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Relaxation dynamics of Kuramoto model with heterogeneous coupling*

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The Landau damping which reveals the characteristic of relaxation dynamics for an equilibrium state is a universal concept in the area of complex system. In this paper, we study the Landau damping in the phase oscillator system by considering two types of coupling heterogeneity in the Kuramoto model. We show that the critical coupling strength for phase transition, which can be obtained analytically through the balanced integral equation, has the same formula for both cases. The Landau damping effects are further explained in the framework of Laplace transform, where the order parameters decay to zero in the long time limit.

Keywords: synchronization, coupled oscillators, Landau damping

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1. Introduction

Collective motion in a large population of interacting units is a common phenomenon in nature, which is widely observed in many practical situations that range from physics to chemistry, biology, engineering, and even social systems.[1,2] Understanding the intrinsic mechanism of such cooperative behavior has become an important issue in the area of complexity science and dynamical system.[3–5]

In 1975, Kuramoto studied the synchronization of limit cycle oscillators by neglecting the amplitude effects and proposed the famous Kuramoto model[6]

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \quad i = 1,2,\ldots,N, \tag{1}$$

where θ_i and ω_i are the phase and natural frequency of the i-th oscillator, respectively, N is the size of the system, and $K > 0$ denotes the global coupling strength. Equation (1) describes an ensemble of coupled phase oscillators. It is useful to define the order parameter $Z(t)$ to symbol the synchronization degree of the system

$$Z(t) = r(t)\,\mathrm{e}^{\mathrm{i}\psi(t)} = \frac{1}{N}\sum_{j=1}^{N}\mathrm{e}^{\mathrm{i}\theta_j(t)}. \tag{2}$$

Here $Z(t)$ is a complex valued vector, $\psi(t)$ is the mean phase, and $r(t)$ is the collective amplitude. When the coupling strength K is small enough, $r(t) = 0$, the system stays in the incoherent state where no oscillators are synchronous and they rotate almost according to their natural frequencies. However, with the increase of K, $r(t) > 0$, the system evolves to an ordered state, where a number of oscillators become phase-locked, forming a macroscopic cluster, in a way that the effective frequencies converge to a common value. When $K \to \infty$, $r(t) = 1$, the system approaches a complete synchronous state where all oscillators are in phase. As a well-known result, Kuramoto pointed out that in the limit case ($N \to \infty$) and the distribution of natural frequency $g(\omega)$ is even and unimodal, a continuous (second-order) phase transition occurs at a critical coupling strength $K_c = \frac{2}{\pi g(0)}$. Near the critical point, the order parameter $r(t)$ satisfies the scaling law $r(t) \propto \sqrt{K - K_c}$. The classical Kuramoto model provides significant insight for studying synchronization relevant problems both theoretically and practically.[7–14]

The Kuramoto model, as a typical coupled phase oscillator system, has been successfully proposed to investigate synchronization problems over the past four decades. Extensive works have carried out on the phase transition towards synchronization and various collective behaviors in generalized Kuramoto models. Apart from different routes to synchronization, relaxation dynamics of coherent states in the Kuramoto-like models have also become important issues, which relate to the general properties of non-equilibrium phase transition. An important topic associated with coupled phase oscillators is about the stability of equilibrium state in the long time limit. In particular, the dynamical stability about incoherent state for the system has become a focus that attracts much attention for

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the past decades. In contrast to intuition, Strogatz pointed out that the eigenvalues of the incoherent state are absent when $K < K_c$ in the thermodynamic limit. However, the collective dynamics (order parameters) decay in exponential form, which resembles the phenomenon of Landau damping.[15,16] The Landau damping, as an important characteristic of the system, reflects the relaxation behavior of the stationary state before phase transition. Also it is closely related to the linear response, susceptibility, critical slowing down, and correlation function in statistical physics.[17,18] Moreover, as a universal result, the Landau damping effect exists even in the presence of heterogeneity in the coupling strength. Importantly, Landau damping effects are related to the resonance poles.[19] Just like the eigenvalue of the Jacobian matrix in finite-dimensional case, the exponential decay for the order parameter corresponds to resonance poles in the left half complex plane when the analytic continuation is permissible. Recently, the rigorous analyses of the Landau damping in Kuramoto model for both the incoherent and partially synchronous states were discussed.[20–22]

In this paper, we study the Landau damping effects in the phase oscillator system. To this end, we investigate the Kuramoto model by considering both the in- and out-heterogeneity of the coupling strength. We prove that the critical point for phase transition is the same in both cases which can be written in a united formula. The mean-field theory and linear stability analysis are implemented for solving K_c. Furthermore, we illustrate that the Landau damping indeed exists in the presence of coupling heterogeneity, where the order parameters decay in exponential form in the long time limit. In particular, we find that the original order parameter is determined by a weighted order parameter.

The paper is organized as follows, in Section 2 we introduce the dynamical model and obtain the critical coupling strength by the mean-field theory and perform linear stability analysis. In Section 3, the Landau damping effects are further discussed in term of Laplace transform. Section 4 gives the conclusion of the paper.

2. Dynamical model and critical point for phase transition

The extended Kuramoto model with heterogeneous coupling has the following form:

$$\dot{\theta}_i = \omega_i + \frac{K_i}{N}\sum_{j=1}^{N} G_j \sin(\theta_j - \theta_i), \quad i = 1,2,\ldots,N, \tag{3}$$

where K_i and G_j are the out- and in- coupling strengthes, respectively, that depend on each oscillator. The classical Kuramoto model corresponds to a simple case where $K_i = K$ and $G_j = 1$. There are many ways to choose K_i and G_j according to specific problems. Without losing generality, we focus on the frequency-weighted coupling where K_i and G_j are proportional to the natural frequency (i.e., $K_i = K\omega_i$, $G_j = K\omega_j$).[23,24] On the one hand, the out- or in- frequency-weighted coupling is a straight forward extension of the positive and negative model where the coupling strength $K_i = \pm K$ with different choices of proportion.[25] On the other hand, the advantage of such choice ensures an analytical treatment that makes it convenient to study phase transition of the system.[26–32] In what follows, we will first conduct the mean-field theory investigation and perform linear stability analysis of the incoherent state to obtain the critical point where the phase transition occurs.

2.1. The in-frequency-weighted case

In this case, we set $K_i = 1$ and $G_j = K\omega_j$. The mean-field theory is useful to investigate the stationary solutions of the system in the long time limit, and the basic idea is to conduct self-consistency equations and mean-field frequency. In addition to the traditional order parameter $Z(t)$, a weighted order parameter is necessary, which is defined as

$$D(t) = |D|\,\mathrm{e}^{\mathrm{i}\phi(t)} = \frac{1}{N}\sum_{j=1}^{N} \omega_j\ \mathrm{e}^{\mathrm{i}\theta_j(t)}. \tag{4}$$

With this definition, dynamical equation (3) can be rewritten in the mean-field form

$$\dot{\theta}_i = \omega_i + K|D|\sin(\phi - \theta_i). \tag{5}$$

Here $K|D|$ can be interpreted as an effective coupling strength. Equation (5) reflects that the interactions between all oscillators are proportional to the coupling with coefficient D. To study the steady states of the system, the self-consistent method turns out to be effective. We assume equation (5) approaches to a stationary state in a long time limit. At this point, the collective amplitude $|D|$ is time independent and the mean-field phase ϕ rotates uniformly with a frequency Ω. We introduce the phase difference

$$\varphi_i = \theta_i - \phi. \tag{6}$$

The equation of motion of the phase difference oscillator can be transformed into

$$\dot{\varphi}_i = \omega_i - \Omega - K|D|\sin\varphi_i. \tag{7}$$

In the thermodynamical limit $N \to \infty$, we introduce the probability density distribution function as $\rho(\varphi,\omega,t)$, which denotes the probability distribution of state variables (φ,ω,t) and satisfies the normalization condition $\int_0^{2\pi}\rho(\varphi,\omega,t)\,\mathrm{d}\varphi = 1$ and periodic condition $\rho(\varphi+2\pi,\omega,t) = \rho(\varphi,\omega,t)$. Consequently, equation (7) is equivalent to the following continuity equation:

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial\varphi}(\rho\cdot(\omega - \Omega - K|D|\sin\varphi)) = 0. \tag{8}$$

The stationary solutions of Eq. (8) can be divided into phase-locked case ($|\omega-\Omega|<K|D|$) and drifting case ($|\omega-\Omega|>K|D|$). Considering both contributions of the phase-locked and drifting oscillators to D, and separating the real and imaginary parts of the amplitude of D, we obtain the following self-consistency equations corresponding to the real and imaginary parts of $|D|$:

$$|D| = \int_{-\infty}^{\infty} \mathrm{d}\omega\, g(\omega)\omega\sqrt{1-\left(\frac{\omega-\Omega}{K|D|}\right)^2} \times\Theta\left(1-\left|\frac{\omega-\Omega}{K|D|}\right|\right), \tag{9}$$

$$0 = \int_{-\infty}^{\infty} \mathrm{d}\omega g(\omega)\omega\frac{\omega-\Omega}{K|D|} - \int_{-\infty}^{\infty} \mathrm{d}\omega g(\omega)\omega\frac{\omega-\Omega}{K|D|} \times\sqrt{1-\left(\frac{K|D|}{\omega-\Omega}\right)^2}\Theta\left(\left|\frac{\omega-\Omega}{K|D|}\right|-1\right). \tag{10}$$

Here $\Theta(x)$ is the Heaviside function that makes it positive in the radical formula. Theoretically, all steady states of the system can be calculated through Eqs. (9) and (10). In particular, in the limit case ($|D|\to 0$, $\Omega\to\Omega_{\mathrm{c}}$), the critical point K_{c} for phase tradition can be obtained analytically as

$$K_{\mathrm{c}} = \frac{2}{\pi\, g(\Omega_{\mathrm{c}})\Omega_{\mathrm{c}}}, \tag{11}$$

and the critical mean-field frequency Ω_{c} satisfies the balance equation

$$\mathrm{P}\int_{-\infty}^{\infty}\frac{\omega g(\omega)}{\omega-\Omega_{\mathrm{c}}}\mathrm{d}\omega = 0. \tag{12}$$

The symbol P means the principal-value integral along the real ω.[19]

Another way to get the critical points K_{c} is to perform the linear stability analysis of the incoherent state. With the density distribution function $F(\theta,\omega,t)=\rho(\varphi+\Omega t,\omega,t)$, the velocity in the continuity equation should be written as

$$v_\theta = \omega + \frac{K}{2\mathrm{i}}(D(t)\,\mathrm{e}^{-\mathrm{i}\theta} - D^*(t)\,\mathrm{e}^{\mathrm{i}\theta}). \tag{13}$$

The order parameter $D(t)$ can be expressed as

$$D(t) = \int_{-\infty}^{\infty}\mathrm{d}\omega\int_0^{2\pi}\omega\,\mathrm{e}^{\mathrm{i}\theta}g(\omega)F(\theta,\omega,t)\mathrm{d}\theta. \tag{14}$$

Let

$$Z_n(\omega,t) = \int_0^{2\pi}\mathrm{e}^{ni\theta}F(\theta,\omega,t)\mathrm{d}\theta, \quad n=0,1,2,\ldots, \tag{15}$$

be the n-th Fourier coefficient of $F(\theta,\omega,t)$. Substituting it into the continuity equation, we can obtain

$$\frac{\mathrm{d}Z_n}{\mathrm{d}t} = ni\omega Z_n + \frac{nK}{2}(D(t)Z_{n-1} - D^*(t)Z_{n+1}), \tag{16}$$

which must be solved self-consistently with the equation,

$$D(t) = \int_{-\infty}^{\infty} Z_1(t,\omega)\omega g(\omega)\mathrm{d}\omega. \tag{17}$$

It is obvious that the incoherent state ($|D|=0$) implies that $Z_n=0$ for all n ($n\neq 0$). Linearizing Eq. (16) around this fixed point, we have

$$\frac{\mathrm{d}\delta Z_1}{\mathrm{d}t} = \mathrm{i}\omega\delta Z_1 + \frac{K}{2}\delta D(t). \tag{18}$$

The linearized equation (18) can be transformed into eigenvalue equation

$$1 = \frac{K}{2}\int_{-\infty}^{\infty}\frac{\omega}{\lambda-\mathrm{i}\omega}g(\omega)\mathrm{d}\omega. \tag{19}$$

It is convenient to investigate eigenvalue equation into Cartesian coordinate system by setting $\lambda = x+\mathrm{i}y$. The sign of x determines the stability of the incoherent state. As a well known result, the eigenvalue λ does not exist when the coupling strength K is small enough provided that $\omega g(\omega)$ has no singularity. Once $K>K_{\mathrm{c}}$, λ appears with $x>0$ and the incoherent state loses its stability. The critical point K_{c} can be obtained by imposing the conditions $x\to 0^+$, $y\to\Omega_{\mathrm{c}}$, which has the same formula as Eqs. (11) and (12).

2.2. The out-frequency-weighted case

In this situation, we set $K_i = K\omega_i$ and $G_j = 1$ that corresponds to the conformist and contrarian model studied in Ref. [34]. Conduct self-consistency equations for r and mean-field frequency Ω, which are

$$r = \int_{-\infty}^{\infty}\mathrm{d}\omega g(\omega)\mathrm{sgn}(\omega)\sqrt{1-\left(\frac{\omega-\Omega}{Kr\omega}\right)^2} \times\Theta\left(1-\left|\frac{\omega-\Omega}{Kr\omega}\right|\right), \tag{20}$$

$$0 = \int_{-\infty}^{\infty}\mathrm{d}\omega g(\omega)\frac{\omega-\Omega}{Kr\omega} - \int_{-\infty}^{\infty}\mathrm{d}\omega g(\omega)\frac{\omega-\Omega}{Kr\omega} \times\sqrt{1-\left(\frac{Kr\omega}{\omega-\Omega}\right)^2}\Theta\left(\left|\frac{\omega-\Omega}{Kr\omega}\right|-1\right), \tag{21}$$

where $\mathrm{sgn}(\omega)$ is the sign function. In the limit case $r\to 0^+$, $\Omega\to\Omega_{\mathrm{c}}$, considering the Taylor series of Eqs. (20) and (21) and preserving the first order term, one obtains the same results as Eqs. (11) and (12).

Following the same procedure, the linear stability analysis of the incoherent state can be performed by changing the variable $D_n=\omega Z_n$. Once again, the equation of eigenvalue $\lambda=x+\mathrm{i}y$ for δZ_1 has the same result as Eq. (19). Letting $x\to 0^+$, $y\to\Omega_{\mathrm{c}}$, we can obtain the same formula for K_{c} as Eqs. (11) and (12). Therefore, we conclude that the position of the heterogeneity of the coupling strength (in or out) does not influence the critical point for the system where a phase transition takes place and the incoherent state loses its stability in the limit $N\to\infty$.

3. Landau damping effects

Based on the discussion above, we can see that the incoherent state loses its stability once $K > K_c$ regardless of the heterogeneity introduced in the system. However, as Strogatz pointed out that the order parameter in the regime ($K < K_c$) actually decays to 0 for the long time limit with an exponential form in the classical Kuramoto model.[15,16] This decay mechanism is termed as Landau damping which resembles the phenomenon in plasma. In this section, we will show that such effects exist even in the presence of heterogeneity in the coupling strength, and the order parameter $Z(t)$ and weighted order parameter $D(t)$ decay in different forms.

3.1. The Landau damping of the out-frequency-weighted case

It is clear that the linearized equation can be solved by making Laplace transform $\delta Z_1(t,\omega) \to \delta Z_1(s,\omega)$, we have

$$\mathrm{i}\omega\delta Z_1(s,\omega) + \frac{K\omega}{2}\delta Z(s) = s\delta Z_1(s,\omega) - \delta Z_1(t=0,\omega). \quad (22)$$

Solving it, one obtains

$$\delta Z_1(s,\omega) = \frac{\frac{K\omega}{2}\delta Z(s) + \delta Z_1(t=0,\omega)}{s - \mathrm{i}\omega}. \quad (23)$$

Integrating it with respective to ω for both sides, we obtain

$$\begin{aligned}&\int_{-\infty}^{\infty} g(\omega)\delta Z_1(s,\omega)\,\mathrm{d}\omega \\ &= \int_{-\infty}^{\infty} g(\omega)\frac{\frac{K\omega}{2}\delta Z(s) + \delta Z_1(t=0,\omega)}{s - \mathrm{i}\omega}\,\mathrm{d}\omega. \end{aligned} \quad (24)$$

Considering the definition of $\delta Z(t)$, equation (24) becomes

$$\begin{aligned}\delta Z(s) =& \frac{K}{2}\delta Z(s)\int_{-\infty}^{\infty} g(\omega)\frac{\omega}{s-\mathrm{i}\omega}\,\mathrm{d}\omega \\ &+ \int_{-\infty}^{\infty} g(\omega)\frac{\delta Z_1(t=0,\omega)}{s-\mathrm{i}\omega}\,\mathrm{d}\omega. \end{aligned} \quad (25)$$

After simplification, we have

$$\delta Z(s) = \frac{\int_{-\infty}^{\infty} g(\omega)\frac{\delta Z_1(t=0,\omega)}{s-\mathrm{i}\omega}\,\mathrm{d}\omega}{1 - \frac{K}{2}\int_{-\infty}^{\infty} g(\omega)\frac{\omega}{s-\mathrm{i}\omega}\,\mathrm{d}\omega}. \quad (26)$$

For convenience, we assume that $\delta Z_1(t=0,\omega)$ is a constant, i.e., $\delta Z_1(t=0,\omega) = 1$, which means that the initial phases of all oscillators are identical, and $g(\omega)$ follows the Lorentzian distribution

$$g(\omega) = \frac{\gamma}{\pi[(\omega-\Delta)^2+\gamma^2]}, \quad (27)$$

where γ denotes the width of the distribution, and Δ stands for the center. We have

$$\int_{-\infty}^{\infty} \frac{\gamma}{\pi[(\omega-\Delta)^2+\gamma^2]}\frac{1}{s-\mathrm{i}\omega}\,\mathrm{d}\omega = \frac{1}{-\mathrm{i}\Delta+\gamma+s}, \quad (28)$$

$$\int_{-\infty}^{\infty} \frac{\gamma}{\pi[(\omega-\Delta)^2+\gamma^2]}\frac{\omega}{s-\mathrm{i}\omega}\,\mathrm{d}\omega = \frac{\Delta+\mathrm{i}\gamma}{-\mathrm{i}\Delta+\gamma+s}. \quad (29)$$

The perturbed order parameter $\delta Z(t)$ can be determined through the inverse Laplace transform of $\delta Z(s)$, which reads

$$\delta Z(t) = \exp\left\{-\left[\left(\gamma - \frac{\Delta K}{2}\right) - \mathrm{i}\left(\Delta + \frac{K\gamma}{2}\right)\right]t\right\}. \quad (30)$$

Obviously, the critical coupling strength is

$$K_c = \frac{2\gamma}{\Delta}, \quad (31)$$

which is consistent with the result obtained by the mean-field theory and linear stability analysis as shown in Fig. 1. Notice that the order parameter decays in the exponential form while the factor $(\Delta K/2 - \gamma, \Delta + K\gamma/2)$ corresponds to the resonance pole in Ref. [19]. Moreover, it is consistent with the result in Ref. [34].

We emphasize that the mean value Δ controls the property of the system. When $\Delta > 0$, the attractive coupling dominates the system and the incoherent state becomes unstable until $K > K_c$. While $\Delta < 0$, the repulsive term governs the system, the incoherent state is always stable once $K > 0$. Particularly, $\Delta = 0$ is a critical case which leads to $K_c \to \infty$.

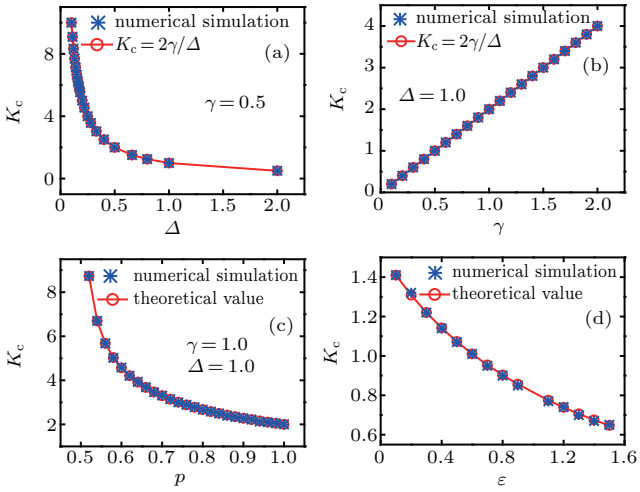


Fig. 1. The critical point for phase transition K_c vs. parameters in heterogeneous case. The natural frequency obeys (a), (b) $g(\omega) = \frac{\gamma}{\pi[(\omega-\Delta)^2+\gamma^2]}$; (c) $g(\omega) = \frac{\gamma}{\pi}[\frac{p}{(\omega-\Delta)^2+\gamma^2} + \frac{1-p}{(\omega+\Delta)^2+\gamma^2}]$, where p represents the height of peak at Δ, with $0 < p < 0.5$; (d) $g(\omega) = 1/2$, where $\omega \in (\varepsilon-1, \varepsilon+1)$ and $\varepsilon > 0$ is a bias of uniform distribution.

3.2. The Landau damping of the in-frequency-weighted case

In this situation, there are two order parameters that should be determined respectively. Considering the variable substitution $\delta D_1(t=0,\omega) = \omega\delta Z_1(t=0,\omega)$, the expression of $\delta D(s)$ can be obtained directly from Eq. (26)

$$\delta D(s) = \frac{\int_{-\infty}^{\infty} g(\omega)\frac{\delta D_1(t=0,\omega)}{s-\mathrm{i}\omega}\,\mathrm{d}\omega}{1 - \frac{K}{2}\int_{-\infty}^{\infty} g(\omega)\frac{\omega}{s-\mathrm{i}\omega}\,\mathrm{d}\omega}. \quad (32)$$

Applying Laplace transform for Eq. (18), as $\delta Z_1(t,\omega) \to \delta Z_1(s,\omega)$, we can obtain

$$\mathrm{i}\omega\delta Z_1(s,\omega)+\frac{K}{2}\delta D(s)=s\delta Z_1(s,\omega)-\delta Z_1(t=0,\omega), \quad (33)$$

hence

$$\delta Z_1(s,\omega)=\frac{\frac{K}{2}\delta D(s)+\delta Z_1(t=0,\omega)}{s-\mathrm{i}\omega}. \quad (34)$$

Using the same method, we can obtain

$$\delta Z(s) = \int_{-\infty}^{\infty} g(\omega)\frac{\frac{K}{2}\delta D(s)+\delta Z_1(t=0,\omega)}{s-\mathrm{i}\omega}\mathrm{d}\omega. \quad (35)$$

Considering the expression of $\delta D(s)$, we can obtain the final expression of $\delta Z(s)$. Assuming $\delta D_1(t=0,\omega)=a$, $\delta Z_1(t=0,\omega)=b$, where a and b are constants (when the initial phases are identical, $b=1$ and a is the mean value of the natural frequency), and ω obeys the Lorentzian distribution as Eq. (27). Taking Eqs. (28) and (29) into consideration and making the inverse Laplace transform, we obtain the weighted order parameter $\delta D(t)$ as

$$\delta D(t)=a\exp\left\{-\left[\left(\gamma-\frac{\Delta K}{2}\right)-\mathrm{i}\left(\Delta+\frac{K\gamma}{2}\right)\right]t\right\}, \quad (36)$$

and the original order parameter $\delta Z(t)$ as

$$\delta Z(t) = \frac{a}{\Delta+\mathrm{i}\gamma}\exp\left\{-\left[\left(\gamma-\frac{\Delta K}{2}\right)-\mathrm{i}\left(\Delta+\frac{K\gamma}{2}\right)\right]t\right\} + \frac{-a+b\Delta+\mathrm{i}b\gamma}{\Delta+\mathrm{i}\gamma}\exp\{-(\gamma-\mathrm{i}\Delta)t\}. \quad (37)$$

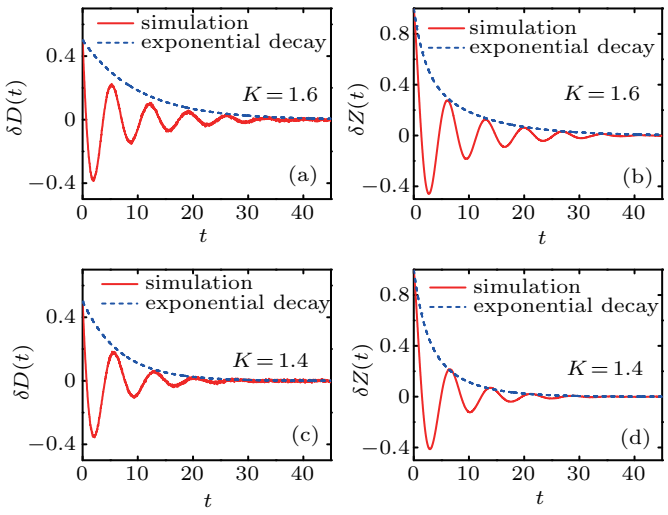


Fig. 2. The decay of $\delta D(t)$ and $\delta Z(t)$ in the in-frequency-weighted case with different K ($K<K_c$). The solid lines are the direct numerical solutions of Eq. (3) (here $K_i=1$ and $G_j=K\omega_j$) and the dashed lines are the fitted curves with exponent as Eqs. (36) and (37). The natural frequency obeys $g(\omega)=\frac{\gamma}{\pi[(\omega-\Delta)^2+\gamma^2]}$. (a) and (b) $K=1.6$, $\Delta=0.5$, $\gamma=0.5$. (c) and (d) $K=1.4$, $\Delta=0.5$, $\gamma=0.5$. In the numerical simulations, we set the total number of oscillators to $N=100000$, and the initial phases for all oscillators are identical.

Obviously, we obtain the critical coupling strength as Eq. (31). The above results show that the Landau damping effect is a common phenomenon, as shown in Fig. 2. The real parts of resonance poles control the exponential relaxation rates of the order parameters. We clarify that the weighted order parameter decays in the exponential form that is the same as the out-weighted case while the original order parameter is controlled by $\delta D(t)$. It should be pointed out that, the exponential decay of the order parameter is just a special case which is caused by the regularity of $g(\omega)$ and identical initial condition. The general decay of the order parameter is possible according to the specifical perturbations and $g(\omega)$.[20–22]

4. Conclusion

To summarize, heterogeneous couplings among interacting elements are ubiquitous in real systems. The Kuramoto model, as a prototype for synchronization, becomes a focus that can present a lot of interesting phenomena such as the Landau damping. In this paper, we consider specific heterogeneous coupling schemes in two different phase oscillator systems by using the mean-field theory and linear stability analysis to extend the frequency-weighted Kuramoto model. The results can further enhance understanding of synchronization in complex system. The explicit expression for the critical point of phase transition can be extended to a united formula which is invariant under the exchange of the weighted factor. In this sense, the Landau damping effects are further studied by using Laplace transform, which can help us understand synchronous transformation in a general network with heterogeneous coupling.

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