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Improving entanglement of assistance by local parity–time symmetric operation*

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We investigate entanglement of assistance without and with decoherence using a local non-Hermitian operation, i.e., parity–time ($\mathscr{P}\mathscr{T}$) symmetric operation. First we give the explicit expressions of entanglement of assistance for a general W-like state of a three-qubit system under a local parity–time symmetric operation. Then for a famous W state without decoherence, we find that entanglement of assistance shared by two parties can be obviously enhanced with the assistance of the third party by a local parity–time symmetric operation. For the decoherence case, we provide two schemes to show the effects of local parity–time symmetric operation on improvement of entanglement of assistance against amplitude damping noise. We find that for the larger amplitude damping case the scheme of $\mathscr{P}\mathscr{T}$ symmetric operation performed on one of two parties with the influence of noise is superior to that of $\mathscr{P}\mathscr{T}$ symmetric operation performed on the third party without the influence of noise in suppressing amplitude damping noise. However, for the smaller amplitude damping case the opposite result is given. The obtained results imply that the local $\mathscr{P}\mathscr{T}$ symmetric operation method may have potential applications in quantum decoherence control.

Keywords: entanglement of assistance, local parity–time symmetric operation, amplitude damping noise

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1. Introduction

It is well known that the Hermiticity of any physical observables is one of basic demands in standard quantum mechanics. For example, the Hamiltonian operator of a closed system must be a Hermitian operator, which means that the eigenvalues (or called energy spectra) of the Hamiltonian is real and the time evolution of the system is unitary. However, in Ref. [1] Bender and his co-workers showed that a non-Hermitian operator, i.e., parity–time ($\mathscr{P}\mathscr{T}$) symmetric Hamiltonian, may still have real energy spectra under some conditions, which is a novel result differed from the conventional viewpoint. At present, non-Hermitian systems attract a great deal of interest, which is no longer only of academic interest since it is becoming experimentally accessible in the optical system.[2–7] The study of non-hermitian systems was generalized to various fields, i.e., quantum thermodynamics and quantum information science. It has been shown that for a special class of non-hermitian systems, namely in the $\mathscr{P}\mathscr{T}$-symmetric systems, the quantum Jarzynski equality and some fluctuation relations still hold.[8–11] Moreover, in Ref. [12] the authors have shown that the evolution time between two quantum states under the $\mathscr{P}\mathscr{T}$-symmetric operation can be arbitrarily small. In Ref. [13] the authors found that the no-signalling principle can be violated when applying the local $\mathscr{P}\mathscr{T}$-symmetric operation on one of the entangled particles. More recently, it has also been shown that $\mathscr{P}\mathscr{T}$-symmetric operation can be used to suppress the decoherence of amplitude damping model,[14–20] and slow down the decoherence of pure phase damping model.[21]

On the other hand, the entanglement of assistance (EOA)[22,23] as a special case of the localizable entanglement[24] has recently attracted intense attention,[25–31] which quantifies the entanglement that can be generated between two parties, Alice and Bob, given assistance from a third party, Charlie, when three of them share a tripartite state and where the assistance consists of Charlie initially performing a measurement on his share and communicating the result to Alice and Bob through a one-way classical channel. Inspired by the $\mathscr{P}\mathscr{T}$ symmetric operation with non-trace preserving feature, in the present paper we study the influence of a $\mathscr{P}\mathscr{T}$ symmetric operation on the EOA of a type of W-like state without and with decoherence. By investigation, we derive some interesting results. First we obtain the explicit expressions of the EOA of the $\mathscr{P}\mathscr{T}$ symmetric operation for both the cases without and with decoherence. Further, for the unconsidered decoherence case we find that the EOA shared by two parties can be obviously enhanced with the assistance of the third party by a local $\mathscr{P}\mathscr{T}$ symmetric operation. For the

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considered decoherence case we find that in the larger amplitude damping case the scheme of $\mathscr{P}\mathscr{T}$ symmetric operation performed on one of two parties with the influence of noise is superior to that of $\mathscr{P}\mathscr{T}$ symmetric operation performed on the third party without the influence of noise in suppressing amplitude damping noise. However, for the smaller amplitude damping case the opposite result is obtained. Hence local $\mathscr{P}\mathscr{T}$ symmetric operation may provide an alternative method to realize effective tripartite-to-bipartite entanglement localization.

The paper is organized as follows. In Section 2, we give a brief reviews of $\mathscr{P}\mathscr{T}$ symmetric operation and EOA. In Sections 3 and 4, we investigate the influences of $\mathscr{P}\mathscr{T}$ symmetric operation on the EOA without and with decoherence. Finally, we give the conclusion of our results in Section 5.

2. A brief reviews of $\mathscr{P}\mathscr{T}$ symmetric operation and EOA

To begin with, we would like to give a brief review of the so-called $\mathscr{P}\mathscr{T}$-symmetric operation adopted in the present paper. From the result of Ref. [32], we learn that for a qubit system a famous $\mathscr{P}\mathscr{T}$-symmetric Hamiltonian can be given by

$$H_{\mathscr{P}\mathscr{T}} = s\begin{pmatrix} \mathrm{i}\sin\theta & 1 \\ 1 & -\mathrm{i}\sin\theta \end{pmatrix}, \tag{1}$$

where the real number s is a scaling constant, and the real number θ is a parameter characterizing the non-Hermiticity of $H_{\mathscr{P}\mathscr{T}}$, i.e., the condition $|\theta| < \pi/2$ ensures the eigenvalues of $H_{\mathscr{P}\mathscr{T}}$ real as $\mathscr{P}\mathscr{T}$ symmetry. Obviously, the condition $\theta = 0$ means that $H_{\mathscr{P}\mathscr{T}}$ is Hermitian. When $\theta \to \pm\frac{\pi}{2}$, $H_{\mathscr{P}\mathscr{T}}$ is strongly non-Hermitian. Note that such a non-Hermitian Hamiltonian can be simulated in a nuclear magnetic resonance quantum system.[4]

Equivalently, the $\mathscr{P}\mathscr{T}$-symmetric Hamiltonian can also be rewritten as

$$\begin{aligned} H_{\mathscr{P}\mathscr{T}} &= s\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + s\begin{pmatrix} \mathrm{i}\sin\theta & 0 \\ 0 & -\mathrm{i}\sin\theta \end{pmatrix} \\ &= H_+ + H_-. \end{aligned} \tag{2}$$

The corresponding time evolution operator of Hamiltonian $H_{\mathscr{P}\mathscr{T}}$ can be obtained as follows:[12,13]

$$U(t) = \frac{1}{\cos\theta}\begin{pmatrix} \cos(t'-\theta) & -\mathrm{i}\sin t' \\ -\mathrm{i}\sin t' & \cos(t'+\theta) \end{pmatrix}, \tag{3}$$

where $t' = t\Delta E$, with $\Delta E \equiv (E_+ - E_-)/2 = s\cos\theta$. Here, $E_\pm = \pm s\cos\theta$ are the eigenvalues of the $H_{\mathscr{P}\mathscr{T}}$.

Generally speaking, the time evolution of a system with a non-Hermitian Hamiltonian is not trace preserving, namely

$$\frac{\partial}{\partial t}\mathrm{Tr}(\rho) = \frac{2}{\mathrm{i}\hbar}\mathrm{Tr}(\rho H_-) \neq 0. \tag{4}$$

To this end, the density operator of the system has to be renormalized by

$$\tilde{\rho}(t) = \frac{U(t)\rho' U^\dagger(t)}{\mathrm{Tr}[U(t)\rho' U^\dagger(t)]}. \tag{5}$$

It is worth noting that the non-trace preserving feature is a characterization of a system with non-Hermitian Hamiltonian, which may correspond to a nonlocal effect, rather than a genuine local operation as explained by Ref. [14]. Naturally, from the point of view of the non-trace preserving feature of non-Hermitian Hamiltonian, we may define the success probability of realizing various tasks via non-Hermitian operation as

$$P_t = \mathrm{Tr}\left[U(t)\rho' U^\dagger(t)\right]. \tag{6}$$

As for the entanglement of assistance, we take a W-like tripartite state as an example. In what follows, we assume a W-like tripartite state as

$$|W\rangle_{123} = \alpha|100\rangle_{123} + \beta|010\rangle_{123} + \gamma|001\rangle_{123} \tag{7}$$

shared by three parties referred to as Alice (subscript 1), Bob (subscript 2), and Charlie (subscript 3), where $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$. Then Charlie implements a measurement on his qubit to yield a known bipartite entangled state shared by Alice and Bob, which leads to the concept of the entanglement of assistance (EOA). For a pure $2\times2\times n$ state $|\psi\rangle_{123}$ a popular measure of EOA[33] is defined as

$$E^a(|\psi\rangle_{123}) \equiv E^a(\rho_{12}) \equiv \max\sum_i p_i C(|\varphi_i\rangle_{12}), \tag{8}$$

where $\rho_{12} = \mathrm{Tr}_3(|\psi\rangle_{123}\langle\psi|)$, and $C(|\varphi_i\rangle_{12})$ is the famous concurrence defined by Ref. [34]. The maximization is taken over all decompositions, $\rho_{12} = \sum_i p_i|\varphi_i\rangle_{12}\langle\varphi| = \mathrm{Tr}_3(|\psi\rangle_{123}\langle\psi|)$. Equation (8) can be adopted to quantity the maximal average entanglement between Alice and Bob with the assistance of Charlie by LOCC. Especially, for a pure $2\times2\times n$ state Thomas Laustsen *et al.*[35] have obtained an analytical expression for the EOA as

$$E^a(|\psi\rangle_{123}) = \mathrm{Tr}\sqrt{\sqrt{\rho_{12}}\tilde{\rho}_{12}\sqrt{\rho_{12}}} = \sum_i \lambda_i, \tag{9}$$

where $\tilde{\rho}_{12} = \sigma_y\otimes\sigma_y\rho_{12}^*\sigma_y\otimes\sigma_y$ with σ_y Pauli matrix, and λ_i are the square roots of the eigenvalues of $\rho_{12}\tilde{\rho}_{12}$. More importantly, it has been proved that equation (1) is an entanglement monotone for $2\times2\times n$-dimensional pure states. Clearly, for here considered state as Eq. (7) the EOA with the assistance of Charlie by local measurements can be obtained as $E^a(|\psi\rangle_{123}) = 2|\alpha||\beta|$. Moreover, if Charlie makes a projective measurement in the computational basis $\{|0\rangle, |1\rangle\}$ an general EPR pair between Alice and Bob can be obtained by the projective measurement with a certain success probability as $|\alpha|^2 + |\beta|^2$. It should be noted that authors in Ref. [36] found that weak measurements and their reversal can be useful for probabilistically increasing the EOA between Alice and Bob with the assistance of Charlie.

3. Generation of entanglement of assistance without decoherence via local $\mathscr{PT}$ symmetric operation

In this section, we discuss the influence of $\mathscr{PT}$ symmetric operation on the EOA without decoherence. To more conveniently compute the effect of $\mathscr{PT}$ symmetric operation as shown in Eq. (3), we first give some interesting relationships for $\mathscr{PT}$ symmetric operation on $|i\rangle\,(i=0,1)$ as

$$\begin{aligned} U(t)|0\rangle &= \frac{1}{\cos\theta}\left(\cos\left(t'-\theta\right)|0\rangle - \mathrm{i}\sin t'|1\rangle\right), \\ U(t)|1\rangle &= \frac{1}{\cos\theta}\left(-\mathrm{i}\sin t'|0\rangle + \cos\left(t'+\theta\right)|1\rangle\right), \end{aligned} \quad (10)$$

which is used to simplify some calculations. In the following, we give the concrete scheme to show the implement of $\mathscr{PT}$ symmetric operation.

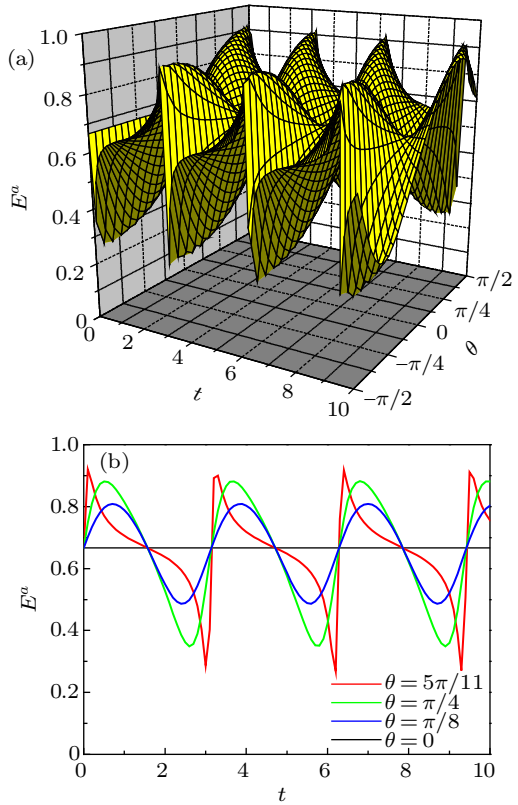


Fig. 1. (a) E^a as functions of time t and parameter θ, and (b) E^a as a function of the time t in the different θ for the W state, i.e., $\alpha=\beta=\gamma=1/\sqrt{3}$ in Eq. (7). In the figure, the time t of the $\mathscr{PT}$-symmetric evolution is in units of $1/\Delta E$.

Scheme Before or after qubits 1 and 2 are sent to Alice and Bob, Charlie carries out a local $\mathscr{PT}$ symmetric operation on his qubit by Eq. (10). The density operator of the total system consisting of Alice and Bob in the basis $\{|00\rangle_{12},|01\rangle_{12},|10\rangle_{12},|11\rangle_{12}\}$ can be expressed as

$$\tilde{\rho}(t)=\frac{1}{g}\begin{pmatrix} a & d & e & 0 \\ d^* & b & f & 0 \\ e^* & f^* & c & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (11)$$

where

$$\begin{aligned} a &= |\gamma|^2\left[\sin^2 t'+\cos^2\left(t'+\theta\right)\right], \\ d &= -\mathrm{i}\sin t'\gamma\beta^*\left[\cos\left(t'+\theta\right)-\cos\left(t'-\theta\right)\right], \\ e &= -\mathrm{i}\sin t'\gamma\alpha^*\left[\cos\left(t'+\theta\right)-\cos\left(t'-\theta\right)\right], \\ b &= |\beta|^2\left[\sin^2 t'+\cos^2\left(t'-\theta\right)\right], \\ c &= |\alpha|^2\left[\sin^2 t'+\cos^2\left(t'-\theta\right)\right], \\ f &= \beta\alpha^*\left[\sin^2 t'+\cos^2\left(t'-\theta\right)\right], \\ g &= a+b+c. \end{aligned}$$

According to Eq. (9), it is easy to obtain the EOA of the state Eq. (7) under $\mathscr{PT}$ symmetric operation as

$$E^a=\begin{cases} 2\sqrt{bc}/g, & \text{for} \quad \sqrt{bc}\geq|f|, \\ 2|f|/g, & \text{for} \quad \sqrt{bc}<|f|. \end{cases} \quad (12)$$

In what follows, using Eq. (12) we plot E^a as functions of time t and parameter θ for famous W state, i.e., $\alpha=\beta=\gamma=1/\sqrt{3}$ in Eq. (7) as shown in Fig. 1. First, E^a of the W state without local $\mathscr{PT}$ symmetric operation can be easily obtained as $E^a=2/3$. From Fig. 1(a) we can clearly see that if only $\theta\neq 0$ (because $\theta=0$ corresponds to a local Hermitian operation, which does not alter the EOA) the E^a of using local $\mathscr{PT}$ symmetric operation in the certain time intervals will exceed the original $E^a=2/3$ without local $\mathscr{PT}$ symmetric operation, which can be further revealed in Fig. 1(b) (see the blue line→green line→red line, i.e., the local $\mathscr{PT}$ symmetric operation case and the black line-local Hermitian symmetric operation case, but the EOA cannot be altered). By simple calculation, we also find that the maximal E^a can reach $E^a\approx 0.91995$, which is greater than $E^a=2/3$ without the local $\mathscr{PT}$ symmetric operation case. This may seem counterintuitive, since in the known entanglement theory one of important axioms is that local operation and classical communication can not increase quantum entanglement. However, this is not the case here. the non-trace preserving feature is a characterization of a system with non-Hermitian Hamiltonian, i.e., $\mathscr{PT}$ symmetric Hamiltonian as Eq. (4). The renormalization procedure as Eq. (5) induces a nonlocal effect, and the entanglement monotonicity is violated as explained by Ref. [14]. Further, it should be noted that the enhancement of the EOA comes at the expense of the success rate. Similar to the weak measurement method in suppressing amplitude-damping decoherence,[37] there is a trade-off between success probability as Eq. (6) and the amount of entanglement as Eq. (12). Moreover, from Figs. 1(a) and 1(b) we can find that E^a shows a periodic oscillation feature with time t for different parameters θ, which means that the EOA between two parties can be restored when we apply the $\mathscr{PT}$-symmetric operation on the third party. Especially, at the $\theta\to\pm\pi/2$ (corresponding to the strongly non-Hermitian case), i.e., $\theta=5\pi/11$ the E^a exhibits a strangely periodic evolution with time t, namely a change will suddenly happen (similar to quantum phase transition)

at some special time. Moreover, from Fig. 1(b) we can also find that the maximal E^a between two parties is also improved with the increase of $\mathscr{P}\mathscr{T}$-symmetric operation strength (i.e., $\theta = 0 \to \pi/8 \to \pi/4 \to 5\pi/11$). From the above analysis, we can learn that entanglement of assistance shared by two parties can be obviously enhanced with the assistance of the third party by local $\mathscr{P}\mathscr{T}$ symmetric operation in some proper time.

4. Protection of entanglement of assistance with decoherence via local $\mathscr{P}\mathscr{T}$ symmetric operation

In this section, we further discuss the influence of $\mathscr{P}\mathscr{T}$ symmetric operation on the EOA with decoherence. In the following, we provide two schemes to show the implementation of $\mathscr{P}\mathscr{T}$ symmetric operation.

Scheme I First, qubits 1 and 2 are sent to Alice and Bob. Then two qubits of Alice and Bob individually undergoe a zero-temperature amplitude damping channel, the density operator evolution is governed by

$$\rho = \sum_{i,j=0}^{1} E_{ij}\rho_0 E_{ij}^{\dagger}. \tag{13}$$

Obviously, the Kraus operator of total system E_{ij} can be expressed as a tensor product of the Kraus operator of each qubit system, i.e., $E_{ij} = E_i \otimes E_j$. For a single qubit system, there are

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & \sqrt{1-|d|^2} \\ 0 & 0 \end{pmatrix},$$

and d denotes the amplitude of the upper level of a qubit. At certain time, Charlie performs a local $\mathscr{P}\mathscr{T}$ symmetric operation on his qubit by Eq. (10). After two qubits of Alice and Bob going through two independent zero-temperature amplitude damping channels the combined state of Alice, Bob, and Charlie at time t_c can be given by

$$|W\rangle_{123}(t_c) = |W'\rangle_{123}\langle W'| + \mu\,|000\rangle_{123}\langle 000|, \tag{14}$$

where $|W'\rangle_{123} = \alpha d_1\,|100\rangle_{123} + \beta d_2\,|010\rangle_{123} + \gamma|001\rangle_{123}$, $\mu = |\alpha|^2\left(1-|d_1|^2\right) + |\beta|^2\left(1-|d_2|^2\right)$, d_1 and d_2 are the amplitudes of the upper two qubits for Alice and Bob, respectively. Then at time t_c, Charlie performs a local $\mathscr{P}\mathscr{T}$ symmetric operation and the corresponding state of Alice and Bob can represented as

$$\begin{aligned}\tilde{\rho}(t) &= \frac{\mathrm{Tr}_C\left[(I_A\otimes I_B\otimes U_C(t))|W\rangle_{1,2,3}(t_c)_{12,3}\langle W|(I_A\otimes I_B\otimes U_C(t))^{\dagger}\right]}{\mathrm{Tr}\left[(I_A\otimes I_B\otimes U_C(t))|W\rangle_{1,2,3}(t_c)_{12,3}\langle W|(I_A\otimes I_B\otimes U_C(t))^{\dagger}\right]} \\ &= \frac{1}{G}\begin{pmatrix} A & D & E & 0 \\ D^* & B & F & 0 \\ E^* & F^* & C & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\end{aligned} \tag{15}$$

where

$$\begin{aligned} A &= |\gamma|^2\left[\sin^2 t' + \cos^2\left(t'+\theta\right)\right] + \mu\left[\sin^2 t' + \cos^2\left(t'-\theta\right)\right], \\ D &= \mathrm{i}\sin t'\gamma\beta^* d_2^* \times \left[\cos\left(t'+\theta\right) - \cos\left(t'-\theta\right)\right], \\ E &= \mathrm{i}\sin t'\gamma\alpha^* d_1^* \left[\cos\left(t'+\theta\right) - \cos\left(t'-\theta\right)\right], \\ B &= |\beta|^2|d_2|^2 \times \left[\sin^2 t' + \cos^2\left(t'-\theta\right)\right], \\ C &= |\alpha|^2|d_1|^2\left[\sin^2 t' + \cos^2\left(t'-\theta\right)\right], \\ F &= \alpha^*\beta d_1^* d_2\left[\sin^2 t' + \cos^2\left(t'-\theta\right)\right], \\ G &= A+B+C = \left(|\alpha|^2+|\beta|^2\right)\left[\sin^2 t' + \cos^2\left(t'-\theta\right)\right] \\ &\quad + |\gamma|^2\left[\sin^2 t' + \cos^2\left(t'+\theta\right)\right]. \end{aligned}$$

Here G is the same as g of Eq. (11). In view of the same form of Eqs. (11) and (15), the corresponding EOA has a similar solution to

$$E^a = \begin{cases} 2\sqrt{BC}/g, & \text{for} \quad \sqrt{BC} \geq |F|, \\ 2|F|/g, & \text{for} \quad \sqrt{BC} < |F|. \end{cases} \tag{16}$$

Equivalently, equation (16) can be rewritten as

$$E^a = \begin{cases} \frac{2\sqrt{bc}}{g}|d_1||d_2|, & \text{for} \ \sqrt{bc} \geq |f|, \ |d_1|\cdot|d_2| \neq 0, \\ \frac{2|f|}{g}|d_1||d_2,| & \text{for} \ \sqrt{bc} < |f|, \ |d_1|\cdot|d_2| \neq 0. \end{cases} \tag{17}$$

Interestingly, equation (17) shows that after the sequence of two amplitude damping channels and a local $\mathscr{P}\mathscr{T}$ symmetric operation, E^a of the W-like state can be expressed as the products of the EOA under a local $\mathscr{P}\mathscr{T}$ symmetric operation, i.e., Eq. (12) and the channels' action on the maximally entangled state, i.e., $|d_1|$, $|d_2|$. Moreover, when we only consider two decoherence channels but no local $\mathscr{P}\mathscr{T}$ symmetric operation (i.e., $\theta = 0$ or $t' = 0$), E^a has a simple form as $E^a = 2|\alpha||\beta||d_1||d_2|$. Recall that for the case without considering amplitude damping channel there are $E^a = 2|\alpha||\beta|$ and Eq. (12). Therefore, in this case the improvement of a local $\mathscr{P}\mathscr{T}$ symmetric operation with two amplitude damping channels is same as that of having no amplitude damping channel. Here it is worth noting that Ref. [38] has derived the similar factorization law of entanglement evolution of two qubits under noisy environments as Eq. (17). Instead, in the present paper we consider the EOA involved in a three-qubit system and the implementation of local $\mathscr{P}\mathscr{T}$ symmetric operation.

Scheme II In scheme I, after qubits 1 and 2 of Alice and Bob undergoing two amplitude damping channels a local $\mathscr{P}\mathscr{T}$ symmetric operation is performed on the qubit of Charlie without the influence of noise. An interesting question naturally arises: What would happen if the local $\mathscr{P}\mathscr{T}$ symmetric operation is applied to the qubit of Alice or Bob with the influence of noise? For two qubits of Alice and Bob individually undergoing a zero-temperture amplitude damping channel as given by Eq. (14), at time t_c Alice makes a local $\mathscr{P}\mathscr{T}$ symmetric operation on her qubit. After the sequence of two amplitude damping channels and a local $\mathscr{P}\mathscr{T}$ symmetric operation, by some simple calculations the reduced density operator of two qubits

of Alice and Bob in the basis $\{|00\rangle_{12}, |01\rangle_{12}, |10\rangle_{12}, |11\rangle_{12}\}$ can be written as

$$\tilde{\rho}(t) = \frac{1}{G}\begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix}, \tag{18}$$

where

$$\begin{aligned}
\rho_{11} &= \sin^2 t\,|\alpha|^2\,|d_1|^2 + \cos^2(t'-\theta)\left(\mu + |\gamma|^2\right), \\
\rho_{12} &= -\mathrm{i}\sin t'\cos(t'-\theta)\,\alpha\beta^* d_1 d_2^*, \\
\rho_{13} &= -\mathrm{i}\sin t'\cos(t'+\theta)\,|\alpha|^2\,|d_1|^2 \\
&\quad +\mathrm{i}\sin t'\cos(t'-\theta)\left(\mu + |\gamma|^2\right), \\
\rho_{14} &= \sin^2 t'\alpha\beta^* d_1 d_2^*, \\
\rho_{22} &= \cos^2(t'-\theta)\,|\beta|^2\,|d_2|^2, \\
\rho_{23} &= \cos(t'-\theta)\cos(t'+\theta)\,\alpha^*\beta d_2 d_1^*, \\
\rho_{24} &= \mathrm{i}\sin t'\cos(t'-\theta)\,|\beta|^2\,|d_2|^2, \\
\rho_{33} &= \cos^2(t'+\theta)\,|\alpha|^2\,|d_1|^2 + \sin^2 t'\left(\mu + |\gamma|^2\right), \\
\rho_{34} &= \mathrm{i}\sin t'\cos(t'+\theta)\,\alpha\beta^* d_1 d_2^*, \\
\rho_{44} &= \sin^2 t'\,|\beta|^2\,|d_2|^2,
\end{aligned}$$

$$\begin{aligned}
G &= \rho_{11} + \rho_{22} + \rho_{33} + \rho_{44} \\
&= \cos^2(t'-\theta) + \sin^2 t' + |\alpha|^2\,|d_1|^2\left(\cos^2(t'+\theta)\right. \\
&\quad \left. -\cos^2(t'-\theta)\right), \\
\rho_{ij} &= \rho_{ji}^*, \\
\mu &= |\alpha|^2\left(1-|d_1|^2\right) + |\beta|^2\left(1-|d_2|^2\right).
\end{aligned}$$

Using Eq. (9), it can be found that the corresponding EOA of two qubits of Alice and Bob can be expressed as

$$E^a = \frac{2\left|\sin^2 t' + \cos(t'-\theta)\cos(t'+\theta)\right|\,|\alpha|\,|\beta|\,|d_1|\,|d_2|}{\cos^2(t'-\theta) + \sin^2 t' + |\alpha|^2|d_1|^2(\cos^2(t'+\theta) - \cos^2(t'-\theta))}. \tag{19}$$

As is expected, if we choose $\theta = 0$ and $t' = 0$ in the local $\mathscr{P}\mathscr{T}$ symmetric operation as Eq. (3), E^a will reduce to $2|\alpha|\,|\beta|\,|d_1|\,|d_2|$, which is just the cases of Hermitian operation and identity matrix, respectively. Now having Eqs. (17) and (19) in mind, we can discuss the local $\mathscr{P}\mathscr{T}$ symmetric operation on the improvement of EOA for both schemes I and Ⅱ.

To show the effects of schemes I and Ⅱ in protecting the EOA against decoherence, in Fig. 2 we plot the comparisons of their E^a as a function of the time t for different d_1 and d_2.

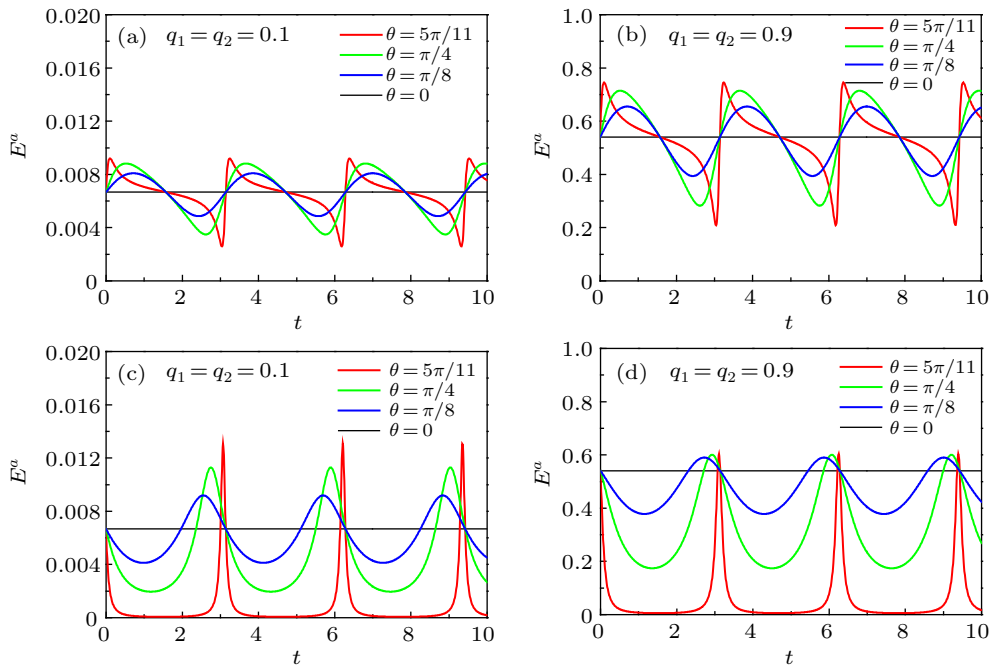


Fig. 2. Comparisons of the E^a in the scheme I (see (a) and (b)) and scheme Ⅱ (see (c) and (d)) as a function of the time t for different d_1 and d_2. In the figure, time t of the $\mathscr{P}\mathscr{T}$-symmetric evolution is in units of $1/\Delta E$.

First figure 2 shows that using the local $\mathscr{P}\mathscr{T}$ symmetric operation (see red, green and blue lines) in certain time can effectively suppress the amplitude damping noise (see the black line-Hermitian operation case) for the non-Hermitian operation applied to both qubit of Charlie without the influence of noise and qubit of Alice with the influence of noise, which can reach the maximum value by choosing a proper time. Further, from Figs. 2(a) and 2(c) we can find that for the larger amplitude damping case (i.e., $d_1 = d_2 = 0.1$) E^a of scheme Ⅱ (see Fig. 2(c)) is better than that of scheme I (see Fig. 2(a)). The reason is that for the larger amplitude damping case, the schemes I and Ⅱ correspond to the local $\mathscr{P}\mathscr{T}$ symmetric operation performed on qubit of Charlie without the influence of noise and qubit of Alice with the influence of noise, re-

spectively. However, for smaller amplitude damping case (i.e., $d_1 = d_2 = 0.9$) the opposite is true, which means that the effect of scheme I (see Fig. 2(b)) is obviously superior to that of scheme Ⅱ (see Fig. 2(d)), which is an interesting phenomenon. Therefore, the different effects of schemes I and Ⅱ on the protection of EOA of two parties may be distinguished by different amplitude damping cases. In addition, it can be seen from Fig. 2 that E^a shows a periodic evolution with period π with the change of time t. To explicitly show the merits of a $\mathscr{P}\mathscr{T}$ symmetric operation including pre-$\mathscr{P}\mathscr{T}$ and post-$\mathscr{P}\mathscr{T}$ symmetric operations on suppression of the amplitude damping noise, we also numerically calculate the maximal E^a with and without the $\mathscr{P}\mathscr{T}$ symmetric operation under concrete parameter conditions (i.e., $\theta = 0 \to \pi/8 \to \pi/4 \to 5\pi/11$), which are listed in Table 1. Before ending the section, it should be noted that although we have focused on the famous W state case, actually for other W-like states with unbalanced coefficients, i.e., $\alpha = \beta = 1/2$, $\gamma = 1/\sqrt{2}$ in Eq. (7), similar effects of the local $\mathscr{P}\mathscr{T}$ symmetric operation can also be obtained, which is not given in the text to avoid repetition.

Table 1. The maximal E^a with and without the $\mathscr{P}\mathscr{T}$ symmetric operation in the specific time for different d_1 and d_2.

Amplitudes d_1, d_2	Without $\mathscr{P}\mathscr{T}$	Scheme I	Scheme Ⅱ
$d_1 = d_2 = 0.1$	0.00067	0.00920	0.01309
$d_1 = d_2 = 0.9$	0.54000	0.74514	0.60381

5. Conclusions

Before ending this paper, it is necessary to discuss the experimental realizations of the preparation of W state and the $\mathscr{P}\mathscr{T}$ symmetric operation in current experimental techniques. First a three-photon polarization-entangled W state can be created by a second-order emission process of type-Ⅱ spontaneous parametric down-conversion (SPDC).[39] The $\mathscr{P}\mathscr{T}$ symmetric operation as shown in Eq. (1) can be implemented in a nuclear magnetic resonance quantum system[4] using the idea of duality quantum computing, in which an ancillary qubit is used as a conventional quantum computer. The concrete procedure of implementation for such a Hamiltonian is as follows:[4] The system we use contains two qubits: an ancillary qubit a and a work qubit e, which are initially in state $|0\rangle_a|0\rangle_e$. First we perform an unitary operation V on the ancillary qubit a. Then we apply two controlled unitary operations. Finally, a Hadamard operation is used on the work qubit e.

In summary, we have studied the effects of the $\mathscr{P}\mathscr{T}$ symmetric operation on the EOA shared by two parties with the assistance of the third party for a general W-like state without and with decoherence. Through an analytical approach, we obtain the explicit expressions of the EOA. For a famous W state without decoherence, we find that the EOA shared by two parties can be obviously enhanced with the assistance of the third party by a local $\mathscr{P}\mathscr{T}$ symmetric operation. Further, for the decoherence case, effective protection of the EOA against decoherence via local $\mathscr{P}\mathscr{T}$ symmetric operation can be also obtained. Especially, we find that for the larger amplitude damping case the scheme of the $\mathscr{P}\mathscr{T}$ symmetric operation performed on one of two parties with the influence of noise is superior to that of the $\mathscr{P}\mathscr{T}$ symmetric operation performed on the third party without the influence of noise in suppressing amplitude damping noise. However, for the smaller amplitude damping case the opposite result is obtained. Therefore, using $\mathscr{P}\mathscr{T}$ symmetric operation method to protect entanglement of assistance against amplitude damping noise could be distinguished by different amplitude damping cases. Moreover, although we have focused on the famous W state case we believe that the similar effect of $\mathscr{P}\mathscr{T}$ symmetric operation also exists in other W-like states with unbalanced coefficients, i.e., $\alpha = \beta = 1/2$, $\gamma = 1/\sqrt{2}$. The result that a local $\mathscr{P}\mathscr{T}$ symmetric operation of the third party can lead to the enhancement of entanglement of other two parties, which may have potential applications in the quantum information processing. Finally, it is worth noting that in Refs. [40] and [41] authors studied the enhancement of intensity-difference squeezing via energy-level modulations in atomic and atomic-like ensemble. Although the effect of the $\mathscr{P}\mathscr{T}$ symmetric operation on quantum state as Eq. (10) has some similarities to the energy-level modulations in the Refs. [40] and [41], the methods for energy-level modulations and subjects for study are completely different.

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