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Generalized Hardy-type tests for hierarchy of multipartite non-locality*

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We propose a family of Hardy-type tests for an arbitrary n-partite system, which can detect different degrees of non-locality ranging from standard to genuine multipartite non-locality. For any non-signaling m-local hidden variable model, the corresponding tests fail, whereas a pass of this type of test indicates that this state is m non-local. We show that any entangled generalized GHZ state exhibits Hardy's non-locality for each rank of multipartite non-locality. Furthermore, for the detection of m non-localities, a family of Bell-type inequalities based on our test is constructed. Numerical results show that it is more efficient than the inequalities proposed in [*Phys. Rev. A* **94** 022110 (2016)].

Keywords: Hardy paradox, Bell non-locality, multipartite non-locality

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1. Introduction

About fifty years ago, Bell demonstrated that there exist quantum correlations based on statistical measurement results which can not be explained by any local hidden variable (LHV) model.[1] After his pioneering work, various experiments have been developed to reveal Bell non-locality[2] including very recently loophole-free Bell experiments.[3–5] Nowadays, non-locality becomes one of the essential resources for quantum information tasks, such as quantum key distribution,[6–8] decreasing the communication complexity,[9] and randomness generation.[10] Non-locality can be revealed via different methods, the most famous one is Bell inequalities. Another way is non-locality without inequalities. Unlike Bell's inequalities which involve many statistical events, non-locality without inequalities can reveal the incompatibility between quantum theory and LHV model via single-run operation. Greenberger *et al.* firstly proposed this type of methods in 1990,[11] famously known as the GHZ paradox. Hardy's paradox that was raised in 1993[12] is also an elegant method without inequalities.

Bipartite correlations can be classified as either local or nonlocal. However, when the number of particles is greater than 2, the structure of non-local correlation becomes more complicated. The general classification schemes are k-subsystem non-local that at most k particles of the system exhibit non-local correlation,[13] and m non-local that the system can not be separated into more than m parts (including m). Our paper deals with the latter case, which is the hierarchy of multipartite non-locality (or m separability).[13] The two extreme cases are genuine and standard multipartite non-locality, which correspond to the strongest and the weakest cases, respectively. There are many Bell-type inequalities for different degrees of multipartite non-locality. To detect the standard multipartite non-locality, Mermin–Ardehali–Belinskii–Klyshko (MABK) inequalities[14–17] and Werner–Wolf–Zukowski–Brukner (WWZB) inequalities[18,19] were proposed. For the cases of genuine multipartite non-locality, Svetlichny firstly introduced the notion of genuine multipartite non-locality and proposed Bell-type inequalities for genuine tripartite non-locality.[20] Later, Svetlichny inequality was generalized to arbitrary n-partite two-dimensional systems[21] and higher-dimensional systems.[22] Between these two extreme cases, a set of Bell-type inequalities have been derived to detect the hierarchy of multipartite non-locality by Wang *et al.*[23]

As an elegant method, Hardy's paradox has been extended to reveal multipartite non-locality. For the standard case, the orginal Hardy's paradox for two-qubit[12] was extended to n-particle systems by Cereceda,[24] and a generalized Hardy's paradox was proposed very recently,[25] the experimental verification followed soon after.[26] For the genuine multipartite non-locality case, a Hardy-type test was constructed by Chen *et al.*[27] based on the marginal probability distribution, and the experimental application has been accomplished in Ref. [28]. Another Hardy-type test for the genuine case was proposed independently in Ref. [29]. While

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the Hardy-type test for hierarchy of multipartite non-locality has not been given. It is interesting to note that the Hardy's theorem is a sufficient condition for the violation of Bell inequalities,[30] which allows us to find Bell inequality though Hardy-type tests. So far, the advantages of Hardy's paradox in some quantum information processing tasks have revealed, which attract many interests recently, such as device-independent dimension witness,[31] temporal non-locality,[32] and device-independent randomness generation.[33]

In this paper, we propose a family of generalized Hardy-type tests for the hierarchy of multipartite non-locality without inequalities for the first time. For standard and genuine multipartite non-locality cases, they reduce to the test in Refs. [24] and [27], respectively. Between these two extreme cases, we obtain several sets of Hardy-type conditions for each rank of multipartite non-locality, and each set of conditions is incompatible with this rank of m-LHV model. We show that any entangled generalized GHZ state exhibits Hardy's non-locality for each rank of multipartite non-locality. Specifically, for a special choice of measurement basis, a concise formula of the corresponding maximal success probability is derived for the GHZ state (the maximally entangled one) of Hardy-type tests when $m \geq 3$. We also derive a family of Bell-type inequalities for hierarchy of multipartite non-locality based on our Hardy-type tests, which includes the inequalities derived in Ref. [23]. More interestingly, our numerical results for the new derived inequalities give lower bound of visibility threshold for the generalized Werner state compare to the inequalities proposed in Ref. [23].

2. Definition of m non-locality

Let us consider a system composed of n space-like separated particles labeled with the index set $I = (1,2,\ldots,n)$. Denote the measurement setting and outcome of the k-th particle ($k \in I$) as M_k and r_k, respectively. In a standard LHV model, the joint probability distribution $P(r_I|M_I)$ with measurement settings $M_I = (M_1,\ldots,M_n)$ and outcomes $r_I = (r_1,\ldots,r_n)$ assumes

$$P(r_I|M_I) = \int \rho(\lambda) \prod_{k=1}^{n} P_k(r_k|M_k,\lambda)\mathrm{d}\lambda, \tag{1}$$

where $P_k(r_k|M_k,\lambda)$ is the probability of observer k ($k \in I$) measuring observable M_k with outcome r_k for a given local hidden variable λ distributed according to $\rho(\lambda)$ that satisfies $\int \rho(\lambda)\mathrm{d}\lambda = 1$. In such a case, there is no correlations between the observers in such measurement setting. However, if there involve three or more observers, some of the observables from different observers may correlated to each other even though the observers are space-like separated. Among the hybrid local-nonlocal models, the most general one is

$$P(r_I|M_I) = \sum_{\gamma\neq\emptyset,\gamma\subset I} \int \rho(\lambda) P_\gamma(r_\gamma|M_\gamma,\lambda) P_{\bar{\gamma}}(r_{\bar{\gamma}}|M_{\bar{\gamma}},\lambda)\mathrm{d}\lambda, \tag{2}$$

where $\bar{\gamma} = I \setminus \gamma$ is also nonempty. The correlations shared among the observers in set γ (or $\bar{\gamma}$) might be nonlocal. If the joint probability distribution for a given system cannot be decomposed in a way given in Eqs. (1) and (2), then we can say such system exhibits standard and genuine multipartite non-locality, respectively. Between these two extreme cases, there are hierarchal kinds of multipartite non-locality, which are called m-partite ($2 \leq m \leq n$) non-locality. The sufficient condition for such non-locality behavior is that the joint probability distribution $P(r_I|M_I)$ will not admit any non-signaling m-local hidden variable (m-LHV) model, which reads

$$P(r_I|M_I) = \sum_{\{\gamma\}} \int \rho(\lambda) \prod_{i=1}^{m} P_{\gamma_i}(r_{\gamma_i}|M_{\gamma_i},\lambda)\mathrm{d}\lambda, \tag{3}$$

where sum over $\{\gamma\}$ means all possibilities by splitting I into m subsets are covered, that $\gamma_i \neq \emptyset$, $\bigcup_{i=1}^{m} \gamma_i = I$, $\gamma_i \cap \gamma_j = \emptyset$ for all $i,j = 1,2,\ldots,m$ and $i \neq j$. Just as Eq. (2), nonlocal correlations may be shared among the observers in γ_i when $|\gamma_i| \geq 2$, which may exclude the standard LHV model and admit a hybrid local-nonlocal model.

Definition 1 If the joint probability distribution $P(r_I|M_I)$ of a system excludes the m-LHV model for any given $\{\gamma\}$, then such system exhibits m non-locality.

It is also important to mention that the superluminal interaction is forbidden, which means the probability distribution $P_{\gamma_i}(r_{\gamma_i}|M_{\gamma_i},\lambda)$ has to be non-signaling ($|\gamma_i| \geq 2$)

$$\begin{aligned} P_{\gamma_i}(r_{\gamma_i\setminus k}|M_{\gamma_i\setminus k},\lambda) &= \sum_{r_k} P_{\gamma_i}(r_{\gamma_i\setminus k} r_k|M_{\gamma_i\setminus k} M'_k,\lambda) \\ &= \sum_{r_k} P_{\gamma_i}(r_{\gamma_i\setminus k} r_k|M_{\gamma_i\setminus k} M_k,\lambda), \end{aligned} \tag{4}$$

i.e., for any $k \in \gamma_i$, the marginal probability distribution of subsystem $\gamma_i \setminus k$ is independent of the inputs on the k-th part.

3. Hardy-type test of the hierarchy of multipartite non-locality

To detect m non-locality ($2 \leq m \leq n$), we construct the following joint probabilities conditions of a qubit system:

$$P(0_I|a_I) = q > 0, \tag{5}$$

$$P(0_I|b_k a_{\bar{k}}) = 0, \quad \forall\, k \in I, \tag{6}$$

$$P(1_{k'\alpha} 0_{\overline{k'\alpha}}|b_{k'\alpha} a_{\overline{k'\alpha}}) = 0, \quad \forall\, |\alpha| = m-1, \tag{7}$$

where $k' \in I$ is fixed, and we denote $a_{\bar{k}} = a_{\forall i\neq k}$, $k'\alpha = \{k' \cup \alpha\}$, $b_\alpha = b_{\forall k\in\alpha}$ for simplicity. The two distinct outcomes of the qubit system are denoted as $\{0,1\}$. The above proposal is called Hardy-type test 1, and $q > 0$ is the success probability.

Theorem 1 Any non-signaling m-LHV model ($2 \leq m \leq n$) cannot satisfy Hardy-type test 1.

Proof Because of linearity of Eq. (3), one only needs to prove that the correlation $\prod_{i=1}^{m} P_{\gamma_i}(r_{\gamma_i}|M_{\gamma_i},\lambda)$ can not satisfy

Hardy-type test 1 for any given $\{\gamma\}$ and λ. Thus, through the left-hand side of Eq. (7), one can find that there always exists at least one condition in which for each $\gamma_i, i \in \{1,\ldots,m\}$, $P_{\gamma_i}(r_{\gamma_i}|M_{\gamma_i},\lambda)$ can only be written as $P_{\gamma_i}(1_{l_i}0_{\gamma_i\setminus l_i}|b_{l_i}a_{\gamma_i\setminus l_i},\lambda)$ or $P_{\gamma_j}(0_{\gamma_j}|a_{\gamma_j},\lambda)$, where $l_i \in \gamma_i$. By combining this kind of condition with Eq. (5), one obtains at least one γ_i that makes $P_{\gamma_i}(1_{l_i}0_{\gamma_i\setminus l_i}|b_{l_i}a_{\gamma_i\setminus l_i},\lambda)=0$. Meanwhile, through Eqs. (5) and (6), one gets $P_{\gamma_i}(0_{l_i}0_{\gamma_i\setminus l_i}|b_{l_i}a_{\gamma_i\setminus l_i},\lambda)=0$ for each γ_i and l_i. From the non-signaling condition, one obtains

$$\begin{aligned} P_{\gamma_i}(0_{\gamma_i}|a_{\gamma_i},\lambda) &\le P_{\gamma_i}(0_{\gamma_i\setminus l_i}|a_{\gamma_i\setminus l_i},\lambda) \\ &= P_{\gamma_i}(1_{l_i}0_{\gamma_i\setminus l_i}|b_{l_i}a_{\gamma_i\setminus l_i},\lambda) \\ &\quad +P_{\gamma_i}(0_{l_i}0_{\gamma_i\setminus l_i}|b_{l_i}a_{\gamma_i\setminus l_i},\lambda) \\ &= 0, \end{aligned}$$

which contradicts with Eq. (5). Thus, any m-LHV model can not satisfy Hardy-type test 1.

While, the quantum state will pass the above tests. Let us find the measurement basis for the generalized GHZ state that satisfies Hardy-type test 1. The generalized n-qubit GHZ state is defined as

$$|\psi_{\rm GHZ}\rangle_n = \cos\theta|00\cdots0\rangle + \sin\theta|11\cdots1\rangle, \tag{8}$$

where $0 \le \theta \le \pi/2$, and the standard GHZ state corresponds to $\theta = \pi/4$.

Quantum mechanically, the measurement settings a_k and b_k are related to non-commute obsevables $\{A_k, B_k\}$, while a_k and b_k with outcome "0" correspond to the eigenvectors $\{|a_k\rangle, |b_k\rangle\}$, and those with outcome "1" correspond to the eigenvectors $\{|\bar{a}_k\rangle, |\bar{b}_k\rangle\}$, that satisfy $\langle a_k|\bar{a}_k\rangle = 0$ and $\langle b_k|\bar{b}_k\rangle = 0$. Then, Hardy-type test 1 becomes

$$\langle a_I|\rho|a_I\rangle = q > 0, \tag{9}$$

$$\langle b_k a_{\bar{k}}|\rho|b_k a_{\bar{k}}\rangle = 0, \quad \forall\, k \in I, \tag{10}$$

$$\langle \bar{b}_{k'\alpha} a_{\overline{k'\alpha}}|\rho|\bar{b}_{k'\alpha} a_{\overline{k'\alpha}}\rangle = 0, \quad \forall\, |\alpha| = m-1. \tag{11}$$

Denote $|\bar{b}_{k'\alpha}\rangle = |\bar{b}_{k'}\rangle \otimes_{\forall k\in\alpha} |\bar{b}_k\rangle$. If the above conditions are fulfilled for any measurement basis, then the state ρ will exclude any m-LHV model, and exhibits m non-locality.

We choose a special measurement basis that

$$\begin{aligned} &|a_1\rangle \ne |a_2\rangle, \quad |b_1\rangle \ne |b_2\rangle, \\ &|a_k\rangle = |a_2\rangle, \quad |b_k\rangle = |b_2\rangle, \quad \forall\, k > 2. \end{aligned} \tag{12}$$

Then we find a concise formula of maximal success probabilities for the GHZ state (the maximally entangled one) when $m \ge 3$,

$$P^{\max}_{m\ge3}(a_{I_n}) = \frac{1}{2^n}\left[1+\cos\left(\frac{\pi}{m-1}\right)\right]. \tag{13}$$

And any generalized entangled GHZ state exhibits all rank of multipartite non-locality. The details are given in Appendix A.

Since the m-LHV model in Eq. (3) is defined for all possible $\{\gamma\}$. By splitting $\{I\setminus k'\}$ into two parts β and $I\setminus\{k'\beta\}$, while $0 \le |\beta| \le n-m$ and $\alpha \subset I\setminus\{k'\beta\}$, we get a generalized Hardy-type test 2

$$P(0_I|a_I) = q > 0, \tag{14}$$

$$P(0_I|b_k a_{\bar{k}}) = 0, \quad \forall\, k \in I, \tag{15}$$

$$P(1_{k'\alpha}0_{\overline{k'\alpha}}|b_{k'\alpha}a_{\overline{k'\alpha}}) = 0, \quad \forall\, |\alpha| = m-1, \tag{16}$$

$$P(1_{k'l}0_{\overline{k'l}}|b_{k'l}a_{\overline{k'l}}) = 0, \quad \forall\, l \in \beta. \tag{17}$$

Theorem 2 In any nonsignaling m-LHV model ($2 \le m \le n$), the joint probabilities can not satisfy Hardy-type test 2.

Proof One can use the same procedure for Theorem 1. The key point is that there at least exists one condition in which for each $\gamma_i, i \in \{1,\ldots,m\}$, $P_{\gamma_i}(r_{\gamma_i}|M_{\gamma_i},\lambda)$ can only be written as $P_{\gamma_i}(1_{l_i}0_{\gamma_i\setminus l_i}|b_{l_i}a_{\gamma_i\setminus l_i},\lambda)$ or $P_{\gamma_j}(0_{\gamma_j}|a_{\gamma_j},\lambda)$ in the last two conditions of Hardy-type test 2. If $\{k'\beta\}$ is a subset of certain γ_i, such condition can be found in Eq. (16), otherwise, it can be found in Eq. (17).

The state given in Eq. (8) also satisfies Hardy-type test 2 under the same measurement basis in Eq. (A1) for some cases. One can see the informal discussions in Appendix B.

For the sake of convenience, denote Hardy-type test 2 as $[n,m;|\beta|]$ scenario. It is obvious that $[n,m;0]$ scenario corresponds to Hardy-type test 1, and $[n,n;0]$ scenario backs to the original standard Hardy's paradox.[24] For genuine multipartite cases, $[n,2;|\beta|]$ scenario reduces to the test proposed in Ref. [27]. Between these two extreme cases, $[n,m;|\beta|]$ scenario gives different set of conditions to detect m nonlocality of different $|\beta|$. Note that the total number of conditions decreases as $|\beta|$ increases, and for the two extreme cases $|\beta| = 0$ and $|\beta| = n-m$, our scenario contains $1+n+C^{m-1}_{n-1}$ and $2n+2-m$ conditions, respectively.

In Ref. [27], it has been proven that all genuine multipartite entangled symmetric pure states can be detected by the Hardy-type test which corresponds to $[n,2;|\beta|]$ scenario, and through numerical results one can conjecture that all genuine multipartite entangled pure states can be detected by this test. As a matter of fact, our Hardy-type tests for hierarchy of multipartite non-locality exclude all m-LHV model description in principle. Here comes an interesting question, could all $(m-1)$-separable entangled states be detected by our Hardy-type tests? However the answer is negative and an illustrating example for a 4-qubit system is given in the following.

Example We consider the case of $n=4$ and $m=3$. Set $k'=1$ for simplicity, the $[4,3;0]$ scenario is

$$\begin{aligned} &P(0,0,0,0|a_1,a_2,a_3,a_4) = q > 0, \\ &P(0,0,0,0|b_1,a_2,a_3,a_4) = 0, \\ &P(0,0,0,0|a_1,b_2,a_3,a_4) = 0, \\ &P(0,0,0,0|a_1,a_2,b_3,a_4) = 0, \\ &P(0,0,0,0|a_1,a_2,a_3,b_4) = 0, \end{aligned}$$

$$P(1,1,1,0|b_1,b_2,b_3,a_4)=0,$$
$$P(1,1,0,1|b_1,b_2,a_3,b_4)=0,$$
$$P(1,0,1,1|b_1,a_2,b_3,b_4)=0,$$

which belongs to Hardy-type test 1. For $[4,3;1]$ scenario, the last three conditions are replaced by the following two:

$$P(1,1,0,0|b_1,b_2,a_3,a_4)=0,$$
$$P(1,0,1,1|b_1,a_2,b_3,b_4)=0.$$

Any non-signaling 3-LHV model can not pass these two scenarios, and a 2-separable entangled state may pass them. So the question is that whether all 2-separable entangled states can be detected by these two scenarios or not. Now we introduce the following state:

$$|\psi\rangle=|\phi\rangle_{12}\otimes|\chi\rangle_{34},$$

where $|\phi\rangle=(|00\rangle+|11\rangle)/\sqrt{2}$, $|\chi\rangle$ is an arbitrary entangled state. For both scenarios, we obtain the same conditions $\langle\phi|\hat{a}_1\hat{a}_2\rangle>0$, $\langle\phi|\hat{b}_1\hat{a}_2\rangle=0$, $\langle\phi|\hat{a}_1\hat{b}_2\rangle=0$, $\langle\phi|\hat{\bar{b}}_1\hat{\bar{b}}_2\rangle=0$. Since the maximally entangled two-qubit state can not pass the original Hardy's paradox,[12] the above four conditions can not fulfill simultaneously. As a result, the state $|\psi\rangle$ can not be detected by these two scenarios.

4. Hierarchy of Bell-type inequalities

From the $[n,m;|\beta|]$ scenario, the corresponding Bell-type inequality can be obtained as

$$P(0_I|a_I)-\sum_{k\in I}P(0_I|b_k a_{\bar{k}})-\sum_{k\in\beta}P(1_{k'k}0_{\overline{kk'}}|b_{k'k}a_{\overline{kk'}})$$
$$-\sum_{|\alpha|=m-1,\alpha\subseteq I\setminus(k'\beta)}P(1_{k'\alpha}0_{\overline{k'\alpha}}|b_{k'\alpha}a_{\overline{k'\alpha}})\leq 0. \qquad (18)$$

Any n-partite state that violates Eq. (18) exhibits m non-locality.

Remark 1 For the proof of Eq. (18), one only needs to prove that the correlation $\prod_{i=1}^{m}P_{\gamma_i}(r_{\gamma_i}|M_{\gamma_i},\lambda)$ can not violate Eq. (18) for any given $\{\gamma_i\}$ and λ. If $\{\beta k'\}$ is a subset of γ_i, one can use the similar method in Ref. [23], otherwise, use the similar method in the supplementary materials of Ref. [27].

One can find that equation (18) reduces to inequality (7) of Ref. [23] when $|\beta|=0$. For the case of $|\beta|>0$, we obtain a new type of inequalities for the m-nonlocal cases. For example, for $[4,3;1]$ scenario, we get a new Bell inequality

$$\langle\hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4\rangle-\langle\hat{b}_1\hat{a}_2\hat{a}_3\hat{a}_4\rangle-\langle\hat{a}_1\hat{b}_2\hat{a}_3\hat{a}_4\rangle-\langle\hat{a}_1\hat{a}_2\hat{b}_3\hat{a}_4\rangle$$
$$-\langle\hat{a}_1\hat{a}_2\hat{a}_3\hat{b}_4\rangle-\langle\hat{\bar{b}}_1\hat{\bar{b}}_2\hat{a}_3\hat{a}_4\rangle-\langle\hat{\bar{b}}_1\hat{a}_2\hat{\bar{b}}_3\hat{\bar{a}}_4\rangle\leq 0. \qquad (19)$$

Consider an n-qubit GHZ state in the presence of white noise, which is the generalized Werner state

$$\rho_{\rm w}=p|{\rm GHZ}\rangle\langle{\rm GHZ}|+(1-p)\frac{\mathbb{I}}{2^n}. \qquad (20)$$

For any given n, combining Eqs. (18) and (20), we can numerically find out the minimal value of visibility threshold $p_{m;|\beta|}$ that for $p>p_{m;|\beta|}$, state $\rho_{\rm w}$ violates the corresponding $[n,m;|\beta|]$-th Bell-type inequality. The lower the $p_{m;|\beta|}$ is, the more noise the corresponding Bell-type inequality can resist. For simplicity, we choose all the measurement settings in the X–Z plane of the Bloch sphere, and set $\hat{a}_k=\hat{a}$ and $\hat{b}_k=\hat{b}$ for all $2\leq k\leq n$. The numerical results are listed in Table 1 for $4\leq n\leq 6$. It recovers the results of Wang *et al.*[23] for $|\beta|=0$. Interestingly, $p_{3;|\beta|>0}$ are always lower than $p_{3;0}$ for any $4\leq n\leq 6$, which means that Bell-type inequalities corresponding to $[n,3;|\beta|>0]$ are always more efficient than $[n,3;|\beta|=0]$ to detect m non-locality. For $m>3$, we also find $p_{4;1}=p_{4;2}<p_{4;0}$ in the case of $n=6$.

Table 1. Numerical results of $p_{m;|\beta|}$ of all $[n,m]$ cases of different $|\beta|$, where $4\leq n\leq 6$. The underlined number is the lowest value of $p_{m;|\beta|}$ for every $[n,m]$ case.

n	$p_{2;0}$	$p_{3;0}$	$p_{3;1}$	$p_{3;2}$	$p_{3;3}$	$p_{4;0}$	$p_{4;1}$	$p_{4;2}$	$p_{5;0}$	$p_{5;1}$	$p_{6;0}$
4	0.948	0.914	0.890			0.822					
5	0.963	0.952	0.949	0.949		0.923	0.926		0.848		
6	0.971	0.969	0.966	0.964	0.961	0.960	0.955	0.955	0.932	0.935	0.867

5. Discussion and conclusion

In conclusion, we have generalized Hardy-type tests to detect hierarchy of multipartite non-locality, both of our proposals can can identify different degrees of multipartite non-localities ranging from standard to genuine cases. We show that any generalized entangled GHZ state exhibits Hardy's non-locality for each rank of multipartite non-locality. Especially, a brief formula of $P_{m\geq 3}^{\max}(a_{I_n})$ for the GHZ state is derived, which decreases exponentially with n, and increases with m.

Furthermore, a family of the Bell-type inequalities to detect m non-locality is derived in Eq. (18), which includes the Bell-type inequality proposed by Wang *et al.*[23] (corresponding to the case of $|\beta|=0$). The numerical results in Table 1 reveal that for some given n and m the lower bound of visibility threshold of the generalized Werner state may be obtained in the case of $|\beta|>0$, i.e., the corresponding $[n,m;|\beta|>0]$-th Bell-type inequality is more efficient than the $[n,m;|\beta|=0]$-th one. In particular, the Bell-type inequalities correspond to $[n,3;|\beta|>0]$ can resist more noise than

$[n,3;|\beta|=0]$ for $4\le n\le 6$.

Appendix A: Solution of Hardy-type test 1

For simplicity, drop the normalization factor, and take the normalization back when considering the maximal success probability, assume

$$\begin{aligned}
&|a_1\rangle \sim |0\rangle + x_1 \mathrm{e}^{\mathrm{i}\theta_{x_1}}|1\rangle,\\
&|a_k\rangle \sim |0\rangle + x_2 \mathrm{e}^{\mathrm{i}\theta_{x_2}}|1\rangle, \quad \forall k\ge 2,\\
&|b_1\rangle \sim |0\rangle + y_1 \mathrm{e}^{\mathrm{i}\theta_{y_1}}|1\rangle,\\
&|b_k\rangle \sim |0\rangle + y_2 \mathrm{e}^{\mathrm{i}\theta_{y_2}}|1\rangle, \quad \forall k\ge 2,
\end{aligned} \tag{A1}$$

and define that $x_i>0, y_i>0$ without loose of generality. Set $k'=1$, the last two conditions of Hardy-type test 1 give the following equations for the generalized GHZ state given in Eq. (8):

$$\begin{aligned}
&y_1 x_2^{n-1}\mathrm{e}^{\mathrm{i}(\theta_{y_1}+(n-1)\theta_{x_2})}\sin\theta+\cos\theta=0,\\
&x_1 x_2^{n-2} y_3 \mathrm{e}^{\mathrm{i}(\theta_{x_1}+\theta_{y_2}+(n-2)\theta_{x_2})}\sin\theta+\cos\theta=0,\\
&y_1 y_2^{m-1}\mathrm{e}^{-\mathrm{i}(\theta_{y_1}+(m-1)\theta_{y_2}-m\pi)}\cos\theta\\
&+x_2^{n-m}\mathrm{e}^{\mathrm{i}(n-m)\theta_{x_2}}\sin\theta=0.
\end{aligned} \tag{A2}$$

While both the imaginary and the real parts should be zero simultaneously. Since $0\le\theta\le\pi/2$ and $x_i>0$, $y_i>0$, the imaginary parts give

$$\begin{aligned}
&\theta_{y_1}+(n-1)\theta_{x_2}=(2k_1+1)\pi,\\
&\theta_{x_1}+\theta_{y_2}+(n-2)\theta_{x_2}=(2k_2+1)\pi,\\
&(n-m)\theta_{x_2}+\theta_{y_1}+(m-1)\theta_{y_2}=(2k_3+1+m)\pi,
\end{aligned} \tag{A3}$$

where k_1,k_2,k_3 are integers. Combining the above equations, we find that

$$\theta_{x_1}+(n-1)\theta_{x_2}=\frac{(2N+1)\pi}{m-1}+2k_2\pi,$$

with $N=k_1-k_3$.

For the real part, one can find that

$$\begin{aligned}
&y_1 x_2^{n-1}\sin\theta=\cos\theta,\\
&x_1 x_2^{n-2} y_2 \sin\theta=\cos\theta,\\
&x_2^{n-m}\sin\theta=y_1 y_2^{m-1}\cos\theta,
\end{aligned}$$

which give the solution

$$\begin{aligned}
&y_2=y_1^{\frac{2n-m-1}{(1-m)(n-1)}}\cot^{\frac{1}{1-n}}\theta,\\
&x_2=y_1^{\frac{2}{1-m}}y_2^{-1},\\
&x_1=x_2 y_1 y_2^{-1}.
\end{aligned} \tag{A4}$$

Then, the success probability becomes

$$P_m(a_{I_n},\theta)=\frac{|\cos\theta+x_1 x_2^{n-1}\mathrm{e}^{\frac{\mathrm{i}(2N+1)\pi}{m-1}}\sin\theta|^2}{(1+x_1^2)(1+x_2^2)^{n-1}}.$$

Thus, one can always find an appropriate measurement basis that makes $P_m(a_{I_n},\theta)>0$, which means any generalized entangled GHZ state exhibits m non-locality.

And because N and m are integers, we find that

$$\cos\left(\frac{(2N+1)\pi}{m-1}\right)\le\cos\left(\frac{\pi}{m-1}\right)$$

for any N, while equal is taken if and only if $N/(m-1)$ is integer or $(N+1)/(m-1)$ is integer. Then, for the GHZ state (the maximally entangled one), the maximal value of success probability reduces to

$$\begin{aligned}
&P_m(a_{I_n})=\frac{1+2y_1^{\kappa}\cos\left(\frac{\pi}{m-1}\right)+y_1^{2\kappa}}{2(1+y_1^{2(1+\kappa)})(1+y_1^{\frac{2}{1-n}})^{n-1}},\\
&\kappa=\frac{2(m-n)}{(1-m)(n-1)}.
\end{aligned} \tag{A5}$$

It is obvious that the GHZ state always passes $[n,2;0]$ scenario except the cases $y_1=1,0,\infty$. When $m\ge 3$, it is easy to prove that

$$\partial_{y_1}P_{m\ge3}(a_{I_n})|_{y_1=1}=0,\quad \partial_{y_1}^2P_{m\ge3}(a_{I_n})|_{y_1=1}<0,$$

which means $P_{m\ge3}(a_{I_n})$ reaches maximum when $y_1=1$. Then, the maximal success probability for $m\ge3$ is

$$P_{m\ge3}^{\max}(a_{I_n})=\frac{1}{2^n}\left[1+\cos\left(\frac{\pi}{m-1}\right)\right]. \tag{A6}$$

When $m=n$, equation (A6) becomes

$$P_{n\ge3}^{\max}(a_{I_n})=\frac{1}{2^n}\left[1+\cos\left(\frac{\pi}{n-1}\right)\right], \tag{A7}$$

which recovers the result in Ref. [24] for the standard case. The maximal value of $P_m(a_{I_n})$ decreases with n, while increases with m.

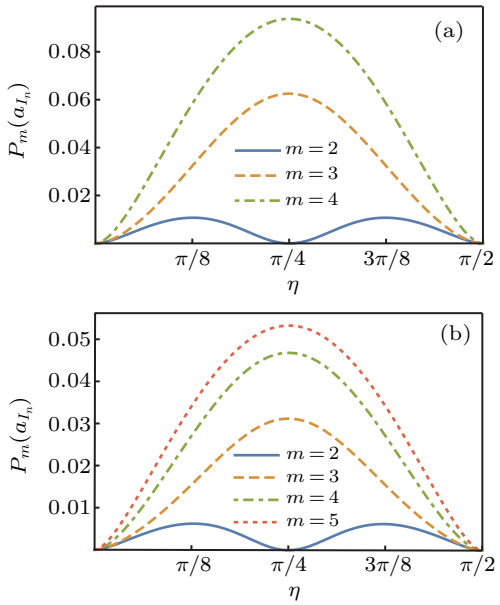


Fig. A1. The success probability $P_m(a_{I_n})$ for GHZ state versus η, where $\eta=\arctan y_1$, and y_1 is the parameter of the measurement basis $|b_1\rangle$. (a) $n=4$, $m=2,3,4$; (b) $n=5$, $m=2,3,4,5$. $P_m(a_{I_n})$ increases with the increase of m for any given measurement bases. The $P_{m\ge3}(a_{I_n})$ reaches maximum when $\eta=\pi/4$, i.e., $y_1=1$.

For the convenience of interpretation, equation (A5) for the cases $n = 4$ and $n = 5$ is represented in Fig. A1 by substituting y_1 as $\tan\eta$.

Appendix B: Solution of Hardy-type test 2

Choose the same measurement basis in Eq. (A1), and set $k' = 1$. For the generalized GHZ state, Hardy-type test 2 gives a new condition compared to Eq. (1)

$$y_1 y_2 \, \mathrm{e}^{-\mathrm{i}(\theta_{y_1}+\theta_{y_2}+(n-2)\theta_{x_2})} \cos\theta + x_2^{n-2} \sin\theta = 0, \qquad \text{(B1)}$$

with the imaginary part

$$\theta_{y_1} + \theta_{y_2} + (n-2)\theta_{x_2} = (2k_4+1)\pi,$$

where k_4 is an integer. Comparing to the conditions of Eq.(A3), one can find that

$$\theta_{x_1} - \theta y_1 = 2(k_2 - k_4)\pi,$$
$$\theta_{y_2} - \theta x_2 = 2(k_2 - k_1)\pi = \frac{2(k_3 - k_4) + m}{m-2},$$

which makes m become even numbers, and also gives

$$y_1 y_2 \cos\theta = x_2^{n-2} \sin\theta.$$

Combined with the solution given in Eq. (A4), the solution for the state given in Eq. (8) is

$$y_1 = \tan^{\frac{m-1}{m-2}} \theta.$$

Since $0 < \theta < \pi/2$ for the generalized entangled GHZ state of Eq. (8), following the discussions in Appendix A, one concludes that any generalized entangled GHZ state exhibits m non-locality of Hardy-type test 2 when m is even.

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