schemes have been reported.[26–28] Entanglement purification protocol (EPP), as one of the key technologies in quantum repeater, is targeted to increase the fidelity of the required state in the mixed state. EPP is different from entanglement concentration,[29–31] which is aimed at increasing the entanglement of partially entangled pure state. In 1996, Bennet *et al.* proposed entanglement purification for the first time based on controlled-NOT (CNOT) gate.[32] In the same year, Deutsch *et al.* proposed an improved EPP by adding a pair of bilateral operations before the CNOT gate.[33] After that, several EPPs were proposed in theory[34–38] and demonstrated in experiment[39] with linear optics. In 2001, Pan *et al.*[37] presented a linear optical EPP for a photon system. In the next two years, their group showed that the linear optical EPP can be improved assisted by hyperentanglement.[38,39] In 2010, Sheng *et al.*[34] and Li[35] independently introduced the one-step deterministic EPP for a photon system using linear optics. The success of EPPs using linear optics depends on the post selection, which may decrease the efficiency of EPPs. To solve this problem, EPPs with quantum non-demolition detection (QND) were presented by using a nonlinear optical medium, such as a nonlinear Kerr medium,[40] a cavity quantum electrodynamics (QED) system containing quantum dots[41] or nitrogen-vacancy (NV) centers.[42–44]

NV center in diamond is a promising candidate in quantum information processing. Experiments have shown that the spin state of the electrons in NV center can be easily initialized,[45] coherently manipulated[46–48] and read out.[49,50] In addition, the electronic spin degree of freedom (DOF) in NV center has an ultra long coherent time

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even at room temperature, which is typically a few microseconds in natural diamond[51] or milliseconds in diamonds with rich ^{12}C.[52] The entanglement between the electron spin state of NV center and the single photon has been demonstrated in experiment,[53] and the interaction between the electron spin state of NV center and the single photon can be enhanced by coupling the NV center with an optical cavity.[54–57] With these natural advantages, NV center in diamond provides a great platform for many quantum information schemes, such as entanglement production and analysis,[50] entanglement purification[42–44] and concentration,[31] quantum communication,[58–60] quantum sensing,[61–63] and quantum computing.[64,65]

In former EPPs using QED with diamond NV center,[42–44] the ideal conditions (e.g. strong coupling strength $g \gg \sqrt{\kappa\gamma}$ and no cavity side leakage $\kappa_s = 0$) are required to get a high fidelity, which is hard to achieve with current technology. In this article, we propose a heralded EPP for a two-photon system using NV center in diamond embedded in a single-sided optical cavity under non-ideal conditions. We first construct a heralded high-fidelity parity-check gate (HH-PCG) by projecting the parity of two-photon polarization state to the electronic spin state of NV center using the input-output process of the NV center-cavity system, where the parity of the two-photon polarization state can be read out by detecting the electronic spin state of the NV center. The photon detector in the HH-PCG can alarm the errors caused by imperfect interaction between the NV center-cavity system and the photon. Then, we use the HH-PCG to purify the nonlocal two-photon polarization mixed state, which can increase the entanglement of the nonlocal two-photon polarization mixed state. In our heralded EPP scheme, the fidelity of the global EPP can be raised considerably in non-ideal conditions due to the heralded function of the HH-PCG. This can relax the technical requirement and enhance the practicality of the heralded EPP in current experiment conditions.

2. Interaction rule of photon and NV center in diamond embedded in optical cavity

A negatively charged NV center is a special defect in diamond, which contains an adjacent vacancy, a substitutional nitrogen atom and six electrons from the nitrogen and three carbon atoms surrounding the vacancy. It has a C_{3v} symmetry structure, which leads to electronic spin triplet ground states with a splitting of 2.88 GHz between the magnetic sublevels $|0\rangle$ ($m_s = 0$) and $|\pm1\rangle$ ($m_s = \pm1$).[66] According to the C_{3v} symmetry structure of the NV center and spin–orbit and spin–spin interactions, there exists six electronic excited states $|A_1\rangle$, $|A_2\rangle$, $|E_x\rangle$, $|E_y\rangle$, $|E_1\rangle$, and $|E_2\rangle$ in the absence of external strain and electric (or magnetic) field.[53] Among these excited states, $|A_2\rangle = \frac{1}{\sqrt{2}}(|E_-\rangle|+1\rangle + |E_+\rangle|-1\rangle)$ is a special state, and it will decay to two ground states $|-1\rangle$ and $|+1\rangle$ by emitting right circularly polarized photon ($|R\rangle$, $\sigma_+ = +1$) and left circularly polarized photon ($|L\rangle$, $\sigma_- = -1$) with equal probability as shown in Fig. 1(b). Here the right (left) circularly polarization direction of photon is defined based on the axis of the NV center. $|E_\pm\rangle$ stand for states with orbit angular momentum projections ±1 along the axis of the NV center, and the orbit angular momentum projection of ground states is 0.

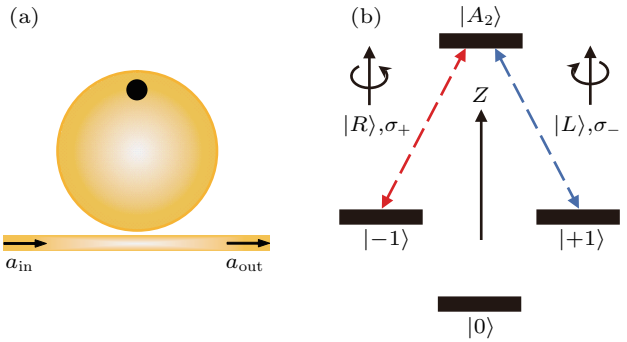


Fig. 1. (a) The NV center in a single-sided optical cavity (the optical cavity coupled to a waveguide). (b) The Λ-type energy levels of the NV center. Z is the axis of the NV center. The right (σ_+) and left (σ_-) circularly polarizations are defined by the axis of the NV center. The optical transitions $|-1\rangle \leftrightarrow |A_2\rangle$ and $|+1\rangle \leftrightarrow |A_2\rangle$ couple to right and left circularly polarized photons, respectively.

The electronic ground states $|\pm1\rangle$ and excited state $|A_2\rangle$ form a Λ-type triplet energy structure.[66] The transitions $|A_2\rangle \leftrightarrow |-1\rangle$ and $|A_2\rangle \leftrightarrow |+1\rangle$ couple to right and left circularly polarized photons with the same frequency, respectively.[53] These optical transitions can be enhanced by coupling the NV center to a single-sided optical cavity as shown in Fig. 1(a). The optical cavity has two modes degenerate in frequency but non-degenerate in polarization.[67] The axis of the NV center points along the propagation direction of the photon. The dynamic process of the NV center-cavity system can be described by the Hamiltonian[68]

$$\begin{aligned}\hat{H} =\ & \hbar\omega_0|A_2\rangle\langle A_2| + \hbar\omega_c\hat{a}_L^\dagger\hat{a}_L + \hbar\omega_c\hat{a}_R^\dagger\hat{a}_R \\ & + \hbar g(\hat{a}_L\hat{\sigma}_-^\dagger + \hat{a}_L^\dagger\hat{\sigma}_- + \hat{a}_R\hat{\sigma}_+^\dagger + \hat{a}_R^\dagger\hat{\sigma}_+),\end{aligned} \tag{1}$$

where g stands for the coupling strength between the NV center and the optical cavity; $\hat{a}_i^\dagger$ and $\hat{a}_i$ ($i = R, L$) are the creation and annihilation operators of the photon in circularly polarization state $|i\rangle$ with cavity field frequency ω_c, respectively; $\hat{\sigma}_- = |+1\rangle\langle A_2|$ and $\hat{\sigma}_+^\dagger = |A_2\rangle\langle+1|$ represent the transitions of the NV center with frequency ω_0.

The input-output property of the NV center-cavity system can be described by the Heisenberg–Langevin equations with the annihilation operator $\hat{a}$ (neglect the subscripts R and L) of the cavity field and the transition operator $\hat{\sigma}$ (neglect the subscripts $\pm$) of the NV center:[69–71]

$$\begin{aligned}\frac{\mathrm{d}\hat{a}}{\mathrm{d}t} &= -\left[\mathrm{i}(\omega_c - \omega_p) + \frac{\kappa}{2} + \frac{\kappa_s}{2}\right]\hat{a} - g\hat{\sigma} - \sqrt{\kappa}\hat{a}_{in} - \sqrt{\kappa_s}\hat{s}_{in}, \\ \frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}t} &= -\left[\mathrm{i}(\omega_0 - \omega_p) + \frac{\gamma}{2}\right]\hat{\sigma} - g\hat{\sigma}_z\hat{a} + \sqrt{\gamma}\hat{\sigma}_z + \hat{N},\end{aligned}$$

$$\hat{a}_{\rm out} = \hat{a}_{\rm in} + \sqrt{\kappa}\hat{a}, \tag{2}$$

where κ and $\kappa_{\rm s}$ are the decay rate and side leakage rate of the cavity field; $\hat{\sigma}_z$ ($\hat{\sigma}_z = \hat{\sigma}^\dagger\hat{\sigma} - \hat{\sigma}\hat{\sigma}^\dagger$) and $\gamma/2$ are the inversion operator and decay rate of the NV center; $\hat{N}$ stands for the noise bath related to the spontaneous emission of the NV center; $\omega_{\rm p}$ is the frequency of input photon; $\hat{a}_{\rm in}$, $\hat{a}_{\rm out}$, and $\hat{s}_{\rm in}$ are the annihilation operators of input, output field and side-leakage operator of cavity field, respectively. Assuming that κ is large enough to get a weak excitation limitation where the NV center is dominantly in the ground state and $\langle\hat{\sigma}_z\rangle = -1$, the reflection coefficient of the NV center-cavity system ($r(\omega_{\rm p}) = \hat{a}_{\rm out}/\hat{a}_{\rm in}$) can be written as

$$r(\omega_{\rm p}) = \frac{\left[{\rm i}(\omega_{\rm c}-\omega_{\rm p}) - \frac{\kappa}{2} + \frac{\kappa_{\rm s}}{2}\right]\left[{\rm i}(\omega_0-\omega_{\rm p}) + \frac{\gamma}{2}\right] + g^2}{\left[{\rm i}(\omega_{\rm c}-\omega_{\rm p}) + \frac{\kappa}{2} + \frac{\kappa_{\rm s}}{2}\right]\left[{\rm i}(\omega_0-\omega_{\rm p}) + \frac{\gamma}{2}\right] + g^2}. \tag{3}$$

If $\omega_{\rm c} = \omega_{\rm p} = \omega_0$, the reflection coefficient can be expressed as

$$r(\omega_{\rm p}) = \frac{(\kappa_{\rm s}-\kappa)\gamma + 4g^2}{(\kappa_{\rm s}+\kappa)\gamma + 4g^2}. \tag{4}$$

In the case of $g = 0$, the reflection coefficient changes to

$$r_0(\omega_{\rm p}) = \frac{\kappa_{\rm s}-\kappa}{\kappa_{\rm s}+\kappa}. \tag{5}$$

In general, the interaction rule of circularly polarized photon and the NV center-cavity system is listed as follows:

$$\begin{aligned} &|R\rangle|-1\rangle \Rightarrow r|R\rangle|-1\rangle, \quad |R\rangle|+1\rangle \Rightarrow r_0|R\rangle|+1\rangle, \\ &|L\rangle|-1\rangle \Rightarrow r_0|L\rangle|-1\rangle, \quad |L\rangle|+1\rangle \Rightarrow r|L\rangle|+1\rangle. \end{aligned} \tag{6}$$

3. Heralded two-photon entanglement purification protocol

3.1. Heralded high-fidelity parity-check gate for two photons in polarization DOF

The quantum circuit of the HH-PCG for polarization DOF of a two-photon system is shown in Fig. 2(a). The subscripts 1 and 2 stand for the first photon and the second photon. The state of the NV center in the HH-PCG is initialized to $|\varphi_N^+\rangle$ ($|\varphi_N^\pm\rangle \equiv (|-1\rangle \pm |+1\rangle)/\sqrt{2}$). Two photons 1 and 2 in the state $(\alpha_i|R_i\rangle + \beta_i|L_i\rangle)$ $(i = 1, 2)$ are imported into the quantum circuit (shown in Fig. 2(a)) in sequence. The initial state of the system comprised of two photons and the NV center is

$$|\Psi_0\rangle = (\alpha_1|R_1\rangle + \beta_1|L_1\rangle)(\alpha_2|R_2\rangle + \beta_2|L_2\rangle)|\varphi_N^+\rangle. \tag{7}$$

First, photon 1 is imported into the quantum circuit using the optical switch SW_1. After photon 1 passes through the circularly polarization beam splitter ($CPBS_1$), which transmits the right circularly polarized component ($|R\rangle$) and reflects the left circularly polarized component ($|L\rangle$), the two circularly polarized components of photon 1 are split into two spatial modes. The state of the system comprised of two photons and the NV center is changed to

$$|\Psi_1\rangle = (\alpha_1|R_1'\rangle + \beta_1|L_1\rangle)(\alpha_2|R_2\rangle + \beta_2|L_2\rangle)|\varphi_N^+\rangle. \tag{8}$$

Then a Hadamard operation H_1, which makes $|R'\rangle \to (|R'\rangle + |L'\rangle)/\sqrt{2}$, is operated on the component $|R'\rangle$. An attenuator A with transmission rate κ is performed on the component $|L\rangle$ at the same time. The state of the system comprised of two photons and the NV center will be transformed to

$$\begin{aligned} |\Psi_2\rangle = &\left[\frac{1}{\sqrt{2}}\alpha_1(|R_1'\rangle + |L_1'\rangle) + \kappa\beta_1|L_1\rangle\right] \\ &\otimes(\alpha_2|R_2\rangle + \beta_2|L_2\rangle)|\varphi_N^+\rangle. \end{aligned} \tag{9}$$

Subsequently, the component $\frac{1}{\sqrt{2}}\alpha_1(|R_1'\rangle + |L_1'\rangle)$ interacts with the NV center-cavity system. After the interaction, another Hadamard operation H_2 and a bit-flip operation X, which makes $|R\rangle \leftrightarrow |L\rangle$, are performed on photon 1. At this time, the state of the system comprised of two photons and the NV center is changed to

$$\begin{aligned} |\Psi_3\rangle = &\left\{\frac{1}{2}\alpha_1[(r+r_0)|L_1'\rangle|\varphi_N^+\rangle + (r-r_0)|R_1'\rangle|\varphi_N^-\rangle]\right. \\ &\left. + \kappa\beta_1|L_1\rangle|\varphi_N^+\rangle\right\}(\alpha_2|R_2\rangle + \beta_2|L_2\rangle). \end{aligned} \tag{10}$$

When photon 1 passes through $CPBS_2$, the component $|L'\rangle$ will be reflected to the output port with single-photon detector D, and the components $|R'\rangle$ and $|L\rangle$ will reunite and be transmitted to the output port with SW_2. The click of the detector D is an error alarm signal, and it denotes that the state of the system has collapsed in the wrong state $\frac{1}{2}\alpha_1(r+r_0)|L_1'\rangle(\alpha_2|R_2\rangle + \beta_2|L_2\rangle)|\varphi_N^+\rangle$. If the detector D does not click, the state of the system comprised of two photons and the NV center will be

$$\begin{aligned} |\Psi_4\rangle = &\left[\frac{(r-r_0)}{2}\alpha_1|R_1'\rangle|\varphi_N^-\rangle + \kappa\beta_1|L_1\rangle|\varphi_N^+\rangle\right] \\ &\otimes(\alpha_2|R_2\rangle + \beta_2|L_2\rangle). \end{aligned} \tag{11}$$

To balance the amplitudes of the components $|R'\rangle$ and $|L\rangle$, the transmission rate of A should be set as $\kappa = (r-r_0)/2$, and the state of the system comprised of two photons and the NV center is expressed as

$$\begin{aligned} |\Psi_5\rangle = &\kappa\alpha_1|R_1\rangle(\alpha_2|R_2\rangle + \beta_2|L_2\rangle)|\varphi_N^-\rangle \\ &+ \kappa\beta_1|L_1\rangle(\alpha_2|R_2\rangle + \beta_2|L_2\rangle)|\varphi_N^+\rangle. \end{aligned} \tag{12}$$

If the detector D clicks, this parity-check process is failed without affecting the states of the NV center and photon 2, and another photon can be put into the quantum circuit to begin another parity-check process.

After photon 1 passes through the quantum circuit without triggering the detector D, photon 2 is imported into the quantum circuit using the optical switch SW_1. After photon 2 passes through the quantum circuit shown in Fig. 2(a) without triggering the detector D, the state of the system comprised of two photons and the NV center can be expressed as

$$|\Psi_6\rangle = \kappa^2(\alpha_1\alpha_2|R_1\rangle|R_2\rangle + \beta_1\beta_2|L_1\rangle|L_2\rangle)|\varphi_N^+\rangle$$

$$+\kappa^2(\alpha_1\beta_2|R_1\rangle|L_2\rangle+\beta_1\alpha_2|L_1\rangle|R_2\rangle)|\varphi_N^-\rangle. \quad (13)$$

Then the state of the NV center is measured on the basis $\{|\varphi_N^\pm\rangle\}$. If the measurement result is $|\varphi_N^+\rangle$, the two photons are in even parity mode (i.e. $|R_1\rangle|R_2\rangle$ and $|L_1\rangle|L_2\rangle$). If the measurement result is $|\varphi_N^-\rangle$, the two photons are in odd parity mode (i.e. $|R_1\rangle|L_2\rangle$ and $|L_1\rangle|R_2\rangle$). Now, we have obtained the result of the HH-PCG, which is the essential component in the heralded EPP scheme for the photon system.

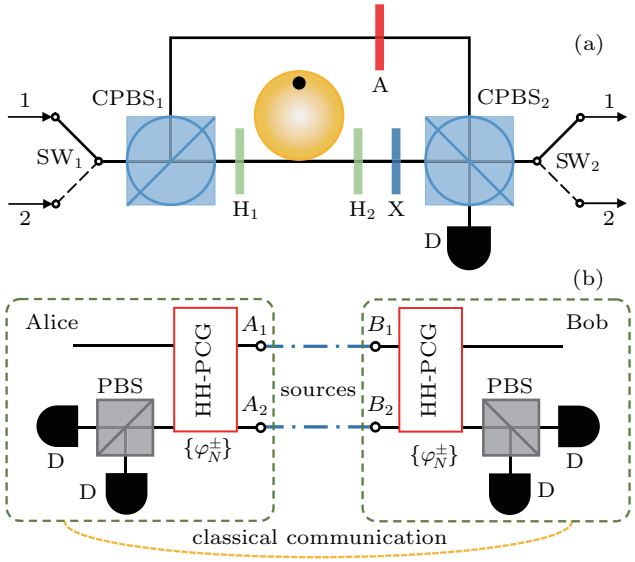


Fig. 2. (a) Scheme of the HH-PCG. Here 1 and 2 are two input ports for photon 1 and photon 2, respectively. CPBS$_i$ ($i=1,2$) is the circularly polarization beam splitter that transmits right circularly polarized component $|R\rangle$ and reflects left circularly polarized component $|L\rangle$. SW$_j$ ($j=1,2$) is an optical switch, which is used to control the path that photons pass through. H$_k$ ($k=1,2$) is a half-wave plate which performs the Hadamard operation on the photon system in polarization DOF ($|R\rangle\rightarrow\frac{1}{\sqrt{2}}(|R\rangle+|L\rangle)$, $|L\rangle\rightarrow\frac{1}{\sqrt{2}}(|R\rangle-|L\rangle)$). X is another half-wave plate that performs the bit flip operation on the photon system in polarization DOF ($|R\rangle\leftrightarrow|L\rangle$). A is an attenuator with transmission rate κ, which balances the amplitudes of two components in different spatial modes. D is the single-photon detector. (b) Scheme of the heralded EPP. The photons A_1A_2 and B_1B_2 belong to Alice and Bob, respectively. Here $\{\varphi_N^\pm\}$ represents the measurement of the NV center on the basis $\{|\varphi_N^+\rangle,|\varphi_N^-\rangle\}$. PBS is the polarization beam splitter that transmits the horizontal polarized component $|H\rangle=(|R\rangle+|L\rangle)/\sqrt{2}$ and reflects the vertical polarized component $|V\rangle=(|R\rangle-|L\rangle)/(\sqrt{2}\mathrm{i})$.

3.2. Heralded EPP scheme for two-photon system

The photon systems are initially prepared in the maximally entangled Bell state $|\phi^+\rangle=(|R_{A_i}R_{B_i}\rangle+|L_{A_i}L_{B_i}\rangle)/\sqrt{2}$, $(i=1,2,\cdots,n)$. Alice preserves the photon A_i and sends the photon B_i to Bob. After passing through the noisy communication channel, the photon B_i may suffer from bit-flip error and phase-flip error. The bit-flip error will cause a bit flip ($|R\rangle\leftrightarrow|L\rangle$) on the photonic qubit, and the phase-flip error will cause a phase flip ($|R\rangle\rightarrow|R\rangle,|L\rangle\rightarrow-|L\rangle$) on the photonic qubit. If the photon B_i passes through the noisy communication channel with bit-flip error, the state of the photon system will be changed from the maximally entangled Bell state to the mixed state

$$\rho_{A_iB_i}=F|\phi^+\rangle_{A_iB_i}\langle\phi^+|+(1-F)|\psi^+\rangle_{A_iB_i}\langle\psi^+|, \quad (14)$$

where $|\psi^+\rangle=(|R_{A_i}L_{B_i}\rangle+|L_{A_i}R_{B_i}\rangle)/\sqrt{2}$. F is the fidelity of the maximally entangled Bell state $|\phi^+\rangle$ in the mixed state. In two-photon Bell state, the phase-flip error can be transformed to bit-flip error by bilateral unitary operations.

In order to increase the fidelity of the maximally entangled Bell state $|\phi^+\rangle$ in the mixed state, a heralded entanglement purification protocol is introduced. The quantum circuit of the heralded EPP scheme for the two-photon system with bit-flip error is shown in Fig. 2(b), where the two remote users, Alice and Bob, both operate HH-PCGs on their photon pairs. The NV centers of HH-PCGs are initially prepared in the state $|\varphi_N^+\rangle$. Two nonlocal entangled photon pairs A_1B_1 and A_2B_2 are required in the heralded EPP scheme, and they are initially in the state

$$\begin{aligned}\rho &= \rho_{A_1B_1}\otimes\rho_{A_2B_2}\\ &= [F|\phi^+\rangle_{A_1B_1}\langle\phi^+|+(1-F)|\psi^+\rangle_{A_1B_1}\langle\psi^+|]\\ &\quad\otimes[F|\phi^+\rangle_{A_2B_2}\langle\phi^+|+(1-F)|\psi^+\rangle_{A_2B_2}\langle\psi^+|]. \quad (15)\end{aligned}$$

That is, the four-photon system is in the state with four constituents: $|\phi^+\rangle_{A_1B_1}|\phi^+\rangle_{A_2B_2}$ with possibility F^2, $|\phi^+\rangle_{A_1B_1}|\psi^+\rangle_{A_2B_2}$ with possibility $F(1-F)$, $|\psi^+\rangle_{A_1B_1}|\phi^+\rangle_{A_2B_2}$ with possibility $F(1-F)$, and $|\psi^+\rangle_{A_1B_1}|\psi^+\rangle_{A_2B_2}$ with possibility $(1-F)^2$.

(1) For the first constituent, the four-photon system $A_1B_1A_2B_2$ is in the state $|\phi^+\rangle_{A_1B_1}|\phi^+\rangle_{A_2B_2}$ with possibility F^2. The initial state of the system composed of the four-photon system and two NV centers of HH-PCGs is

$$\begin{aligned}|\Psi_0\rangle_{\phi^+\phi^+} &= \frac{1}{2}(|R_{A_1}\rangle|R_{B_1}\rangle+|L_{A_1}\rangle|L_{B_1}\rangle)(|R_{A_2}\rangle|R_{B_2}\rangle\\ &\quad+|L_{A_2}\rangle|L_{B_2}\rangle)\otimes|\varphi_N^+\rangle_A|\varphi_N^+\rangle_B. \quad (16)\end{aligned}$$

After the two photons A_1B_1 pass through the two HH-PCGs respectively (shown in Fig. 2(b)), the state of the system composed of the four-photon system and two NV centers is changed to

$$\begin{aligned}|\Psi_0'\rangle_{\phi^+\phi^+} &= \frac{1}{2}\Big\{\frac{1}{4}[(r+r_0)|L_{A_1}'\rangle|\varphi_N^+\rangle_A+(r-r_0)|R_{A_1}'\rangle|\varphi_N^-\rangle_A]\\ &\quad\otimes[(r+r_0)|L_{B_1}'\rangle|\varphi_N^+\rangle_B+(r-r_0)|R_{B_1}'\rangle|\varphi_N^-\rangle_B]\\ &\quad+\kappa^2|L_{A_1}\rangle|L_{B_1}\rangle|\varphi_N^+\rangle_A|\varphi_N^+\rangle_B]\Big\}(|R_{A_2}\rangle|R_{B_2}\rangle\\ &\quad+|L_{A_2}\rangle|L_{B_2}\rangle). \quad (17)\end{aligned}$$

If the detectors D of HH-PCGs do not click, the state of the system composed of the four-photon system and two NV centers will be

$$\begin{aligned}|\Psi_1\rangle_{\phi^+\phi^+} &= \frac{\kappa^2}{2}|R_{A_1}\rangle|R_{B_1}\rangle(|R_{A_2}\rangle|R_{B_2}\rangle\\ &\quad+|L_{A_2}\rangle|L_{B_2}\rangle)|\varphi_N^-\rangle_A|\varphi_N^-\rangle_B\\ &\quad+\frac{\kappa^2}{2}|L_{A_1}\rangle|L_{B_1}\rangle(|R_{A_2}\rangle|R_{B_2}\rangle\\ &\quad+|L_{A_2}\rangle|L_{B_2}\rangle)|\varphi_N^+\rangle_A|\varphi_N^+\rangle_B. \quad (18)\end{aligned}$$

Here $\kappa=(r-r_0)/2$. After the two photons A_2B_2 pass through the two HH-PCGs, the state of the system composed of the four-photon system and two NV centers is changed to

$$\begin{aligned}|\Psi_2\rangle_{\phi^+\phi^+} =& \frac{\kappa^4}{2}(|R_{A_1}\rangle|R_{B_1}\rangle|R_{A_2}\rangle|R_{B_2}\rangle\\ &+|L_{A_1}\rangle|L_{B_1}\rangle|L_{A_2}\rangle|L_{B_2}\rangle)|\varphi_N^+\rangle_A|\varphi_N^+\rangle_B\\ &+\frac{\kappa^4}{2}(|R_{A_1}\rangle|R_{B_1}\rangle|L_{A_2}\rangle|L_{B_2}\rangle\\ &+|L_{A_1}\rangle|L_{B_1}\rangle|R_{A_2}\rangle|R_{B_2}\rangle)|\varphi_N^-\rangle_A|\varphi_N^-\rangle_B, \quad (19)\end{aligned}$$

if the detectors D of HH-PCGs do not click.

After the four photons pass through the two HH-PCGs, the two photons A_2B_2 will pass through PBSs, which can measure the polarization state of photon on the basis $\{|H\rangle,|V\rangle\}$. Here $|H\rangle=(|R\rangle+|L\rangle)/\sqrt{2}$ and $|V\rangle=(|R\rangle-|L\rangle)/(\sqrt{2}\mathrm{i})$. The state of the system composed of the four-photon system and two NV centers can be rewritten as

$$\begin{aligned}&|\Psi_2\rangle_{\phi^+\phi^+}\\ =&\frac{\kappa^4}{4}[(|R_{A_1}\rangle|R_{B_1}\rangle+|L_{A_1}\rangle|L_{B_1}\rangle)(|H_{A_2}\rangle|H_{B_2}\rangle-|V_{A_2}\rangle|V_{B_2}\rangle)\\ &+\mathrm{i}(|R_{A_1}\rangle|R_{B_1}\rangle-|L_{A_1}\rangle|L_{B_1}\rangle)(|H_{A_2}\rangle|V_{B_2}\rangle\\ &+|V_{A_2}\rangle|H_{B_2}\rangle)]|\varphi_N^+\rangle_A|\varphi_N^+\rangle_B\\ &+\frac{\kappa^4}{4}[(|L_{A_1}\rangle|L_{B_1}\rangle+|R_{A_1}\rangle|R_{B_1}\rangle)(|H_{A_2}\rangle|H_{B_2}\rangle-|V_{A_2}\rangle|V_{B_2}\rangle)\\ &+\mathrm{i}(|L_{A_1}\rangle|L_{B_1}\rangle-|R_{A_1}\rangle|R_{B_1}\rangle)(|H_{A_2}\rangle|V_{B_2}\rangle\\ &+|V_{A_2}\rangle|H_{B_2}\rangle)]|\varphi_N^-\rangle_A|\varphi_N^-\rangle_B. \quad (20)\end{aligned}$$

Alice and Bob measure the states of photon pair A_2B_2 and two NV centers on the bases $\{|H\rangle,|V\rangle\}$ and $\{|\varphi_N^\pm\rangle\}$, respectively. If the two NV centers are both in the state $|\varphi_N^+\rangle$ (or $|\varphi_N^-\rangle$) and the two photons A_2B_2 are in the state $|H_{A_2}\rangle|H_{B_2}\rangle$ (or $|V_{A_2}\rangle|V_{B_2}\rangle$), the two-photon system A_1B_1 will be in the state $|\phi^+\rangle_{A_1B_1}$. If the two NV centers are both in the state $|\varphi_N^+\rangle$ (or $|\varphi_N^-\rangle$) and the two photons A_2B_2 are in the state $|H_{A_2}\rangle|V_{B_2}\rangle$ (or $|V_{A_2}\rangle|H_{B_2}\rangle$), the two-photon system A_1B_1 will be in the state $|\phi^-\rangle_{A_1B_1}=(|R_{A_1}R_{B_1}\rangle-|L_{A_1}L_{B_1}\rangle)/\sqrt{2}$, and a phase flip operation $|R\rangle\rightarrow|R\rangle,|L\rangle\rightarrow-|L\rangle$ is performed on photon A_1 (or B_1) to obtain the state $|\phi^+\rangle_{A_1B_1}$.

If the detectors D of HH-PCGs click, the state of the system composed of the four-photon system and two NV centers is in the wrong term, where two photons A_1 (or A_2) and B_1 (or B_2) are in the states $|L_A'\rangle$ and $|L_B'\rangle$ and detected by the detectors D. In this case, the states of the other photon pair and two NV centers are not affected, and another nonlocal photon pair A_3B_3 can be imported into quantum circuit in Fig. 2(b) to implement the heralded EPP scheme.

(2) For the second constituent, the four-photon system $A_1B_1A_2B_2$ is in the state $|\phi^+\rangle_{A_1B_1}|\psi^+\rangle_{A_2B_2}$ with possibility $F(1-F)$. The initial state of the system composed of the four-photon system and two NV centers of HH-PCGs is

$$\begin{aligned}|\Psi_0\rangle_{\phi^+\psi^+} =& \frac{1}{2}(|R_{A_1}\rangle|R_{B_1}\rangle+|L_{A_1}\rangle|L_{B_1}\rangle)(|R_{A_2}\rangle|L_{B_2}\rangle\\ &+|L_{A_2}\rangle|R_{B_2}\rangle)\otimes|\varphi_N^+\rangle_A|\varphi_N^+\rangle_B. \quad (21)\end{aligned}$$

After two photon pairs pass through the HH-PCGs and PBSs, the state of the system composed of the four-photon system and two NV centers of HH-PCGs will be changed to

$$\begin{aligned}&|\Psi_2\rangle_{\phi^+\psi^+}\\ =&\frac{\kappa^4}{4}[(|R_{A_1}\rangle|R_{B_1}\rangle+|L_{A_1}\rangle|L_{B_1}\rangle)(|H_{A_2}\rangle|H_{B_2}\rangle+|V_{A_2}\rangle|V_{B_2}\rangle)\\ &+\mathrm{i}(|R_{A_1}\rangle|R_{B_1}\rangle-|L_{A_1}\rangle|L_{B_1}\rangle)(|V_{A_2}\rangle|H_{B_2}\rangle\\ &-|H_{A_2}\rangle|V_{B_2}\rangle)]|\varphi_N^+\rangle_A|\varphi_N^-\rangle_B\\ &+\frac{\kappa^4}{4}[(|R_{A_1}\rangle|R_{B_1}\rangle+|L_{A_1}\rangle|L_{B_1}\rangle)(|H_{A_2}\rangle|H_{B_2}\rangle+|V_{A_2}\rangle|V_{B_2}\rangle)\\ &+\mathrm{i}(|R_{A_1}\rangle|R_{B_1}\rangle-|L_{A_1}\rangle|L_{B_1}\rangle)(|H_{A_2}\rangle|V_{B_2}\rangle\\ &-|V_{A_2}\rangle|H_{B_2}\rangle)]|\varphi_N^-\rangle_A|\varphi_N^+\rangle_B, \quad (22)\end{aligned}$$

if the detectors D of HH-PCGs do not click.

For the third constituent, the four-photon system $A_1B_1A_2B_2$ is in the state $|\psi^+\rangle_{A_1B_1}|\phi^+\rangle_{A_2B_2}$ with possibility $F(1-F)$. The initial state of the system composed of the four-photon system and two NV centers of HH-PCGs is

$$\begin{aligned}|\Psi_0\rangle_{\psi^+\phi^+} =& \frac{1}{2}(|R_{A_1}\rangle|L_{B_1}\rangle+|L_{A_1}\rangle|R_{B_1}\rangle)(|R_{A_2}\rangle|R_{B_2}\rangle\\ &+|L_{A_2}\rangle|L_{B_2}\rangle)\otimes|\varphi_N^+\rangle_A|\varphi_N^+\rangle_B. \quad (23)\end{aligned}$$

After two photon pairs pass through the HH-PCGs and PBSs, the state of the system composed of the four-photon system and two NV centers of HH-PCGs will be changed to

$$\begin{aligned}&|\Psi_2\rangle_{\psi^+\phi^+}\\ =&\frac{\kappa^4}{4}[(|R_{A_1}\rangle|L_{B_1}\rangle+|L_{A_1}\rangle|R_{B_1}\rangle)(|H_{A_2}\rangle|H_{B_2}\rangle-|V_{A_2}\rangle|V_{B_2}\rangle)\\ &+\mathrm{i}(|R_{A_1}\rangle|L_{B_1}\rangle-|L_{A_1}\rangle|R_{B_1}\rangle)(|V_{A_2}\rangle|H_{B_2}\rangle\\ &+|H_{A_2}\rangle|V_{B_2}\rangle)]|\varphi_N^+\rangle_A|\varphi_N^-\rangle_B\\ &+\frac{\kappa^4}{4}[(|L_{A_1}\rangle|R_{B_1}\rangle+|R_{A_1}\rangle|L_{B_1}\rangle)(|H_{A_2}\rangle|H_{B_2}\rangle-|V_{A_2}\rangle|V_{B_2}\rangle)\\ &+\mathrm{i}(|L_{A_1}\rangle|R_{B_1}\rangle-|R_{A_1}\rangle|L_{B_1}\rangle)(|H_{A_2}\rangle|V_{B_2}\rangle\\ &+|V_{A_2}\rangle|H_{B_2}\rangle)]|\varphi_N^-\rangle_A|\varphi_N^+\rangle_B, \quad (24)\end{aligned}$$

if the detectors D of HH-PCGs do not click.

Alice and Bob measure two NV centers with the basis $\{|\varphi_N^\pm\rangle\}$, and the measurement results show that the two NV centers are in different states (either $|\varphi_N^+\rangle_A|\varphi_N^-\rangle_B$ or $|\varphi_N^-\rangle_A|\varphi_N^+\rangle_B$). In these two constituents, Alice and Bob cannot identify which photon pair has bit-flip error, so they have to discard the result of these two constituents. If the detectors D of HH-PCGs click, the state of the system composed of the four-photon system and two NV centers will be in the wrong term, and another nonlocal photon pair A_3B_3 can be imported into the quantum circuit in Fig. 2(b) to implement the heralded EPP scheme.

(3) For the fourth constituent, the four-photon system $A_1B_1A_2B_2$ is in the state $|\psi^+\rangle_{A_1B_1}|\psi^+\rangle_{A_2B_2}$ with possibility

$(1-F)^2$. The initial state of the system composed of the four-photon system and two NV centers of HH-PCGs is

$$|\Psi_0\rangle_{\psi^+\psi^+} = \frac{1}{4}(|R_{A_1}\rangle|L_{B_1}\rangle+|L_{A_1}\rangle|R_{B_1}\rangle)(|R_{A_2}\rangle|L_{B_2}\rangle + |L_{A_2}\rangle|R_{B_2}\rangle)\otimes|\varphi_N^+\rangle_A|\varphi_N^+\rangle_B. \quad (25)$$

After two photon pairs pass through the HH-PCGs and PBSs, the state of the system composed of the four-photon system and two NV centers of HH-PCGs will be changed to

$$\begin{aligned}
&|\Psi_2\rangle_{\psi^+\psi^+}\\
&= \frac{\kappa^4}{4}[(|R_{A_1}\rangle|L_{B_1}\rangle+|L_{A_1}\rangle|R_{B_1}\rangle)(|H_{A_2}\rangle|H_{B_2}\rangle+|V_{A_2}\rangle|V_{B_2}\rangle)\\
&+\mathrm{i}(|R_{A_1}\rangle|L_{B_1}\rangle-|L_{A_1}\rangle|R_{B_1}\rangle)(|V_{A_2}\rangle|H_{B_2}\rangle\\
&-|H_{A_2}\rangle|V_{B_2}\rangle)]|\varphi_N^+\rangle_A|\varphi_N^+\rangle_B\\
&+\frac{\kappa^4}{4}[(|R_{A_1}\rangle|L_{B_1}\rangle+|L_{A_1}\rangle|R_{B_1}\rangle)(|H_{A_2}\rangle|H_{B_2}\rangle+|V_{A_2}\rangle|V_{B_2}\rangle)\\
&+\mathrm{i}(|R_{A_1}\rangle|L_{B_1}\rangle-|L_{A_1}\rangle|R_{B_1}\rangle)(|H_{A_2}\rangle|V_{B_2}\rangle\\
&-|V_{A_2}\rangle|H_{B_2}\rangle)]|\varphi_N^-\rangle_A|\varphi_N^-\rangle_B, \quad (26)
\end{aligned}$$

if the detectors D of HH-PCGs do not click.

Alice and Bob measure the states of photon pair A_2B_2 and two NV centers with bases $\{|H\rangle,|V\rangle\}$ and $|\varphi_N^\pm\rangle$, respectively. If the two NV centers are both in the state $|\varphi_N^+\rangle$ (or $|\varphi_N^-\rangle$) and the two photons A_2B_2 are in the state $|H_{A_2}\rangle|H_{B_2}\rangle$ (or $|V_{A_2}\rangle|V_{B_2}\rangle$), the two-photon system A_1B_1 is in the state $|\psi^+\rangle_{A_1B_1}$. If the two NV centers are both in the state $|\varphi_N^+\rangle$ (or $|\varphi_N^-\rangle$) and the two photons A_2B_2 are in the state $|H_{A_2}\rangle|V_{B_2}\rangle$ (or $|V_{A_2}\rangle|H_{B_2}\rangle$), the two-photon system A_1B_1 is in the state $|\psi^-\rangle_{A_1B_1}=\frac{1}{\sqrt{2}}(|R_{A_1}\rangle|L_{B_1}\rangle-|L_{A_1}\rangle|R_{B_1}\rangle)$, and a phase flip operation $|R\rangle\rightarrow|R\rangle,|L\rangle\rightarrow-|L\rangle$ is performed on photon A_1 (or B_1) to obtain the state $|\psi^+\rangle_{A_1B_1}$. If the detectors D of HH-PCGs click, the state of the system composed of the four-photon system and two NV centers will be in the wrong term, and another nonlocal photon pair A_3B_3 can be imported into the quantum circuit in Fig. 2(b) to implement the heralded EPP scheme.

Finally, Alice and Bob can obtain a new mixed state

$$\rho'_{A_1B_1} = F'|\phi^+\rangle_{A_1B_1}\langle\phi^+|+(1-F')|\psi^+\rangle_{A_1B_1}\langle\psi^+|, \quad (27)$$

where $F'=\frac{F^2}{F^2+(1-F)^2}$. When $F>1/2$, the fidelity of state $|\phi^+\rangle_{A_1B_1}$ is increased ($F'>F$). By iterating the heralded EPP for several rounds, the parties will share a subset of two-photon systems with high (nearly unity) fidelity of state $|\phi^+\rangle_{A_iB_i}$. From Fig. 3(a), we can see that the fidelity of state $|\phi^+\rangle_{A_1B_1}$ in the mixed state will approach to 1 quickly in just 3 rounds of heralded EPP process.

4. Discussion

We have presented a heralded two-photon EPP scheme using the HH-PCG based on an NV center embedded in a single-sided optical cavity. The HH-PCG based on cavity QED is the essential part of this heralded EPP, where the interaction between the photon and the NV center-cavity system is the most important mechanics. The NV center in diamond is a promising dipole emitter in cavity QED. The dephasing of the NV center can be ignored, because the electron-spin coherent time of the NV center[51,52] is much longer than the input-output process,[53] the electron-spin manipulation time (about subnanosecond),[48] the electron-spin readout time (about 100 μs),[49] and the photon coherent time. Therefore, the coupling between the NV center and the optical cavity is a main factor for influencing the fidelity of quantum operations.

Under an ideal condition, the coupling strength between the NV center and the optical cavity is $g\gg\sqrt{\kappa\gamma}$, and the side leakage rate of cavity is $\kappa_s=0$. In this case, the reflection coefficients of the photon interacted with the NV center-cavity system are $r\approx1$ and $r_0\approx-1$, according to Eqs. (4) and (5). The result of the PCG under the ideal condition is written as

$$|\varphi_i\rangle = (\alpha_1\alpha_2|R_1\rangle|R_2\rangle+\beta_1\beta_2|L_1\rangle|L_2\rangle)|\varphi_N^+\rangle + (\alpha_1\beta_2|R_1\rangle|L_2\rangle+\beta_1\alpha_2|L_1\rangle|R_2\rangle)|\varphi_N^-\rangle. \quad (28)$$

However, in practical applications, the coupling strength g is limited, and the side leakage rate of optical cavity cannot be ignored, which will reduce the fidelity of the PCG. In order to increase the fidelity of the PCG, we introduce the heralded high-fidelity method. The result of the HH-PCG under practical conditions can be written as

$$|\varphi_r\rangle = \frac{(r-r_0)^2}{4}(\alpha_1\alpha_2|R_1\rangle|R_2\rangle+\beta_1\beta_2|L_1\rangle|L_2\rangle)|\varphi_N^+\rangle + \frac{(r-r_0)^2}{4}(\alpha_1\beta_2|R_1\rangle|L_2\rangle+\beta_1\alpha_2|L_1\rangle|R_2\rangle)|\varphi_N^-\rangle. \quad (29)$$

In order to evaluate the quality of the HH-PCG, the average fidelity and efficiency should be calculated. The average fidelity of the HH-PCG is defined as

$$\overline{F} = \overline{|\langle\varphi_r|\varphi_i\rangle|^2} = \frac{1}{(2\pi)^2}\int_0^{2\pi}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi|\langle\varphi_r|\varphi_i\rangle|^2, \quad (30)$$

where $\cos\theta=\alpha_1$, $\sin\theta=\beta_1$, $\cos\phi=\alpha_2$, and $\sin\phi=\beta_2$, and $|\varphi_r\rangle$ and $|\varphi_i\rangle$ should be normalized. The overline indicates the average value of all possible input (output) states. In our heralded high-fidelity scheme, the average fidelity $\overline{F}$ keeps 100% under near-realistic conditions, because the parameters that lead to the imperfect interaction between the NV center-cavity system and the photon with finite line-width are concentrated to a global coefficient $(r-r_0)^2/4$ of the final state as shown in Eq. (29). However, the imperfect interaction will decrease the average efficiency of our HH-PCG. The average efficiency of the HH-PCG is defined as the ratio between the numbers of output and input photons, which is

$$\overline{\eta} = \frac{1}{(2\pi)^2}\int_0^{2\pi}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\frac{n_{\mathrm{out}}}{n_{\mathrm{in}}} = \frac{1}{16}(r-r_0)^4$$

$$= \frac{1}{16}\cdot\left[\frac{8g^2}{\kappa\gamma}\frac{1}{(\frac{\kappa_s}{\kappa}+1)(\frac{4g^2}{\kappa\gamma}+\frac{\kappa_s}{\kappa}+1)}\right]^4. \quad (31)$$

Figure 3(b) shows that the average efficiency of the HH-PCG varies with the parameter $g/\sqrt{\kappa\gamma}$ in different side leakage situations κ_s/κ. One can see that the average efficiency of the HH-PCG is increased with the raise of $g/\sqrt{\kappa\gamma}$. Till now, several experiments have been demonstrated for observing the phenomenon of NV center coupled to optical cavity.[54–57] When an NV center is coupled to a chip-based micro-cavity, the parameters $[g,\kappa,\gamma,\gamma_{ZPL}]/2\pi = [0.30, 26, 0.013, 0.0004]$ GHz[55] are obtained. Under the condition of $\omega_0 = \omega_p = \omega_c$, it can be calculated that $g/\sqrt{\kappa\gamma} \approx 2.94$. With the side leakage being $\kappa_s/\kappa = 0$, $\kappa_s/\kappa = 0.03$, or $\kappa_s/\kappa = 0.06$, the average efficiency is calculated to be 89.23%, 78.02%, or 70.21%, respectively. This shows that the side leakage κ_s/κ of the optical cavity has great impact in weak coupling situation, thus it affects the average efficiency of the HH-PCG.

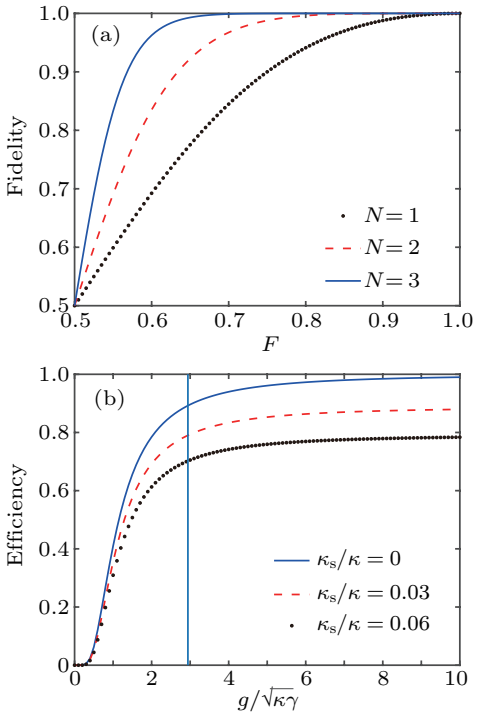


Fig. 3. (a) The fidelity of state $|\phi^+\rangle_{A_1B_1}$ in the mixed state $\rho'_{A_1B_1}$ as a function of initial fidelity F with different iterating times. N ($N = 1$, 2, 3) is the number of iterations. (b) The average efficiency of the HH-PCG varies with the parameter $g/\sqrt{\kappa\gamma}$ in different side leakage situations κ_s/κ. The vertical line stands for $g/\sqrt{\kappa\gamma} \approx 2.94$.

The efficiency of the EPP scheme is defined as the probability that the members can obtain a higher-fidelity entangled two-photon system from a pair of lower-fidelity entangled two-photon systems transmitted through a noisy channel without photon loss. The efficiency Y of our heralded EPP scheme is calculated mainly by two parts. The first part comes from the two preserved constituents discussed in subsection 3.2 (i.e., the first constituent $|\phi^+\rangle_{A_1B_1}|\phi^+\rangle_{A_2B_2}$ and fourth constituent $|\psi^+\rangle_{A_1B_1}|\psi^+\rangle_{A_2B_2}$), where the second and the third constituents are discarded. So the first part of the efficiency is $Y_1 = F^2 + (1-F)^2$. The second part of the efficiency is constructed by the efficiencies of two HH-PCGs, and it is easy to find that the second part of the efficiency is $Y_2 = \overline{\eta}^2$, where $\overline{\eta}$ is the average efficiency of the HH-PCG shown in Fig. 3(b). Assuming that the influences of the photon detectors and linear optical elements are ignored, the efficiency of our heralded EPP scheme can be written as

$$Y = \overline{\eta}^2[F^2 + (1-F)^2]. \quad (32)$$

Figure 4 shows the efficiency Y of our heralded EPP as a function of the initial fidelity F and the parameter $g/\sqrt{\kappa\gamma}$. The side leakage is taken as $\kappa_s/\kappa = 0.03$. From Fig. 4, it can be seen intuitively that the efficiency of our heralded EPP is decreased with the decrease of coupling strength $g/\sqrt{\kappa\gamma}$, which is similar to the average efficiency of the HH-PCG. That is, in our scheme, the fidelity is increased at cost of efficiency. In the weak coupling regime, the fidelity of our HH-PCG is robust with lower efficiency, which is useful when the efficiency requirement is not too high. If the high fidelity and high efficiency are both expected, strong coupling regime is required. Strong coupling has been observed in different systems recently,[72–75] and the EPP scheme with high fidelity and high efficiency may be considerable in the future.

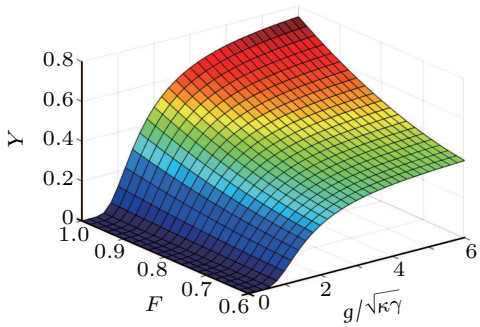


Fig. 4. The efficiency Y of the heralded EPP as a function of the initial fidelity F and the parameter $g/\sqrt{\kappa\gamma}$, where side leakage of the cavity is taken as $\kappa_s/\kappa = 0.03$. The influences of the photon detectors and linear optical elements are ignored.

5. Summary

In summary, we have proposed a heralded two-photon EPP scheme using an HH-PCG based on the input-output process of an NV center embedded in a single-sided optical cavity. In the HH-PCG, the parity of the two-photon polarization state is read out by the state of the NV center. Using the HH-PCG to purify the nonlocal two-photon polarization mixed state, the entanglement of the nonlocal two-photon polarization mixed state can be increased. In a near-realistic situation, as the unwanted components are filtrated by the photon detector in the HH-PCG, the fidelity of this heralded scheme is robust to the imperfect interaction between the NV center-cavity system and a photon at the cost of lower efficiency. Thus our heralded EPP scheme is feasible in near-realistic situations and useful for increasing the fidelity of quantum communication.

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