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Coherence measures based on sandwiched Rényi relative entropy*

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Coherence is a fundamental ingredient for quantum physics and a key resource for quantum information theory. Baumgratz, Cramer and Plenio established a rigorous framework (BCP framework) for quantifying coherence [Baumgratz T, Cramer M and Plenio M B *Phys. Rev. Lett.* **113** 140401 (2014)]. In the present paper, under the BCP framework we provide two classes of coherence measures based on the sandwiched Rényi relative entropy. We also prove that we cannot get a new coherence measure $f(C(\cdot))$ by a function f acting on a given coherence measure C.

Keywords: quantum coherence, coherence measure, sandwiched Rényi relative entropy

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1. Introduction

Quantum coherence is one of the most fundamental features of quantum physics. Recently, Baumgratz, Cramer, and Plenio established a rigorous framework (BCP framework) for quantifying coherence.[1] The BCP framework has been widely accepted and triggered rapidly growing research for quantifying coherence (recent reviews see Refs. [2–4]).

To construct a coherence measure, we first define the incoherent states and incoherent operations as in Ref. [1]. For a fixed orthonormal basis $\{|j\rangle\}_{j=1}^{d}$ of the d-dimensional Hilbert space H, a quantum state σ on H is named to be incoherent with respect to $\{|j\rangle\}_{j=1}^{d}$ if σ is diagonal when expressed in $\{|j\rangle\}_{j=1}^{d}$. We denote the set of all incoherent states by $\mathcal{I}$, and the set of density operators by $\mathcal{D}$. A quantum operation Φ, or called a CPTP (completely positive trace preserving) map, can be expressed by a set of Kraus operators $\{K_n\}_n$ satisfying $\sum_n K_n^\dagger K_n = I$, where I being the identity operator on H, and operate a state ρ as $\Phi(\rho) = \sum_n K_n \rho K_n^\dagger$. A quantum operation Φ will be called an incoherent operation (ICPTP) Φ_I if it admits that a set of Kraus operators $\{K_n\}_n$ and $K_n \sigma K_n^\dagger$ is diagonal for any n and any incoherent state σ. Notice that the definitions of incoherent state, incoherent operation, and the coherence measure all depend on the fixed orthonormal basis $\{|j\rangle\}_{j=1}^{d}$, we call this basis the reference basis.

The BCP framework consists of the following postulates (C1)–(C4) that any quantifier of coherence C should fulfill:

(C1) **Non-negativity**

$$C(\rho) \geq 0,\ C(\rho) = 0 \Leftrightarrow \rho \in \mathcal{I}. \tag{1}$$

(C2) **Monotonicity** C does not increase under the operation of any incoherent operation Φ_I,

$$C[\Phi_I(\rho)] \leq C(\rho). \tag{2}$$

(C3) **Strong monotonicity** for any incoherent operation $\Phi_I = \{K_n\}_n$ with $K_n \sigma K_n^\dagger$ being diagonal for any n and any incoherent state σ,

$$\sum_n \mathrm{tr}(K_n \rho K_n^\dagger) C\left[\frac{K_n \rho K_n^\dagger}{\mathrm{tr}(K_n \rho K_n^\dagger)}\right] \leq C(\rho). \tag{3}$$

(C4) **Convexity** C is a convex function of the state, i.e.,

$$\sum_n p_n C(\rho_n) \geq C\Big(\sum_n p_n \rho_n\Big), \tag{4}$$

where $p_n > 0$, $\sum_n p_n = 1$, $\rho_1, \rho_2 \in \mathcal{D}$.

We call a quantifier C satisfying (C1)–(C4) together a coherence measure. Note that (C3) + (C4) implies (C2).[1]

Yu, Zhang, Xu and Tong proposed condition (C5), and showed that (C1)–(C4) is equivalent to (C1) + (C2) + (C5).[5]

(C5) **Additivity on block-diagonal states**

$$C(p_1\rho_1 \oplus p_2\rho_2) = p_1 C(\rho_1) + p_2 C(\rho_2), \tag{5}$$

where $p_1 > 0$, $p_2 > 0$, $p_1 + p_2 = 1$, $\rho_1, \rho_2 \in \mathcal{D}$.

BCP framework draws strong attention and many discussions, but it is not the unique framework for quantifying coherence, and other potential candidates have been investigated (see, e.g., Refs. [6–16]).

So far, some coherence measures under the BCP framework have been found out for different applications and backgrounds, such as relative entropy of coherence,[1] the l_1 norm of coherence,[1] geometric measure of coherence,[17] modified trace norm of coherence,[5] robustness of coherence,[18,19] coherence measure via quantum skew information,[20] coherence measures based on Tsallis relative entropy,[21–24] coherence weight.[25] For a coherence measure defined only for all pure states, it can be extended to mixed states via the convex roof construction.[6,12,26,27] Also, a coherence measure defined on all pure states is determined by its majorization property on

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the modulus square of coefficients of pure states.[7,8,28–30] Although the convex roof construction and majorization on pure states together provide a powerful way to construct coherence measures, the coherence measures obtained in such a way generally speaking are only in the form of optimization and it is difficult to obtain analytical expressions.[2]

Coherence measures have many fruitful applications in exploring the properties of coherence, such as the freezing of coherence[4,31–34] and coherence estimating based on experimental data.[35]

In this paper, we provide two classes of coherence measures based on the sandwiched Rényi relative entropy in Section 2, the geometric measure of coherence[17] is a special case of them. Also in Section 3, we discuss whether or not one can get a new coherence measure through a function of a given coherence measure.

2. Coherence measures based on sandwiched Rényi relative entropy

In this section, we propose two classes of coherence measures based on the sandwiched Rényi relative entropy.

Theorem 1 For $\alpha \in [\frac{1}{2}, 1)$, $\rho \in \mathcal{D}$,

$$C_{s1,\alpha}(\rho) = 1 - \max_{\sigma\in\mathcal{I}}(\{\mathrm{tr}[(\rho^{\frac{1-\alpha}{2\alpha}}\sigma\rho^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\}^{\frac{1}{1-\alpha}}) \tag{6}$$

is a coherence measure.

Proof For $\alpha \in [\frac{1}{2}, 1)$, $\sigma, \rho \in \mathcal{D}$, the sandwiched Rényi relative entropy is defined as[36,37]

$$F_{\alpha}(\sigma||\rho) = \frac{\ln \mathrm{tr}[(\rho^{\frac{1-\alpha}{2\alpha}}\sigma\rho^{\frac{1-\alpha}{2\alpha}})^{\alpha}]}{\alpha - 1}. \tag{7}$$

Note that $F_{\alpha}(\sigma||\rho)$ can be defined for $\alpha > 0$,[36,37] but in Theorem 1 we only consider the case of $\alpha \in [\frac{1}{2}, 1)$.

It is shown that[37,38] for $\alpha \in [\frac{1}{2}, 1)$,

$$F_{\alpha}(\sigma||\rho) \geq 0, \quad \text{and equality iff} \quad \sigma = \rho. \tag{8}$$

This is equivalent to

$$\mathrm{tr}[(\rho^{\frac{1-\alpha}{2\alpha}}\sigma\rho^{\frac{1-\alpha}{2\alpha}})^{\alpha} \leq 1, \quad \text{and equality iff} \quad \sigma = \rho, \tag{9}$$

and further equivalent to

$$\{\mathrm{tr}[(\rho^{\frac{1-\alpha}{2\alpha}}\sigma\rho^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\}^{\frac{1}{1-\alpha}} \leq 1, \quad \text{and equality iff} \quad \sigma = \rho. \tag{10}$$

This says that $C_{s1,\alpha}(\rho)$ satisfies (C1).

For $\alpha \in [\frac{1}{2}, 1)$, it has been shown that[39] for $\sigma, \rho \in \mathcal{D}$ and for any CPTP map Φ,

$$F_{\alpha}(\Phi(\sigma)||\Phi(\rho)) \leq F_{\alpha}(\sigma||\rho). \tag{11}$$

This implies

$$\begin{aligned}
&\mathrm{tr}[(\Phi(\rho))^{\frac{1-\alpha}{2\alpha}}\Phi(\sigma)(\Phi(\rho))^{\frac{1-\alpha}{2\alpha}})^{\alpha}] \\
&\geq \mathrm{tr}[(\rho^{\frac{1-\alpha}{2\alpha}}\sigma\rho^{\frac{1-\alpha}{2\alpha}})^{\alpha}], \qquad (12)\\
&\quad \{\mathrm{tr}[(\Phi(\rho))^{\frac{1-\alpha}{2\alpha}}\Phi(\sigma)(\Phi(\rho))^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\}^{\frac{1}{1-\alpha}} \\
&\geq \{\mathrm{tr}[(\rho^{\frac{1-\alpha}{2\alpha}}\sigma\rho^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\}^{\frac{1}{1-\alpha}}. \qquad (13)
\end{aligned}$$

For any ICPTP map Φ_I, there exists $\sigma^* \in \mathcal{I}$ such that

$$\begin{aligned}
&\max_{\sigma\in\mathcal{I}}\{\mathrm{tr}[(\rho^{\frac{1-\alpha}{2\alpha}}\sigma\rho^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\}^{\frac{1}{1-\alpha}} \\
&= \{\mathrm{tr}[(\rho^{\frac{1-\alpha}{2\alpha}}\sigma^*\rho^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\}^{\frac{1}{1-\alpha}} \\
&\leq \{\mathrm{tr}[(\Phi_I(\rho))^{\frac{1-\alpha}{2\alpha}}\Phi_I(\sigma^*)(\Phi_I(\rho))^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\}^{\frac{1}{1-\alpha}} \\
&\leq \max_{\sigma\in\mathcal{I}}\{\mathrm{tr}[(\Phi_I(\rho))^{\frac{1-\alpha}{2\alpha}}\sigma(\Phi_I(\rho))^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\}^{\frac{1}{1-\alpha}}. \qquad (14)
\end{aligned}$$

This proves that $C_{s1,\alpha}(\rho)$ satisfies (C2).

Next we prove $C_{s1,\alpha}(\rho)$ satisfies (C5). Suppose that ρ is block-diagonal in the reference basis $\{|j\rangle\}_{j=1}^{d}$,

$$\rho = p_1\rho_1 \otimes p_2\rho_2, \tag{15}$$

with $p_1 > 0$, $p_2 > 0$, $p_1 + p_2 = 1$; ρ_1 and ρ_2 are density operators. For any diagonal state σ, we can write it as

$$\sigma = q_1\sigma_1 \otimes q_2\sigma_2, \tag{16}$$

with the diagonal states σ_1 and σ_2 having the same rows (columns) with ρ_1 and ρ_2, respectively, $q_1 \geq 0$, $q_2 \geq 0$, $q_1 + q_2 = 1$.

It follows

$$\begin{aligned}
&\max_{\sigma\in\mathcal{I}}\mathrm{tr}[(\rho^{\frac{1-\alpha}{2\alpha}}\sigma\rho^{\frac{1-\alpha}{2\alpha}})^{\alpha}] \\
&= \max_{q_1,q_2}\{(p_1^{1-\alpha}q_1^{\alpha})\max_{\sigma_1}\mathrm{tr}[(\rho_1^{\frac{1-\alpha}{2\alpha}}\sigma_1\rho_1^{\frac{1-\alpha}{2\alpha}})^{\alpha}] \\
&\quad +(p_2^{1-\alpha}q_2^{\alpha})\max_{\sigma_2}\mathrm{tr}[(\rho_2^{\frac{1-\alpha}{2\alpha}}\sigma_2\rho_2^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\} \\
&= \max_{q_1,q_2}\{p_1^{1-\alpha}q_1^{\alpha}t_1 + p_2^{1-\alpha}q_2^{\alpha}t_2\} \\
&= p_1^{1-\alpha}p_2^{1-\alpha}t_1t_2(p_1^{-1}t_1^{\frac{1}{\alpha-1}} + p_2^{-1}t_2^{\frac{1}{\alpha-1}})^{1-\alpha}, \qquad (17)
\end{aligned}$$

where we have denoted

$$t_1 = \max_{\sigma_1}\mathrm{tr}[(\rho_1^{\frac{1-\alpha}{2\alpha}}\sigma_1\rho_1^{\frac{1-\alpha}{2\alpha}})^{\alpha}], \tag{18}$$

$$t_2 = \max_{\sigma_2}\mathrm{tr}[(\rho_2^{\frac{1-\alpha}{2\alpha}}\sigma_2\rho_2^{\frac{1-\alpha}{2\alpha}})^{\alpha}], \tag{19}$$

and have used the Hölder inequality in Appendix A (note that $t_1 > 0$, $t_2 > 0$) such that

$$\begin{aligned}
&q_1^{\alpha}(p_1^{1-\alpha}t_1) + q_2^{\alpha}(p_2^{1-\alpha}t_2) \\
&\leq [(q_1^{\alpha})^{\frac{1}{\alpha}} + (q_2^{\alpha})^{\frac{1}{\alpha}}]^{\alpha}[(p_1^{1-\alpha}t_1)^{\frac{1}{1-\alpha}} + (p_2^{1-\alpha}t_2)^{\frac{1}{1-\alpha}}]^{1-\alpha} \\
&= [p_1t_1^{\frac{1}{1-\alpha}} + p_2t_2^{\frac{1}{1-\alpha}}]^{1-\alpha}, \qquad (20)
\end{aligned}$$

the equality holds when

$$\frac{q_1}{q_2} = \frac{p_1t_1^{\frac{1}{1-\alpha}}}{p_2t_2^{\frac{1}{1-\alpha}}}. \tag{21}$$

Consequently,

$$\begin{aligned}&\max_{\sigma\in\mathcal{I}}(\{\mathrm{tr}[(\rho^{\frac{1-\alpha}{2\alpha}}\sigma\rho^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\}^{\frac{1}{1-\alpha}})\\ &=\{\max_{\sigma\in\mathcal{I}}\mathrm{tr}[(\rho^{\frac{1-\alpha}{2\alpha}}\sigma\rho^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\}^{\frac{1}{1-\alpha}}\\ &=p_1p_2t_1^{\frac{1}{1-\alpha}}t_2^{\frac{1}{1-\alpha}}(p_1^{-1}t_1^{\frac{1}{\alpha-1}}+p_2^{-1}t_2^{\frac{1}{\alpha-1}})\\ &=p_1t_1^{\frac{1}{1-\alpha}}+p_2t_2^{\frac{1}{1-\alpha}}.\end{aligned}\tag{22}$$

This shows that $C_{s1,\alpha}(\rho)$ satisfies (C5), and then we complete this proof.

We remark that when $\alpha=\frac{1}{2}, C_{s1,\frac{1}{2}}(\rho)$ corresponds to the geometric measure of coherence.[17]

Example 1 For pure state $\rho=|\psi\rangle\langle\psi|$,

$$C_{s1,\alpha}(|\psi\rangle)=1-\max_j\{|\langle j|\psi\rangle|^{\frac{2\alpha}{1-\alpha}}\}.\tag{23}$$

Proof Suppose that $\sigma=\sum_j\sigma_j|j\rangle\langle j|$ is an incoherent state,

$$\begin{aligned}\mathrm{tr}[(\rho^{\frac{1-\alpha}{2\alpha}}\sigma\rho^{\frac{1-\alpha}{2\alpha}})^{\alpha}]&=\mathrm{tr}\left[(|\psi\rangle\langle\psi|\sum_j\sigma_j|j\rangle\langle j|\psi\rangle\langle\psi|)^{\alpha}\right]\\ &=\left(\sum_j\sigma_j|\langle j|\psi\rangle|^2\right)^{\alpha},\end{aligned}\tag{24}$$

then we can reach the result.

Theorem 2 For $\alpha\in[\frac{1}{2},1)\cup(1,\infty)$,

$$\begin{aligned}C_{s,\alpha}(\rho)&=\min_{\sigma\in\mathcal{I}}\frac{\{\mathrm{tr}[(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\}^{\frac{1}{\alpha}}-1}{\alpha-1},\\ &\mathrm{supp}(\rho)\subset\mathrm{supp}(\sigma)\quad\text{when}\quad\alpha>1,\end{aligned}\tag{25}$$

is a coherence measure.

Proof The proof of Theorem 2 is similar to the proof of Theorem 1, we postpone it to Appendix B.

Note that since the quantum fidelity

$$F(\rho,\sigma)=\mathrm{tr}[(\sigma^{\frac{1}{2}}\rho\sigma^{\frac{1}{2}})^{\frac{1}{2}}]=\mathrm{tr}[(\rho^{\frac{1}{2}}\sigma\rho^{\frac{1}{2}})^{\frac{1}{2}}],\tag{26}$$

then $C_{s,\frac{1}{2}}(\rho)=C_{s1,\frac{1}{2}}(\rho)$ again corresponds to the geometric measure of coherence.[17]

We remark that a coherence quantifier based on sandwiched Rényi relative entropy was also investigated in Refs. [7,15], but the quantifier is not a coherence measure in the sense of satisfying (C1)–(C4), i.e., under the BCP framework, we will give a proof for this assertion in Example 3.

Example 2 For pure state $\rho=|\psi\rangle\langle\psi|$,

$$C_{s,\alpha}(|\psi\rangle)=\frac{\left(\sum_j|\langle\psi|j\rangle|^{\frac{2\alpha}{2\alpha-1}}\right)^{\frac{2\alpha-1}{\alpha}}-1}{\alpha-1}.\tag{27}$$

Proof Suppose that $\sigma=\sum_j\sigma_j|j\rangle\langle j|$ is an incoherent state,

$$\begin{aligned}&\mathrm{tr}[(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\\ &=\mathrm{tr}[(\sigma^{\frac{1-\alpha}{2\alpha}}|\psi\rangle\langle\psi|\sigma^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\\ &=(\langle\psi|\sigma^{\frac{1-\alpha}{\alpha}}|\psi\rangle)^{\alpha}\\ &=\left(\sum_j\sigma_j^{\frac{1-\alpha}{\alpha}}|\langle\psi|j\rangle|^2\right)^{\alpha}.\end{aligned}\tag{28}$$

Using the Hölder inequality (Appendix A), we then reach the result.

Examples 1 and 2 show that $C_{s,\alpha}$ and $C_{s1,\beta}$ will not be equivalent even if $\alpha=\beta$.

3. Linearization theorem of coherence measures

Constructing a coherence measure under the BCP framework in general is not an easy task because of the stringent requirements of (C1)–(C4). (C5) is more easy to check than (C3) + (C4) in many cases such as in the proof of Theorem 1 in the present work. Different coherence measures have their advantages in different applications. An attractive idea for constructing new coherence measures is to ask whether or not there exists a function f such that $f[C(\rho)]$ still is a coherence measure for a given coherence measure C. The answer of this question is essentially negative. In fact we have the following theorem.

Theorem 3 (Linearization theorem) Given a coherence measure C, the function $f:[0,\infty)\longrightarrow[0,\infty)$ will make $f(C(\rho))$ also a coherence measure, if and only if there exists $\lambda>0$ such that $f(x)=\lambda x$.

Proof Since $C(\rho)$ and $f(C(\rho))$ are all coherence measures, then (C1) leads to

$$f(x)\geq0\quad\text{and}\quad f(0)=0\quad\text{iff}\quad x=0.\tag{29}$$

Suppose the dimension of system $d>2$, for the state

$$\rho=p_1\rho_1\otimes p_2\rho_2,\tag{30}$$

with $p_1>0$, $p_2>0$, $p_1+p_2=1$; ρ_1 and ρ_2 are density operators, $\dim\rho_1\geq1$, $\dim\rho_2\geq1$, $\dim\rho_1+\dim\rho_2=d$. Since $C(\rho)$ is coherence measure, (C5) leads to

$$f[C(p_1\rho_1\oplus p_2\rho_2)]=f[p_1C(\rho_1)+p_2C(\rho_2)].\tag{31}$$

Similarly, $f(C(\rho))$ is also a coherence measure, then (C5) leads to

$$f[C(p_1\rho_1\oplus p_2\rho_2)]=p_1f[C(\rho_1)]+p_2f[C(\rho_2)].\tag{32}$$

As a result, we obtain

$$f[p_1C(\rho_1)+p_2C(\rho_2)]=p_1f[C(\rho_1)]+p_2f[C(\rho_2)].\tag{33}$$

Without loss of generality, suppose $\dim\rho_2\geq2$, then there exist ρ_1 and ρ_2 such that $C(\rho_1)=0$, $C(\rho_2)=\mu>0$. The above equation yields

$$f(p_2\mu)=p_2f(\mu),\tag{34}$$

with $f(\mu)>0$.

Let $p_2\mu = x$, then

$$f(x) = x\frac{f(\mu)}{\mu} = \lambda x, \lambda > 0. \tag{35}$$

We complete this proof.

We make a note that Eq. (33) is a functional equation (see, e.g., Ref. [40]).

Since coherence measures C and λC ($\lambda > 0$) have no essential difference, from Theorem 3 we can say that it is impossible to obtain a new coherence measure $f(C(\rho))$ by a function f.

Note that in the proof of Theorem 3 we only use $C(\rho)$ and $f(C(\rho))$ satisfying (C1)+(C5) while (C2) is unnecessary.

Example 3 Consider the coherence quantifier in Refs. [7, 15],

$$\overline{C}_\alpha(\rho) = \min_{\sigma\in\mathcal{I}} F_\alpha(\rho||\sigma), \alpha \in [\frac{1}{2}, 1)\cup(1,\infty), \tag{36}$$

$$F_\alpha(\rho||\sigma) = \frac{\ln \mathrm{tr}[(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}})^\alpha]}{\alpha-1}, \tag{37}$$

$$\mathrm{supp}(\rho) \subset \mathrm{supp}(\sigma) \quad \text{when} \quad \alpha > 1. \tag{38}$$

We now show that $\overline{C}_\alpha(\rho)$ is not a coherence measure under the BCP framework.

Denote

$$g(\rho) = \begin{cases} \max_{\sigma\in\mathcal{I}} \mathrm{tr}[(\rho^{\frac{1-\alpha}{2\alpha}}\sigma\rho^{\frac{1-\alpha}{2\alpha}})^\alpha], & \alpha \in [\frac{1}{2}, 1), \\ \min_{\sigma\in\mathcal{I}} \mathrm{tr}[(\rho^{\frac{1-\alpha}{2\alpha}}\sigma\rho^{\frac{1-\alpha}{2\alpha}})^\alpha], & \alpha \in (1,\infty), \end{cases} \tag{39}$$

then from the definitions of $C_{s,\alpha}(\rho)$ and $\overline{C}_\alpha(\rho)$ we have

$$C_{s,\alpha}(\rho) = \frac{[g(\rho)]^{\frac{1}{\alpha}} - 1}{\alpha - 1}, \tag{40}$$

$$\overline{C}_\alpha(\rho) = \frac{\ln g(\rho)}{\alpha-1}, \tag{41}$$

$$\overline{C}_\alpha(\rho) = \frac{\alpha}{\alpha-1}\ln[(\alpha-1)C_{s,\alpha}(\rho)+1]. \tag{42}$$

From Theorem 3, we can realize that $\overline{C}_\alpha(\rho)$ is not a coherence measure under the BCP framework. From the proof of Theorem 2 (Appendix B) we can see that $\overline{C}_\alpha$ satisfies (C1) and (C2), then Example 3 implies that $\overline{C}_\alpha$ violates (C5) (hence violates (C3) or (C4)).

4. Conclusions

In summary, under the BCP framework for quantifying coherence, we have proposed two classes of coherence measures based on sandwiched Rényi relative entropy. We also prove that it is essentially impossible to obtain a new coherence measure $f(C(\rho))$ by a function f acting on a given coherence measure C.

There are many open questions for future investigations after the present work. For example, the monotonicity of $C_{s1,\alpha}$ and $C_{s,\alpha}$ in α, the ordering of magnitude for them and other coherence measures, the operational interpretations for them, potential applications in quantum information processing, and also the counterparts for quantifying coherence of Gaussian states as performed in Refs. [41,42].

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Appendix A: Hölder inequality

Suppose that $\{a_j\}_{j=1}^d$ and $\{b_j\}_{j=1}^d$ are both finite positive real numbers, p and q are finite real numbers, $\frac{1}{p}+\frac{1}{q}=1$, then

(1) when $p > 1$,

$$\sum_{j=1}^d a_j b_j \le \left(\sum_{j=1}^d a_j^p\right)^{\frac{1}{p}}\left(\sum_{j=1}^d b_j^q\right)^{\frac{1}{q}}, \tag{A1}$$

and the equality will appear iff there exists positive number C such that $a_j^p = Cb_j^q$ for any j;

(2) when $0 < p < 1$,

$$\sum_{j=1}^d a_j b_j \ge \left(\sum_{j=1}^d a_j^p\right)^{\frac{1}{p}}\left(\sum_{j=1}^d b_j^q\right)^{\frac{1}{q}}, \tag{A2}$$

and the equality will appear iff there exists positive number C such that $a_j^p = Cb_j^q$ for any j.

Appendix B: Proof of Theorem 2

Part 1 $\alpha \in [\frac{1}{2}, 1)$

For $\alpha \in [\frac{1}{2}, 1)$, $\rho, \sigma \in \mathcal{D}$, the sandwiched Rényi relative entropy is defined as[36,37]

$$F_\alpha(\rho||\sigma) = \frac{\ln \mathrm{tr}[(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}})^\alpha]}{\alpha-1}. \tag{B1}$$

Note that $F_\alpha(\rho||\sigma)$ can be defined for $\alpha > 0$,[36,37] but in Part 1 we only consider the case of $\alpha \in [\frac{1}{2}, 1)$.

It is shown that[37,38] for $\alpha \in [\frac{1}{2}, 1)$,

$$F_\alpha(\rho||\sigma) \ge 0, \quad \text{and equality iff} \quad \rho = \sigma. \tag{B2}$$

This is equivalent to

$$\mathrm{tr}[(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}})^\alpha \le 1, \quad \text{and equality iff} \quad \rho = \sigma, \tag{B3}$$

and further equivalent to

$$\{\mathrm{tr}[(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}})^\alpha]\}^{\frac{1}{\alpha}} \le 1, \quad \text{and equality iff} \quad \rho = \sigma. \tag{B4}$$

This says that $C_{s,\alpha}(\rho)$ satisfies (C1).

For $\alpha \in [\frac{1}{2}, 1)$, it has been shown that[39] for $\rho, \sigma \in \mathcal{D}$, and any CPTP map Φ,

$$F_\alpha(\Phi(\rho)||\Phi(\sigma)) \le F_\alpha(\rho||\sigma). \tag{B5}$$

This implies

$$\begin{aligned}&\mathrm{tr}[(\Phi(\sigma))^{\frac{1-\alpha}{2\alpha}}\Phi(\rho)(\Phi(\sigma))^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\\ &\geq \mathrm{tr}[(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}})^{\alpha}], \qquad \text{(B6)}\\ &\{\mathrm{tr}[(\Phi(\sigma))^{\frac{1-\alpha}{2\alpha}}\Phi(\rho)(\Phi(\sigma))^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\}^{\frac{1}{\alpha}}\\ &\geq \{\mathrm{tr}[(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\}^{\frac{1}{\alpha}}. \qquad \text{(B7)}\end{aligned}$$

For any ICPTP map Φ_I, there exists $\sigma^*\in\mathcal{I}$ such that

$$\begin{aligned}&\max_{\sigma\in\mathcal{I}}\{\mathrm{tr}[(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\}^{\frac{1}{\alpha}}\\ &=\{\mathrm{tr}[((\sigma^*)^{\frac{1-\alpha}{2\alpha}}\rho(\sigma^*)^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\}^{\frac{1}{\alpha}}\\ &\leq\{\mathrm{tr}[(\Phi_I(\sigma^*))^{\frac{1-\alpha}{2\alpha}}\Phi_I(\rho)(\Phi_I(\sigma^*))^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\}^{\frac{1}{\alpha}}\\ &\leq\max_{\sigma\in\mathcal{I}}\{\mathrm{tr}[(\sigma^{\frac{1-\alpha}{2\alpha}}\Phi_I(\rho)\sigma^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\}^{\frac{1}{\alpha}}. \qquad \text{(B8)}\end{aligned}$$

This proves that $C_{s,\alpha}(\rho)$ satisfies (C2).

Next we prove $C_{s,\alpha}(\rho)$ satisfies (C5). Suppose that ρ is block-diagonal in the reference basis $\{|j\rangle\}_{j=1}^{d}$,

$$\rho=p_1\rho_1\otimes p_2\rho_2, \qquad \text{(B9)}$$

with $p_1>0,\ p_2>0,\ p_1+p_2=1$; ρ_1 and ρ_2 are density operators. For any diagonal state σ, we can write it as

$$\sigma=p_1\sigma_1\otimes p_2\sigma_2, \qquad \text{(B10)}$$

with the diagonal states σ_1 and σ_2 having the same rows (columns) with ρ_1 and ρ_2 respectively, $q_1\geq 0,\ q_2\geq 0,\ q_1+q_2=1$.

It follows that

$$\begin{aligned}&\max_{\sigma\in\mathcal{I}}\mathrm{tr}[(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\\ &=\max_{\sigma\in\mathcal{I}}\mathrm{tr}[((q_1\sigma_1\oplus q_2\sigma_2)^{\frac{1-\alpha}{2\alpha}}\\ &\quad\times(p_1\rho_1\oplus p_2\rho_2)(q_1\sigma_1\oplus q_2\sigma_2)^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\\ &=\max_{q_1,q_2}\{(p_1^{\alpha}q_1^{1-\alpha})\max_{\sigma_1}\mathrm{tr}[(\sigma_1^{\frac{1-\alpha}{2\alpha}}\rho_1\sigma_1^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\\ &\quad+(p_2^{\alpha}q_2^{1-\alpha})\max_{\sigma_2}\mathrm{tr}[(\sigma_2^{\frac{1-\alpha}{2\alpha}}\rho_2\sigma_2^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\}\\ &=\max_{q_1,q_2}\{p_1^{\alpha}q_1^{1-\alpha}t_1+p_2^{\alpha}q_2^{1-\alpha}t_2\}\\ &=[p_1t_1^{\frac{1}{\alpha}}+p_2t_2^{\frac{1}{\alpha}}]^{\alpha}, \qquad \text{(B11)}\end{aligned}$$

where we have denoted

$$t_1=\max_{\sigma_1}\mathrm{tr}[(\sigma_1^{\frac{1-\alpha}{2\alpha}}\rho_1\sigma_1^{\frac{1-\alpha}{2\alpha}})^{\alpha}], \qquad \text{(B12)}$$

$$t_2=\max_{\sigma_2}\mathrm{tr}[(\sigma_2^{\frac{1-\alpha}{2\alpha}}\rho_2\sigma_2^{\frac{1-\alpha}{2\alpha}})^{\alpha}], \qquad \text{(B13)}$$

and have used the Hölder inequality in Appendix A (note that $t_1>0,\ t_2>0$) such that

$$\begin{aligned}&q_1^{1-\alpha}p_1^{\alpha}t_1+q_2^{1-\alpha}p_2^{\alpha}t_2\\ &\leq[(q_1^{1-\alpha})^{\frac{1}{1-\alpha}}+(q_2^{1-\alpha})^{\frac{1}{1-\alpha}}]^{1-\alpha}[(p_1^{\alpha}t_1)^{\frac{1}{\alpha}}+(p_2^{\alpha}t_2)^{\frac{1}{\alpha}}]^{\alpha}\\ &=[p_1t_1^{\frac{1}{\alpha}}+p_2t_2^{\frac{1}{\alpha}}]^{\alpha}, \qquad \text{(B14)}\end{aligned}$$

the equality holds when

$$\frac{q_1}{q_2}=\frac{p_1t_1^{1/\alpha}}{P_2t_2^{1/2\alpha}}. \qquad \text{(B15)}$$

Consequently,

$$\begin{aligned}&\max_{\sigma\in\mathcal{I}}(\{\mathrm{tr}[(\rho^{\frac{1-\alpha}{2\alpha}}\sigma\rho^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\}^{\frac{1}{\alpha}})\\ &=\{\max_{\sigma\in\mathcal{I}}\mathrm{tr}[(\rho^{\frac{1-\alpha}{2\alpha}}\sigma\rho^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\}^{\frac{1}{\alpha}}\\ &=p_1t_1^{\frac{1}{\alpha}}+p_2t_2^{\frac{1}{\alpha}}. \qquad \text{(B16)}\end{aligned}$$

This shows that $C_{s,\alpha}(\rho)$ satisfies (C5), and we complete this proof.

Part 2 $\alpha>1$

For $\alpha>1$, $\rho,\sigma\in\mathcal{D}$, the sandwiched Rényi relative entropy is defined as[36,37]

$$F_{\alpha}(\rho||\sigma)=\frac{\ln\mathrm{tr}[(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}})^{\alpha}]}{\alpha-1}, \qquad \text{(B17)}$$

with $\mathrm{supp}(\rho)\subset\mathrm{supp}(\sigma)$. Note that $F_{\alpha}(\rho||\sigma)$ can be defined for $\alpha>0$,[36,37] but in Part 2 we only consider the case of $\alpha>1$.

It is shown that[37,38] for $\alpha>1$,

$$F_{\alpha}(\rho||\sigma)\geq 0,\quad \text{and equality iff}\quad \rho=\sigma. \qquad \text{(B18)}$$

This is equivalent to

$$\mathrm{tr}[(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}})^{\alpha}\geq 1,\quad \text{and equality iff}\quad \rho=\sigma, \qquad \text{(B19)}$$

and further equivalent to

$$\{\mathrm{tr}[(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\}^{\frac{1}{\alpha}}\geq 1,\quad \text{and equality iff}\quad \rho=\sigma. \qquad \text{(B20)}$$

This says that $C_{s,\alpha}(\rho)$ satisfies (C1).

For $\alpha>1$, it has been shown that[38,39] for $\rho,\sigma\in\mathcal{D}$, and any CPTP map Φ,

$$F_{\alpha}(\Phi(\rho)||\Phi(\sigma))\leq F_{\alpha}(\rho||\sigma). \qquad \text{(B21)}$$

This implies

$$\begin{aligned}&\mathrm{tr}[(\Phi(\sigma))^{\frac{1-\alpha}{2\alpha}}\Phi(\rho)(\Phi(\sigma))^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\\ &\leq\mathrm{tr}[(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}})^{\alpha}], \qquad \text{(B22)}\\ &\{\mathrm{tr}[(\Phi(\sigma))^{\frac{1-\alpha}{2\alpha}}\Phi(\rho)(\Phi(\sigma))^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\}^{\frac{1}{\alpha}}\\ &\leq\{\mathrm{tr}[(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\}^{\frac{1}{\alpha}}. \qquad \text{(B23)}\end{aligned}$$

For any ICPTP map Φ_I, there exists $\sigma^*\in\mathcal{I}$ such that

$$\begin{aligned}&\min_{\sigma\in\mathcal{I}}\{\mathrm{tr}[(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\}^{\frac{1}{\alpha}}\\ &=\{\mathrm{tr}[((\sigma^*)^{\frac{1-\alpha}{2\alpha}}\rho(\sigma^*)^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\}^{\frac{1}{\alpha}}\\ &\geq\{\mathrm{tr}[(\Phi_I(\sigma^*))^{\frac{1-\alpha}{2\alpha}}\Phi_I(\rho)(\Phi_I(\sigma^*))^{\frac{1-\alpha}{2\alpha}})^{\alpha}]\}^{\frac{1}{\alpha}}\end{aligned}$$

$$\geq \min_{\sigma\in\mathcal{I}}\{\mathrm{tr}[(\sigma^{\frac{1-\alpha}{2\alpha}}\Phi_I(\rho)\sigma^{\frac{1-\alpha}{2\alpha}})^\alpha]\}^{\frac{1}{\alpha}}. \tag{B24}$$

This proves that $C_{s,\alpha}(\rho)$ satisfies (C2).

Next we prove that $C_{s,\alpha}(\rho)$ satisfies (C5). Suppose that ρ is block-diagonal in the reference basis $\{|j\rangle\}_{j=1}^d$,

$$\rho = p_1\rho_1 \oplus p_2\rho_2, \tag{B25}$$

with $p_1>0,\ p_2>0,\ p_1+p_2=1$; ρ_1 and ρ_2 are density operators. For any diagonal state σ, we can write it as

$$\sigma = q_1\sigma_1 \oplus q_2\sigma_2, \tag{B26}$$

with the diagonal states σ_1 and σ_2 having the same rows (columns) with ρ_1 and ρ_2, respectively, $q_1\geq 0,\ q_2\geq 0,\ q_1+q_2=1$.

It follows that

$$\begin{aligned}
&\min_{\sigma\in\mathcal{I}}\mathrm{tr}[(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}})^\alpha]\\
&=\min_{\sigma\in\mathcal{I}}\mathrm{tr}[((q_1\sigma_1\oplus q_2\sigma_2)^{\frac{1-\alpha}{2\alpha}}\\
&\quad\times(p_1\rho_1\oplus p_2\rho_2)(q_1\sigma_1\oplus q_2\sigma_2)^{\frac{1-\alpha}{2\alpha}})^\alpha]\\
&=\min_{q_1,q_2}\{(p_1^\alpha q_1^{1-\alpha})\min_{\sigma_1}\mathrm{tr}[(\sigma_1^{\frac{1-\alpha}{2\alpha}}\rho_1\sigma_1^{\frac{1-\alpha}{2\alpha}})^\alpha]\\
&\quad+(p_2^\alpha q_2^{1-\alpha})\min_{\sigma_2}\mathrm{tr}[(\sigma_2^{\frac{1-\alpha}{2\alpha}}\rho_2\sigma_2^{\frac{1-\alpha}{2\alpha}})^\alpha]\}\\
&=\min_{q_1,q_2}\{p_1^\alpha q_1^{1-\alpha}t_1+p_2^\alpha q_2^{1-\alpha}t_2\}\\
&=[p_1t_1^{\frac{1}{\alpha}}+p_2t_2^{\frac{1}{\alpha}}]^\alpha,
\end{aligned} \tag{B27}$$

where we have denoted

$$t_1=\min_{\sigma_1}\mathrm{tr}[(\sigma_1^{\frac{1-\alpha}{2\alpha}}\rho_1\sigma_1^{\frac{1-\alpha}{2\alpha}})^\alpha], \tag{B28}$$

$$t_2=\min_{\sigma_2}\mathrm{tr}[(\sigma_2^{\frac{1-\alpha}{2\alpha}}\rho_2\sigma_2^{\frac{1-\alpha}{2\alpha}})^\alpha], \tag{B29}$$

and have used the Hölder inequality in Appendix A (note that $t_1>0,\ t_2>0$) such that

$$\begin{aligned}
&q_1^{1-\alpha}p_1^\alpha t_1+q_2^{1-\alpha}p_2^\alpha t_2\\
&\geq[(q_1^{1-\alpha})^{\frac{1}{1-\alpha}}+(q_2^{1-\alpha})^{\frac{1}{1-\alpha}}]^{1-\alpha}[(p_1^\alpha t_1)^{\frac{1}{\alpha}}+(p_2^\alpha t_2)^{\frac{1}{\alpha}}]^\alpha\\
&=[p_1t_1^{\frac{1}{\alpha}}+p_2t_2^{\frac{1}{\alpha}}]^\alpha,
\end{aligned} \tag{B30}$$

the equality holds when

$$\frac{q_1}{q_2}=\frac{p_1t_1^{\frac{1}{\alpha}}}{p_2t_2^{\frac{1}{\alpha}}}. \tag{B31}$$

Consequently,

$$\begin{aligned}
&\min_{\sigma\in\mathcal{I}}(\{\mathrm{tr}[(\rho^{\frac{1-\alpha}{2\alpha}}\sigma\rho^{\frac{1-\alpha}{2\alpha}})^\alpha]\}^{\frac{1}{\alpha}})\\
&=\{\min_{\sigma\in\mathcal{I}}\mathrm{tr}[(\rho^{\frac{1-\alpha}{2\alpha}}\sigma\rho^{\frac{1-\alpha}{2\alpha}})^\alpha]\}^{\frac{1}{\alpha}}\\
&=p_1t_1^{\frac{1}{\alpha}}+p_2t_2^{\frac{1}{\alpha}}.
\end{aligned} \tag{B32}$$

This shows that $C_{s,\alpha}(\rho)$ satisfies (C5), and we complete this proof.

References

[1] Baumgratz T, Cramer M and Plenio M B 2014 *Phys. Rev. Lett.* **113** 140401

[2] Streltsov A, Adesso G and Plenio M B 2017 *Rev. Mod. Phys.* **89** 041003

[3] Hu M L, Hu X, Wang J, Peng Y, Zhang Y and Fan H 2018 *Phys. Rep.* **762–764** 1

[4] Li B M, Hu M L and Fan H 2019 *Acta Phys. Sin.* **68** 030304 (in Chinese)

[5] Yu X D, Zhang D J, Xu G F and Tong D M 2016 *Phys. Rev. A* **94** 060302

[6] Aberg J 2006 arXiv:0612146[quant-ph]

[7] Chitambar E and Gour G 2016 *Phys. Rev. Lett.* **117** 030401

[8] Chitambar E and Gour G 2016 *Phys. Rev. A* **94** 052336

[9] Marvian I and Spekkens R W 2016 *Phys. Rev. A* **94** 052324

[10] de Vicente J I and Streltsov A 2017 *J. Phys. A: Math. Theor.* **50** 045301

[11] Liu Z W, Hu X and Lloyd S 2017 *Phys. Rev. Lett.* **118** 060502

[12] Winter A and Yang D 2016 *Phys. Rev. Lett.* **116** 120404

[13] Yadin B, Ma J, Girolami D, Gu M and Vedral V 2016 *Phys. Rev. X* **6** 041028

[14] Shao L H, Li Y M, Luo Y and Xi Z J 2017 *Commun. Theor. Phys.* **67** 631

[15] Zhu H, Hayashi M and Chen L 2017 *J. Phys. A: Math. Theor.* **50** 475303

[16] Bu K, Singh U, Fei S M, Pati A K and Wu J 2017 *Phys. Rev. Lett.* **119** 150405

[17] Streltsov A, Singh U, Dhar H S, Bera M N and Adesso G 2015 *Phys. Rev. Lett.* **115** 020403

[18] Napoli C, Bromley T R, Cianciaruso M, Piani M, Johnston N and Adesso G 2016 *Phys. Rev. Lett.* **116** 150502

[19] Piani M, Cianciaruso M, Bromley T R, Napoli C, Johnston N and Adesso G 2016 *Phys. Rev. A* **93** 042107

[20] Yu C S 2017 *Phys. Rev. A* **95** 042337

[21] Rastegin A E 2016 *Phys. Rev. A* **93** 032136

[22] Zhao H and Yu C S 2018 *Sci. Rep.* **8** 299

[23] Kollas N K 2018 *Phys. Rev. A* **97** 062344

[24] Xiong C, Kumar A and Wu J 2018 *Phys. Rev. A* **98** 032324

[25] Bu K, Anand N and Singh U 2018 *Phys. Rev. A* **97** 032342

[26] Yuan X, Zhou H, Cao Z and Ma X 2015 *Phys. Rev. A* **92** 022124

[27] Qi X, Gao T and Yan F 2017 *J. Phys. A: Math. Theor.* **50** 285301

[28] Du S, Bai Z and Guo Y 2015 *Phys. Rev. A* **91** 052120

[29] Du S, Bai Z and Guo Y 2017 *Phys. Rev. A* **95** 029901(E)

[30] Zhu H, Ma Z, Cao Z, Fei S M and Vlatko Vedral 2017 *Phys. Rev. A* **96** 032316

[31] Bromley T R, Cianciaruso M and Adesso G 2015 *Phys. Rev. Lett.* **114** 210401

[32] Yu X D, Zhang D J, Liu C L and Tong D M 2016 *Phys. Rev. A* **93** 060303

[33] Hu M L and Fan H 2016 *Sci. Rep.* **6** 29260

[34] Yang L W, Han W and Xia Y J 2018 *Chin. Phys. B* **27** 040302

[35] Zhang D J, Liu C L, Yu X D and Tong D M 2018 *Phys. Rev. Lett.* **120** 170501

[36] Wilde M M, Winter A and Yang D 2014 *Commun. Math. Phys.* **331** 593

[37] Müller-Lennert M, Dupuis F, Szehr O, Fehr S and Tomamichel M 2013 *J. Math. Phys.* **54** 122203

[38] Beigi S 2013 *J. Math. Phys.* **54** 122202

[39] Frank R L and Lieb E H 2013 *J. Math. Phys.* **54** 122201

[40] Wang X D, Rong H W and Li W R 2003 *Functional equations and their applications* (Shanghai: Shanghai Science and Technology Academic Press) (in Chinese)

[41] Xu J 2016 *Phys. Rev. A* **93** 032111

[42] Buono D, Nocerino G, Petrillo G, Torre G, Zonzo G and Illuminati F 2016 arXiv:1609.00913[quant-ph]