ISSN 1674-1056
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Citation: Chin. Phys. B . 2019, 28(12): 120402 . **doi:** 10.1088/1674-1056/ab5210
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A new type of adiabatic invariants for disturbed non-conservative nonholonomic system*

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(Received 18 July 2019; revised manuscript received 27 September 2019; published online 15 November 2019)

According to the Herglotz variational principle and differential variational principle of Herglotz type, we study the adiabatic invariants for a non-conservative nonholonomic system. Firstly, the differential equations of motion of the non-conservative nonholonomic system based upon the generalized variational principle of Herglotz type are given, and the exact invariant for the non-conservative nonholonomic system is introduced. Secondly, a new type of adiabatic invariant for the system under the action of a small perturbation is obtained. Thirdly, the inverse theorem of the adiabatic invariant is given. Finally, an example is given.

Keywords: non-conservative nonholonomic mechanics, adiabatic invariants, differential variational principle of Herglotz type

PACS: 04.20.Fy **DOI:** 10.1088/1674-1056/ab5210

1. Introduction

We encounter many difficulties in describing many important physical progress with classical variational principles. For instance, the famous Hamilton principle for holonomic conservative systems, it encounters great difficulties in generalizing to non-conservative systems. In general, the Hamilton principle for holonomic non-conservative systems can be expressed as $\int_{t_0}^{t}(\delta T+Q_s\delta q_s)\mathrm{d}t=0$, and the formula can not be expressed in the form that the variation of an action functional is equal to zero, so it is no longer a stable action principle. For this purpose, Herglotz[1] proposed a generalized variational principle and its action functional is defined by a differential equation. Different from classical variational principles, it can study non-conservative dynamical systems in a very good way. At the same time, it can handle the problems of conservative and non-conservative systems in a unified way. After that, Georgieva and Gueuther obtained the Noether theorem based on the Herglotz generalized variational principle.[2,3] Torres *et al.* studied the higher-order Herglotz variational problem[4] and Herglotz variational problem with time delay and its Noether theorem.[5] Further, according to the Herglotz variational principle, Zhang studied the Noether's theorem and conserved quantities in phase space,[6] for non-conservative nonholonomic system,[7] for Birkhoffian system,[8–10] and with time delay.[11,12] Recently, many results have been obtained about the fractional Herglotz variational principle.[13–17]

The study of symmetry and invariants of dynamic systems is of great significance. Symmetry is a method to study the conservation of dynamical systems. There are mainly three different concepts of symmetry: Noether symmetry, Lie symmetry, and Mei symmetry. In addition, we can also study conserved quantities by using the differential variational principle. In 1986, Vujanovic studied the conservation laws of dynamical systems based on Jourdain's principle and Gauss's principle.[18] After that, Liu studied the conservation laws of nonholonomic non-conservative dynamical systems based on Jourdain's principle.[19] Mei studied the conservation law of Birkhoffian systems by using Pfaff–Birkhoff–D'Alembert principle.[20] Up to now, few papers on differential variational principle of Herglotz type have been published.

As we all know, an exact invariant can be found by studying conserved quantities. When a dynamic system is subjected to a small perturbation, the original conserved quantity will change. A physical quantity that almost does not change when a certain parameter of the system changes slowly with respect to the change of the parameter is called an adiabatic invariant. In 1917, Burgers[21] first proposed the concept of adiabatic invariants through a special Hamiltonian system. After that, Beulanov,[22] Kruskal,[23] and Djukić[24] obtained many important results. The research on the adiabatic invariants has aroused the interest of scholars and achieved some important results.[25–36] To the author's knowledge, the method of the differential variational principle of Herglotz type has not been applied to study the adiabatic invariant.

The layout of the article is as follows. We give the differential variational principle and exact invariant of Herglotz type in Section 2. In Section 3, we give a new type of adia-

*Project supported by the National Natural Science Foundation of China (Grant Nos. 11572212, 11272227, and 10972151) and the Innovation Program for Postgraduade in Higher Education Institutions of Jiangsu Province, China (Grant No. KYCX18_2548).

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batic invariant for the non-conservative nonholonomic system. Besides, the existence conditions and form of the adiabatic invariant are presented. In Section 4, we give the inverse theorem of adiabatic invariant. In Section 5, we give an example to support our results. In Section 6, we give some conclusions to end the paper.

2. Differential variational principle and exact invariant of Herglotz type

The differential variational principle of Herglotz type for the non-conservative nonholonomic system is[7]

$$\begin{aligned}&\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left[\frac{\partial\tilde{L}}{\partial q_\sigma}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}\right.\\&+\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}\frac{\partial L}{\partial z}+\frac{\partial\tilde{L}}{\partial q_{\varepsilon+\beta}}\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma}-\frac{\partial L}{\partial\dot{q}_{\varepsilon+\beta}}\\&\left.\times\left(\frac{\partial\varphi_\beta}{\partial q_\sigma}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma}+\frac{\partial\varphi_\beta}{\partial q_{\varepsilon+\gamma}}\frac{\partial\varphi_\gamma}{\partial\dot{q}_\sigma}\right)\right]\delta q_\sigma=0,\\&(\sigma=1,2,\ldots,\varepsilon).\end{aligned}\tag{1}$$

Here, $\dot{z}(t)=L(t,q_s(t),\dot{q}_s(t),z(t)),L(t,q_s,\dot{q}_s,z)$ is the Lagrangian of Herglotz type, and q_s $(s=1,2,\ldots,n)$ are the generalized coordinates of the system. The constraint equation is $\dot{q}_{\varepsilon+\beta}=\varphi_\beta(t,q_s,\dot{q}_\sigma)$ with $s=1,2,\ldots,n$; $\beta=1,2,\ldots,g$; $\varepsilon=n-g$; $\sigma=1,2,\ldots,\varepsilon$.

Because of the independence of δq_σ, the differential equations of motion for this system are derived as follows:

$$\begin{aligned}&\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left[\frac{\partial\tilde{L}}{\partial q_\sigma}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}\right.\\&+\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}\frac{\partial L}{\partial z}+\frac{\partial\tilde{L}}{\partial q_{\varepsilon+\beta}}\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma}-\frac{\partial L}{\partial\dot{q}_{\varepsilon+\beta}}\\&\left.\times\left(\frac{\partial\varphi_\beta}{\partial q_\sigma}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma}+\frac{\partial\varphi_\beta}{\partial q_{\varepsilon+\gamma}}\frac{\partial\varphi_\gamma}{\partial\dot{q}_\sigma}\right)\right]=0,\\&(\sigma=1,2,\ldots,\varepsilon).\end{aligned}\tag{2}$$

If F_s^0 is the infinitesimal generator of space and f^0 is the infinitesimal generator of time, then the isochronic variations are[7]

$$\delta q_s(t)=\varepsilon\left[F_s^0(t,q_k,\dot{q}_k,z)-\dot{q}_s f^0(t,q_k,\dot{q}_k,z)\right],\tag{3}$$

where ε is an infinitesimal parameter.

If the infinitesimal generating functions F_s^0 and f^0 satisfy the following conditions:

$$\begin{aligned}&\frac{\partial\tilde{L}}{\partial q_s}F_s^0+\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}\dot{F}_\sigma^0+\left(\tilde{L}-\dot{q}_\sigma\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}\right)\dot{f}^0\\&-\frac{\partial L}{\partial\dot{q}_{\varepsilon+\beta}}\left(\frac{\partial\varphi_\beta}{\partial q_\sigma}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma}+\frac{\partial\varphi_\beta}{\partial q_{\varepsilon+\gamma}}\frac{\partial\varphi_\gamma}{\partial\dot{q}_\sigma}\right)\left(F_\sigma^0-\dot{q}_\sigma f^0\right)\\&+\frac{\partial\tilde{L}}{\partial t}f^0-\frac{\partial L}{\partial z}G^0+\dot{G}^0=0,\end{aligned}\tag{4}$$

$$\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma}\left(F_\sigma^0-\dot{q}_\sigma f^0\right)+\varphi_\beta f^0-F_{\varepsilon+\beta}^0=0,\tag{5}$$

then there exists a conserved quantity for the non-conservative nonholonomic system of Herglotz type

$$\begin{aligned}I&=\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}F_\sigma^0+\left(\tilde{L}-\dot{q}_\sigma\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}\right)f^0+G^0\right)\\&=\text{const.},\end{aligned}\tag{6}$$

where $G^0=G^0(t,q_k,\dot{q}_k,z)$ is a gauge function. If the system is not disturbed, the conserved quantity (6) is an invariant. We can call it an exact invariant.

3. Adiabatic invariants of Herglotz type for disturbed non-conservative nonholonomic system

Assume that the non-conservative nonholonomic system (2) is subjected to a small perturbation εQ_σ, that is,

$$\begin{aligned}&\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left[\frac{\partial\tilde{L}}{\partial q_\sigma}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}\right.\\&+\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}\frac{\partial L}{\partial z}+\frac{\partial\tilde{L}}{\partial q_{\varepsilon+\beta}}\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma}-\frac{\partial L}{\partial\dot{q}_{\varepsilon+\beta}}\\&\left.\times\left(\frac{\partial\varphi_\beta}{\partial q_\sigma}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma}+\frac{\partial\varphi_\beta}{\partial q_{\varepsilon+\gamma}}\frac{\partial\varphi_\gamma}{\partial\dot{q}_\sigma}\right)\right]=\varepsilon Q_\sigma.\end{aligned}\tag{7}$$

Under the influence of εQ_σ, the primary symmetries and conserved quantities of the system will change accordingly. If the infinitesimal generating functions $f(t,q_k,\dot{q}_k)$ and $F_\sigma(t,q_k,\dot{q}_k)$ of the system which are perturbed can be expressed as

$$\begin{aligned}f&=\tau^0+\varepsilon f^1+\varepsilon^2 f^2+\cdots,\\F_\sigma&=F_\sigma^0+\varepsilon F_\sigma^1+\varepsilon^2 F_\sigma^2+\cdots,\end{aligned}\tag{8}$$

and they satisfy

$$\begin{aligned}&\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left[\frac{\partial\tilde{L}}{\partial q_s}F_s+\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}\dot{F}_\sigma+\left(\tilde{L}-\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}\dot{q}_\sigma\right)\dot{f}\right.\\&+\frac{\partial\tilde{L}}{\partial t}f-\frac{\partial L}{\partial\dot{q}_{\varepsilon+\beta}}\left(\frac{\partial\varphi_\beta}{\partial q_\sigma}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma}+\frac{\partial\varphi_\beta}{\partial q_{\varepsilon+\gamma}}\frac{\partial\varphi_\gamma}{\partial\dot{q}_\sigma}\right)\\&\left.\times\left(F_\sigma-\dot{q}_\sigma f\right)-\frac{\partial L}{\partial z}G+\dot{G}\right]=\varepsilon Q_\sigma(F_\sigma-\dot{q}_\sigma f),\end{aligned}\tag{9}$$

where G is a gauge function and satisfies

$$G=G^0+\varepsilon G^1+\varepsilon^2 G^2+\cdots,\tag{10}$$

then we have the following theorem.

Theorem 1 For the non-conservative nonholonomic system (2) which is subjected to a small perturbation εQ_σ, if the infinitesimal generators $f^j(t,q_k,\dot{q}_k)$ and $F_\sigma^j(t,q_k,\dot{q}_k)$ satisfy the following conditions:

$$\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma}(F_\sigma^j-\dot{q}_\sigma f^j)+\varphi_\beta f^j-F_{\varepsilon+\beta}^j=0,\tag{11}$$

$$\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left[\frac{\partial\tilde{L}}{\partial q_s}F_s^j+\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}\dot{F}_\sigma^j+\left(\tilde{L}-\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}\dot{q}_\sigma\right)\dot{f}^j+\frac{\partial\tilde{L}}{\partial t}f^j\right.$$
$$\left.-\frac{\partial L}{\partial\dot{q}_{\varepsilon+\beta}}\left(\frac{\partial\varphi_\beta}{\partial q_\sigma}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma}+\frac{\partial\varphi_\beta}{\partial q_{\varepsilon+\gamma}}\frac{\partial\varphi_\gamma}{\partial\dot{q}_\sigma}\right)\left(F_\sigma^j-\dot{q}_\sigma f^j\right)-\frac{\partial L}{\partial z}G^j+\dot{G}^j\right]=Q_\sigma(F_\sigma^{j-1}-\dot{q}_\sigma f^{j-1}),\tag{12}$$

where G^j is a gauge function, then

$$I_m=\sum_{j=0}^{m}\varepsilon^j\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left[\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}F_\sigma^j+\left(\tilde{L}-\dot{q}_\sigma\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}\right)f^j+G^j\right]\tag{13}$$

is the m-th order adiabatic invariant of the non-conservative nonholonomic system.

Proof According to condition (12) and Eq. (7), we have

$$\begin{aligned}\frac{\mathrm{d}I_m}{\mathrm{d}t}&=\sum_{j=0}^{m}\varepsilon^j\left\{\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left[\left(-\frac{\partial L}{\partial z}\right)\left(\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}F_\sigma^j+\tilde{L}f^j-\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}\dot{q}_\sigma f^j+G^j\right)+\tilde{L}\dot{f}+\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}\left(F_\sigma^j-\dot{q}_\sigma f^j\right)\right.\right.\\&\quad+\left(\frac{\partial\tilde{L}}{\partial t}+\frac{\partial\tilde{L}}{\partial q_\sigma}\dot{q}_\sigma+\frac{\partial\tilde{L}}{\partial q_{\varepsilon+\beta}}\dot{q}_{\varepsilon+\beta}+\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}\ddot{q}_\sigma+\frac{\partial\tilde{L}}{\partial z}L\right)f^j+\dot{G}^j-\frac{\partial\tilde{L}}{\partial q_s}F_s^j\\&\quad+\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}\left(\dot{F}_\sigma^j-\ddot{q}_\sigma f^j-\dot{q}_\sigma\dot{f}^j\right)+\frac{\partial L}{\partial\dot{q}_{\varepsilon+\beta}}\left(\frac{\partial\varphi_\beta}{\partial q_\sigma}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma}+\frac{\partial\varphi_\beta}{\partial q_{\varepsilon+\gamma}}\frac{\partial\varphi_\gamma}{\partial\dot{q}_\sigma}\right)\left(F_\sigma^j-\dot{q}_\sigma f^j\right)\\&\quad\left.\left.-\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}\dot{F}_\sigma^j-\frac{\partial\tilde{L}}{\partial t}f^j-\left(\tilde{L}-\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}\dot{q}_\sigma\right)\dot{f}^j+\frac{\partial L}{\partial z}G^j-\dot{G}^j\right]+Q_\sigma\left(F_\sigma^{j-1}-\dot{q}_\sigma^j f^{j-1}\right)\right\}\\&=\sum_{j=0}^{m}\varepsilon^j\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left\{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}\left(F_\sigma^j-\dot{q}_\sigma f^j\right)+\frac{\partial\tilde{L}}{\partial q_{\varepsilon+\beta}}\left[F_{\varepsilon+\beta}^j-\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma}\left(F_\sigma^j-\dot{q}_\sigma f^j\right)\right]-\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}\frac{\partial L}{\partial z}\left(F_\sigma^j-\dot{q}_\sigma f^j\right)\right.\\&\quad\left.+\frac{\partial L}{\partial\dot{q}_{\varepsilon+\beta}}\left(\frac{\partial\varphi_\beta}{\partial q_\sigma}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma}+\frac{\partial\varphi_\beta}{\partial q_{\varepsilon+\gamma}}\frac{\partial\varphi_\gamma}{\partial\dot{q}_\sigma}\right)\left(F_\sigma^j-\dot{q}_\sigma f^j\right)-\frac{\partial\tilde{L}}{\partial q_s}F_s^j+\frac{\partial L}{\partial q_\sigma}\dot{q}_\sigma f^j\right\}+\sum_{j=0}^{m}\varepsilon^jQ_\sigma\left(F_\sigma^{j-1}-\dot{q}_\sigma^j f^{j-1}\right)\\&=\sum_{j=0}^{m}\varepsilon^j\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}\left(F_\sigma^j-\dot{q}_\sigma f^j\right)-\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}\frac{\partial L}{\partial z}\left(F_\sigma^j-\dot{q}_\sigma f^j\right)-\frac{\partial\tilde{L}}{\partial q_{\varepsilon+\beta}}\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma}\left(F_\sigma^j-\dot{q}_\sigma f^j\right)\right.\\&\quad\left.+\frac{\partial L}{\partial\dot{q}_{\varepsilon+\beta}}\left(\frac{\partial\varphi_\beta}{\partial q_\sigma}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma}+\frac{\partial\varphi_\beta}{\partial q_{\varepsilon+\gamma}}\frac{\partial\varphi_\gamma}{\partial\dot{q}_\sigma}\right)\left(F_\sigma^j-\dot{q}_\sigma f^j\right)-\frac{\partial L}{\partial q_\sigma}\left(F_\sigma^j-\dot{q}_\sigma f^j\right)\right]+\sum_{j=0}^{m}\varepsilon^jQ_\sigma\left(F_\sigma^{j-1}-\dot{q}_\sigma^j f^{j-1}\right)\\&=\sum_{j=0}^{m}\varepsilon^j\left[-\varepsilon Q_\sigma\left(F_\sigma^j-\dot{q}_\sigma f^j\right)+Q_\sigma\left(F_\sigma^{j-1}-\dot{q}_\sigma f^{j-1}\right)\right]=-\varepsilon^{m+1}Q_\sigma\left(F_\sigma^m-\dot{q}_\sigma f^m\right).\end{aligned}\tag{14}$$

Therefore, I_m is the m-th order adiabatic invariant. According to the definition of adiabatic invariants,[21] we can obtain a new type of adiabatic invariant in the form of formula (13).

If the system is not subjected to nonholonomic constraints,the above theorem gives the corresponding results for holonomic non-conservative systems.

Theorem 2 For the non-conservative holonomic system which is subjected to a small perturbation εQ_s, if the infinitesimal generators $f^j(t,q_k,\dot{q}_k)$ and $F_s^j(t,q_k,\dot{q}_k)$ satisfy the following condition:

$$\begin{aligned}&\exp\left(-\int_{t_1}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left[\frac{\partial L}{\partial t}f^m+\frac{\partial L}{\partial q_s}F_s^m+\frac{\partial L}{\partial\dot{q}_s}\dot{F}_s^m\right.\\&\left.+\left(L-\frac{\partial L}{\partial\dot{q}_s}\dot{q}_s\right)\dot{f}^m-\frac{\partial L}{\partial z}G^m+\dot{G}^m\right]\\&=Q_s\left(F_s^{m-1}-\dot{q}_sf^{m-1}\right),\end{aligned}\tag{15}$$

where G^j is a gauge function, then the system has the m-th order adiabatic invariant as follow:

$$I_m=\sum_{j=0}^{m}\varepsilon^j\exp\left(-\int_{t_1}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)$$
$$\times\left[\frac{\partial L}{\partial\dot{q}_s}F_s^j+\left(L-\frac{\partial L}{\partial\dot{q}_s}\dot{q}_s\right)f^j+G^j\right].\tag{16}$$

4. Inverse problem

If the non-conservative nonholonomic system (2) is subjected to a small perturbation εQ_σ, then it has the following adiabatic invariant:

$$I_1=F_0(t,q_s,\dot{q}_\sigma,z)+\varepsilon F_1(t,q_s,\dot{q}_\sigma,z).\tag{17}$$

Because its orbit should satisfy formula (7), we have

$$\begin{aligned}&\left\{\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left[\frac{\partial\tilde{L}}{\partial q_\sigma}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}+\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}\frac{\partial L}{\partial z}\right.\right.\\&+\frac{\partial\tilde{L}}{\partial q_{\varepsilon+\beta}}\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma}-\frac{\partial L}{\partial\dot{q}_{\varepsilon+\beta}}\left(\frac{\partial\varphi_\beta}{\partial q_\sigma}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma}\right.\\&\left.\left.\left.+\frac{\partial\varphi_\beta}{\partial q_{\varepsilon+\gamma}}\frac{\partial\varphi_\gamma}{\partial\dot{q}_\sigma}\right)\right]-\varepsilon Q_\mu\right\}(F_\sigma-\dot{q}_\sigma f)=0.\end{aligned}\tag{18}$$

Considering that

$$\frac{\mathrm{d}I_1}{\mathrm{d}t}=\frac{\partial F_0}{\partial t}+\frac{\partial F_0}{\partial q_\sigma}\dot{q}_\sigma+\frac{\partial F_0}{\partial q_{\varepsilon+\beta}}\varphi_\beta+\frac{\partial F_0}{\partial \dot{q}_\sigma}\ddot{q}_\sigma+\frac{\partial F_0}{\partial z}\dot{z}+\varepsilon\left(\frac{\partial F_1}{\partial t}+\frac{\partial F_1}{\partial q_\sigma}\dot{q}_\sigma+\frac{\partial F_1}{\partial q_{\varepsilon+\beta}}\varphi_\beta+\frac{\partial F_1}{\partial \dot{q}_\sigma}\ddot{q}_\sigma+\frac{\partial F_1}{\partial z}\dot{z}\right), \tag{19}$$

according to formula (14), and considering formulae (18) and (19), we can obtain

$$\begin{aligned}&\frac{\partial F_0}{\partial t}+\frac{\partial F_0}{\partial q_\sigma}\dot{q}_\sigma+\frac{\partial F_0}{\partial q_{\varepsilon+\beta}}\varphi_\beta+\frac{\partial F_0}{\partial \dot{q}_\sigma}\ddot{q}_\sigma+\frac{\partial F_0}{\partial z}\dot{z}+\varepsilon\left(\frac{\partial F_1}{\partial t}+\frac{\partial F_1}{\partial q_\sigma}\dot{q}_\sigma+\frac{\partial F_1}{\partial q_{\varepsilon+\beta}}\varphi_\beta+\frac{\partial F_1}{\partial \dot{q}_\sigma}\ddot{q}_\sigma+\frac{\partial F_1}{\partial z}\dot{z}\right)+\left\{\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\right.\\&\left.\times\left[\frac{\partial\tilde{L}}{\partial q_\sigma}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}+\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}\frac{\partial L}{\partial z}+\frac{\partial\tilde{L}}{\partial q_{\varepsilon+\beta}}\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma}-\frac{\partial L}{\partial\dot{q}_{\varepsilon+\beta}}\left(\frac{\partial\varphi_\beta}{\partial q_\sigma}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma}+\frac{\partial\varphi_\beta}{\partial q_{\varepsilon+\gamma}}\frac{\partial\varphi_\gamma}{\partial\dot{q}_\sigma}\right)\right]-\varepsilon Q_\sigma\right\}(F_\sigma-\dot{q}_\sigma f)\\&=-\varepsilon^2Q_\sigma\left(F_\sigma^1-\dot{q}_\sigma f^1\right),\end{aligned} \tag{20}$$

where $f=f^0+\varepsilon f^1+\varepsilon^2f^2+\cdots$ and $F_\sigma=F_\sigma^0+\varepsilon F_\sigma^1+\varepsilon^2F_\sigma^2+\cdots$. We can rewrite formula (20) as

$$\begin{aligned}&\frac{\partial F_0}{\partial t}+\frac{\partial F_0}{\partial q_\sigma}\dot{q}_\sigma+\frac{\partial F_0}{\partial q_{\varepsilon+\beta}}\varphi_\beta+\frac{\partial F_0}{\partial \dot{q}_\sigma}\ddot{q}_\sigma+\frac{\partial F_0}{\partial z}\dot{z}+\left\{\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\right.\\&\left.\times\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma}\frac{\partial L}{\partial\dot{q}_{\varepsilon+\beta}}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}+\frac{\partial\tilde{L}}{\partial q_\sigma}+\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}\frac{\partial L}{\partial z}+\frac{\partial\tilde{L}}{\partial q_{\varepsilon+\beta}}\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma}-\frac{\partial L}{\partial\dot{q}_{\varepsilon+\beta}}\left(\frac{\partial\varphi_\beta}{\partial q_\sigma}+\frac{\partial\varphi_\beta}{\partial q_{\varepsilon+\gamma}}\frac{\partial\varphi_\gamma}{\partial\dot{q}_\sigma}\right)\right]\right\}(F_\sigma^0-\dot{q}_\sigma f^0)\\&+\varepsilon\left\{\frac{\partial F_1}{\partial t}+\frac{\partial F_1}{\partial q_\sigma}\dot{q}_\sigma+\frac{\partial F_1}{\partial q_{\varepsilon+\beta}}\varphi_\beta+\frac{\partial F_1}{\partial\dot{q}_\sigma}\ddot{q}_\sigma+\frac{\partial F_1}{\partial z}\dot{z}+\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\right.\\&\times\left[\frac{\partial\tilde{L}}{\partial q_\sigma}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}+\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}\frac{\partial L}{\partial z}+\frac{\partial\tilde{L}}{\partial q_{\varepsilon+\beta}}\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma}-\frac{\partial L}{\partial\dot{q}_{\varepsilon+\beta}}\left(\frac{\partial\varphi_\beta}{\partial q_\sigma}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma}+\frac{\partial\varphi_\beta}{\partial q_{\varepsilon+\gamma}}\frac{\partial\varphi_\gamma}{\partial\dot{q}_\sigma}\right)\right]\\&\left.\times\left(F_\sigma^1-\dot{q}_\sigma f^1\right)-Q_\sigma\left(F_\sigma^0-\dot{q}_\sigma f^0\right)\right\}=0.\end{aligned} \tag{21}$$

So, if we have

$$\frac{\partial F_0}{\partial\dot{q}_l}-\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(\frac{\partial^2\tilde{L}}{\partial\dot{q}_l\partial\dot{q}_\sigma}-\frac{\partial^2\varphi_\beta}{\partial\dot{q}_l\partial\dot{q}_\sigma}\frac{\partial L}{\partial\dot{q}_{\varepsilon+\beta}}\right)(F_\sigma^0-\dot{q}_\sigma f^0)=0,\quad(l=1,2,\ldots,n), \tag{22}$$

assuming further

$$F_0=\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left[\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}F_\sigma^0+\left(\tilde{L}-\dot{q}_\sigma\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}\right)f^0+G^0\right], \tag{23}$$

by Eqs. (22) and (23), we can find the generators of time f^0 and space F_σ^0 corresponding to the undisturbed part. Furthermore, we can also obtain

$$\frac{\partial F_1}{\partial\dot{q}_l}-\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(\frac{\partial^2\tilde{L}}{\partial\dot{q}_l\partial\dot{q}_\sigma}-\frac{\partial^2\varphi_\beta}{\partial\dot{q}_l\partial\dot{q}_\sigma}\frac{\partial L}{\partial\dot{q}_{\varepsilon+\beta}}\right)\left(F_\sigma^1-\dot{q}_\sigma f^1\right)=0,\quad(l=1,2,\ldots,n), \tag{24}$$

$$F_1=\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left[\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}F_\sigma^1+\left(\tilde{L}-\dot{q}_\sigma\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}\right)f^1+G^1\right]. \tag{25}$$

Hence, we have the following theorem.

Theorem 3 For the non-conservative nonholonomic system (2) which is subjected to a small perturbation εQ_σ, if the system has a first order adiabatic invariant as Eq. (17), then there are corresponding infinitesimal transformations, the generating functions without perturbation can be determined by formulae (22), (23) and the generating functions with perturbation can be determined by formulae (24), (25).

5. Example

Under the action of non-conservative force $F_x=-x, F_y=-y$, a unit mass particle moves along the horizontal plan xoy. The particle is constrained so that the slope of the trajectory is proportional to time t and the proportional coefficient is an unit. By means of the method expounded in this paper,[7] we try to study the adiabatic invariants of the system.

The Lagrangian of the system of Herglotz type is

$$L=\frac{1}{2}\left(\dot{q}_1^2+\dot{q}_2^2\right)-z, \tag{26}$$

and the nonholonomic constraint is

$$\dot{q}_2-t\dot{q}_1=0. \tag{27}$$

There exists a conserved quantity

$$I_N=\ln\dot{q}_1+t+\frac{1}{2}\ln\left(1+t^2\right)=\text{const}. \tag{28}$$

When the system is not disturbed, formula (28) is also an exact invariant.

Assume that the system is subjected to small perturbations

$$\varepsilon Q_1 = \mathrm{e}^{2t-2t_0}\varepsilon\left(t+t^3\right)\dot{q}_1^2, \quad \varepsilon Q_2 = 0. \tag{29}$$

Equation (12) gives

$$\begin{aligned}&\mathrm{e}^{t-t_0}\left\{\left(1+t^2\right)\dot{q}_1\xi_1^1-\left[\frac{1}{2}\left(1+t^2\right)\dot{q}_1^2+z\right]\tau^1\right.\\&\left.+t\dot{q}_1^2\tau^1+\dot{q}_2\left(\xi_1^1-\dot{q}_1\tau^1\right)+G^1+\dot{G}^1\right\}\\&=\mathrm{e}^{2t-2t_0}\left(t+t^3\right)\dot{q}_1^2\left(\xi_1^0-\dot{q}_1\tau^0\right).\end{aligned} \tag{30}$$

Equation (30) has the following solutions:

$$\tau^1 = 0, \quad \xi_1^1 = 1, \quad G^1 = \mathrm{e}^{-t+t_0}. \tag{31}$$

By Theorem 1, we have

$$I_{N1} = \ln\dot{q}_1 + t + \frac{1}{2}\ln\left(1+t^2\right) + \varepsilon\left[\mathrm{e}^{t-t_0}\left(1+t^2\right)\dot{q}_1+1\right]. \tag{32}$$

Formula (32) is the first order adiabatic invariant for the system. Similarly, we can obtain much higher order adiabatic invariants.

Lastly, we consider the inverse problem. If the system is disturbed by the small disturbance (29), then by formulae (22) and (23), we have

$$\frac{1}{\dot{q}_1} - \mathrm{e}^{t-t_0}\left(1+t^2\right)\left(\xi_1^0-\dot{q}_1\tau^0\right) = 0, \tag{33}$$

$$\begin{aligned}&\mathrm{e}^{t-t_0}\left[\left(1+t^2\right)\dot{q}_1\xi_1^0-\left(\frac{1}{2}\left(1+t^2\right)\dot{q}_1^2+z\right)\tau^0+G^0\right]\\&=\ln\dot{q}_1+t+\frac{1}{2}\ln\left(1+t^2\right).\end{aligned} \tag{34}$$

If we take

$$G^0 = \mathrm{e}^{-t+t_0}\left[\ln\dot{q}_1+t+\frac{1}{2}\ln\left(1+t^2\right)-1\right], \tag{35}$$

then we can obtain

$$\tau^0 = 0, \quad \xi_1^0 = \frac{\mathrm{e}^{-t+t_0}}{\left(1+t^2\right)\dot{q}_1}, \quad \xi_2^0 = \frac{t\,\mathrm{e}^{-t+t_0}}{\left(1+t^2\right)\dot{q}_1}. \tag{36}$$

Similarly, we have

$$\mathrm{e}^{t-t_0}\left(1+t^2\right) - \mathrm{e}^{t-t_0}\left(1+t^2\right)\left(\xi_1^1-\dot{q}_1\tau^1\right) = 0, \tag{37}$$

$$\begin{aligned}&\mathrm{e}^{t-t_0}\left[\left(1+t^2\right)\dot{q}_1\xi_1^1-\left(\frac{1}{2}\left(1+t^2\right)\dot{q}_1^2+z\right)\tau^1+G^1\right]\\&=\mathrm{e}^{t-t_0}\left(1+t^2\right)\dot{q}_1+1.\end{aligned} \tag{38}$$

If we take

$$G^1 = \mathrm{e}^{-t+t_0}, \tag{39}$$

then we can also obtain

$$\tau^1 = 0, \quad \xi_1^1 = 1, \quad \xi_2^1 = t. \tag{40}$$

Therefore, the generators (36) and (40) give the infinitesimal transformations corresponding to the first order adiabatic invariant (32).

6. Conclusion

It is a new way to study adiabatic invariants by using the differential variational problem of Herglotz type. The main work of the article is summarized as follows. (i) The exact invariant (6) of the non-conservative nonholonomic system and its existence conditions (4) and (5) are established. (ii) A new type of adiabatic invariant (13) for the non-conservative nonholonomic system is given and proved. (iii) The inverse problem of adiabatic invariants is discussed. The main results are two theorems and one corollary. The methods and conclusions can be further popularized and applied. For example, we can study the adiabatic invariant for a nonholonomic system on time scales based on Herglotz variational problem.

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