ISSN 1674-1056
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Citation: Chin. Phys. B . 2019, 28(12): 128701 . **doi:** 10.1088/1674-1056/ab4f60
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Effects of refractory period on dynamical range in excitable networks*

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(Received 4 July 2019; revised manuscript received 14 October 2019; published online 15 November 2019)

Effects of refractory period on the dynamical range in excitable networks are studied by computer simulations and theoretical analysis. The first effect is that the maximum or peak of the dynamical range appears when the largest eigenvalue of adjacent matrix is larger than one. We present a modification of the theory of the critical point by considering the correlation between excited nodes and their neighbors, which is brought by the refractory period. Our analysis provides the interpretation for the shift of the peak of the dynamical range. The effect is negligible when the average degree of the network is large. The second effect is that the dynamical range increases as the length of refractory period increases, and it is independent of the average degree. We present the mechanism of the second effect. As the refractory period increases, the saturated response decreases. This makes the bottom boundary of the dynamical range smaller and the dynamical range extend.

Keywords: criticality, branching process, random network, dynamical range

PACS: 87.18.Sn, 05.45.–a, 87.19.lj **DOI:** 10.1088/1674-1056/ab4f60

1. Introduction

Critical avalanche is the characteristic feature of the self-organized criticality and is observed in a variety of natural phenomena.[1–3] The principle of neural network functions has been widely studied using dynamic models,[4–8] and critical avalanches have been found in neural networks.[9] The studied systems in the critical states exhibit several advantages.[9–12] One advantage is that the dynamical range is maximal in the critical state.[11,13,14] The critical states have optimal information processing by responding stimulus spanning several orders of magnitude.[11] The property was proposed in modeling study on networks of coupled excitable elements,[11] which are used widely as the simplified model of neural networks.[10,11,15–19] The property that neural network exhibits critical avalanche and largest dynamical range has been observed in experiments.[13,14] In research of excitable networks, the theory of predicating the critical state and the dynamical range has been proposed. It has been shown that criticality and the maximal dynamical range occur in networks in which the largest eigenvalue of adjacency matrix is one.[17]

The properties of networked systems depend on both the network structure and the dynamics of network elements. The biological realistic dynamic properties of network elements may play an important role in the criticality of the network. The way of enhancing the dynamical range was studied by including biological realistic features in the model.[20,21] For example, the inhibitory element was considered in the study of enhancing the dynamical range in excitable networks.[20]

The refractory period of neurons is an essential dynamical feature in excitable networks. In the refractory period, neurons do not response to signals. The dynamical features have important role in the collective behavior of the network. It was shown that refractory period leads to a wide range of scaling as well as periodic behavior.[22] The refractory period was considered in the theory predicting the criticality.[18] However, the effect of the refractory period on the dynamical range has not been studied.

In the present work, we study how the dynamical range changes with the refractory period. We show that the largest eigenvalue of the network corresponding to the critical point and the maximal dynamic range depends on the averaged degree of the network. In the critical state, the maximal dynamical range is enhanced by the refractory period.

2. Model

The critical branching model[10,11] is used. The network consists of N nodes and each node has $r+2$ states: $s_i = 0$ is the resting state, $s_i = 1$ is the excited state, and $s_i = 2, 3, \ldots, r+1$ are refractory states. Time is discrete $t = 0, 1, \ldots$ The node state is updated at each step. After excitation, the evolution of node state is deterministic. The state s_i is increased by 1 until $s_i = r+1$, then the state changes to $s_i = 0$. The length of refractory state is r steps. If node i is at the resting state $s_i^t = 0$, it can be excited by its neighbor j being in excited state, with

*Project supported by the National Natural Science Foundation of China (Grant No. 11675096), the Fundamental Research Funds for the Central Universities of China (Grant No. GK201702001), and the Fund for the Academic Leaders and Academic Backbones, Shaanxi Normal University of China (Grant No. 16QNGG007).

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probability A_{ij}, or independently by an external stimulus with probability φ, where $\varphi = 1 - \exp(-\eta\Delta t)$, η is the stimulus intensity ($\Delta t = 1$).[17]

We create networks in three steps: First, we create an ER random network with $NK/2$ links. The links are assigned to randomly chosen pairs of nodes. The average connection degree of each node in the network is K. Second, we assign a weight to each link. If node i and node j are connected, the entries of adjacent matrix A_{ij} and A_{ji} will be random numbers drawn from a uniform distribution between 0 and 1. Third, we calculate the largest eigenvalue λ of adjacency matrix, and multiply the adjacency matrix $\boldsymbol{A}$ by a constant to rescale λ to the targeted eigenvalue.[17]

In addition, we calculate the largest eigenvalue of adjacency matrix by

$$\lambda = \frac{\langle d_i^{\text{in}} d_j^{\text{out}} \rangle_{\text{e}}}{\langle d_i^{\text{in}} \rangle_{\text{e}} \langle d_j^{\text{out}} \rangle_{\text{e}}} \frac{\langle d^{\text{in}} d^{\text{out}} \rangle}{\langle d \rangle}, \tag{1}$$

where $d_i^{\text{in}} = \sum_j A_{ij}$ and $d_i^{\text{out}} = \sum_j A_{ji}$ are the input and output weights of node i, $\langle d \rangle = \frac{1}{N}\sum_{ij} A_{ij} = \langle d^{\text{in}} \rangle = \langle d^{\text{out}} \rangle$, which is equal to the average branching ratio in random networks. Here $\langle \cdot \rangle$ is an average over nodes, and $\langle \cdot \rangle_{\text{e}}$ is an average over edges whose two ends are indicated by the indices i and j. The largest eigenvalue governs the stability of fixed point of the network. In the following, we use λ to denote the largest eigenvalue of the adjacency matrix.

The response F is defined as[11]

$$F = \frac{1}{T}\sum_{t=1}^{T} f^t, \tag{2}$$

where T is a large time window ($T = 1000$ in this paper) and f^t is the fraction of excited nodes at time t in the network. The response has three types:[11,17,18] (1) quiescent, in which the network activity is zero for vanishing stimulus, $\lim_{\eta\to 0} F = 0$; (2) active, in which there is self-sustained activity for vanishing stimulus, $\lim_{\eta\to 0} F > 0$; and (3) critical, in which the response is still zero for vanishing stimulus but have a larger variance in the distribution of extinction times of active compared with the quiescent case.[17]

The dynamical range was proposed to measure the ability of a network to robustly code the input stimuli.[11] As a function of the stimulus intensity η, networks have a minimum response F_0 and a maximum response $F_{\max}$. Discarding stimuli that are too weak to be distinguished from F_0 or close to saturation, the dynamical range is defined as

$$\Delta = 10\log_{10}\left(\frac{\eta_{0.9}}{\eta_{0.1}}\right). \tag{3}$$

The range $[\eta_{0.1}, \eta_{0.9}]$ is found from its corresponding response interval $[F_{0.1}, F_{0.9}]$, where $F_\gamma = F_0 + \gamma(F_{\max} - F_0)$. The choice of interval is arbitrary and does not affect the results.[11] Here we use a simplified form of the dynamical range[17]

$$\Delta = 10\log_{10}\left(\frac{1}{\eta_{0.1}}\right). \tag{4}$$

In the formula, the upper boundary of the range is selected as $\eta = 1$.[11,20]

3. Results

We simulate the response of the network to external stimulus and compute the dynamical range of networks with or without the refractory period. Figure 1 shows that the dynamical range changes with the increase of refractory states. It is notable that the value of the maximal dynamical range increases with the number of refractory states. The refractory period promotes the advantage of the neural network in responding to stimulus.

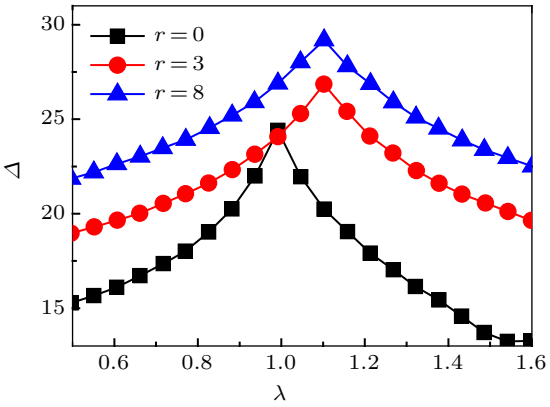


Fig. 1. The dynamical range Δ versus the largest eigenvalue λ of adjacency matrix. Network parameters are $N = 10^5$ and $K = 10$.

It has been shown that criticality exhibits the optimal dynamical range.[11] Here the maximal dynamical range occurs at $\lambda = 1.0$ for networks without refractory period. It is the same as the predicated critical point that the largest eigenvalue of the network adjacent matrix equals one ($\lambda = 1$). However, the maximal dynamical range occurs at $\lambda > 1.0$ for the network with refractory period.

To understand the shift of the maximal dynamical range in the relation between the dynamical range and the largest eigenvalue, we study the the transition of the response of the network. In the absence of external stimulus, sub-critical networks and critical networks do not exhibit self-sustained activity, whereas super-critical networks do. The critical point can be obtained from the transition of network response from zero to nonzero with $\eta \to 0$.[11] In our simulation, the stimulus strength $\eta = 10^{-5}$ is used to study the transition of response. Figure 2 shows that the transition will occur at the largest eigenvalue $\lambda = 1.0$ if there is no refractory state, as the same as the results in the previous works.[17,18] As shown in Fig. 2, the transition of excitable network shifts to $\lambda > 1.0$

when the length of refractory period is $r = 3$ or 8. The shift of the peak of dynamical range is identical to the critical point as the refractory states increase.

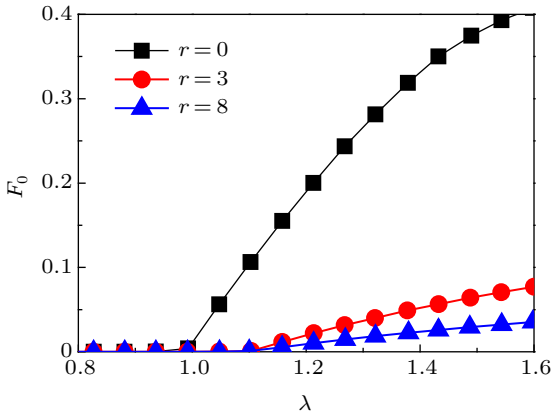


Fig. 2. Response F_0 versus the largest eigenvalue of adjacency matrix λ of random networks. The length of refractory states are $r = 0$, 3, and 8. Network parameters are $N = 10^5$ and $K = 10$.

We next analyze the relation between the critical point and the refractory period. In Ref. [17] the network without refractory period was considered, and it was assumed that no correlation exists between nodes in the excited states and nodes in the resting states. In networks with refractory period, the neighborhood of the excited nodes becomes different from the whole network. We first show this difference by numerical simulations.

The probability that the ith node is in the resting state ($s_i = 0$) at time t is denoted by z_i^t. We use $z_i'^t$ to denote the probability that one node randomly selected from the neighborhood of an excited nodes is in the resting state. Figure 3 shows the simulation results of these two kinds of probability. One can see that the resting state nodes are less in the neighborhood of excited nodes, i.e., $z_i'^t < z_i^t$. We compute the ratio $\alpha = \langle z_i'^t / z_i^t \rangle_t$, where $\langle\cdot\rangle_t$ represents the average over steps at which there are nodes in excited states. In networks without refractory period, as shown in Figs. 3(a) and 3(c), the value is $\alpha \approx 0.999$. The ratios are the same for networks with the average degrees $K = 10$ and 200. In networks with the refractory period $r = 8$, the ratio is distinctly low. The values of ratio are $\alpha = 0.923$ for $K = 10$ and $\alpha = 0.992$ for $K = 200$.

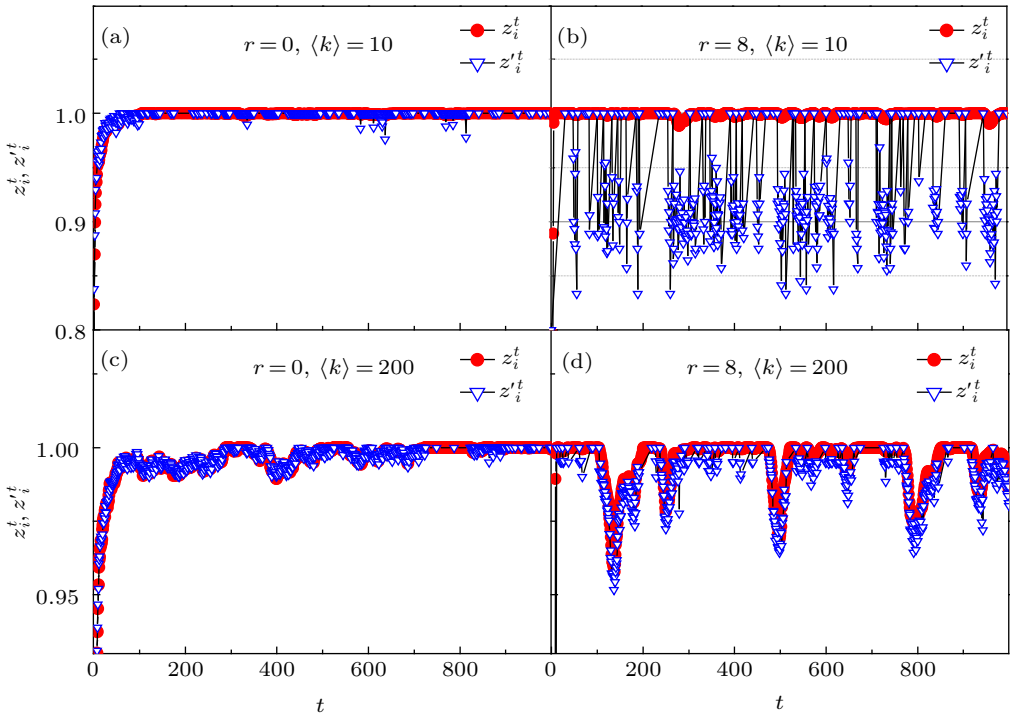


Fig. 3. Circles are the probability that nodes are in resting state in the whole networks. Triangles are the probability that nodes are in resting state among the neighbors of excited nodes. Network parameters are $N = 10^5$, $\eta = 10^{-5}$, and $\lambda = 1$. (a) and (b) Average degree is $K = 10$, refractory periods are $r = 0$ and 8, respectively. (c) and (d) Average degree is $K = 200$, refractory periods are $r = 0$ and 8, respectively.

In networks without refractory period $r = 0$, the probability z satisfies $z = 1 - F_0$, where F_0 represents the fraction of nodes in the excited state under $\eta \to 0$. When the external stimulus is absent, in the network in critical states, the F_0 can be ignored. For deriving z', we assume that the node i is excited. Besides the node i, the number of excited nodes is $NF_0 - 1$. Except for the node i, nodes are in the excited state with the probability $(NF_0 - 1)/(N - 1)$. Assuming that all excited nodes are independent, the probability that a neighbor of node i is in resting state is

$$z' = 1 - (NF_0 - 1)\frac{1}{N-1}. \tag{5}$$

When the network size is infinite, the value of z' tends to $z' = 1 - F_0$, which is the same as z.

In the network with refractory period ($r > 0$), nodes in

excited states are not independent of their neighbor's states. If node i is excited by the signals from nodes in the networks, the predecessor node must be in the refractory state ($s = 2$). Therefore, in the neighborhood of the node i, a node is in the state $s = 2$ with the probability $1/K$. One can randomly select a node from $N-2$ nodes except the node i, and its predecessor is in the excited state with the probability $(NF_0-1)/(N-2)$ or one of r refractory states with the probability $(NF_0r-1)/(N-2)$. In the neighborhood of node i, a node is in the resting state with the probability

$$z' = 1 - \frac{1}{K} - (NF_0(r+1)-2)\frac{1}{N-2}. \tag{6}$$

When N is of infinity, the probability tends to

$$z' = 1 - \frac{1}{K} - F_0(r+1). \tag{7}$$

As the F_0 cannot be ignored, the value of z' depends on both r and K. As the F_0 can be ignored, the value of z' depends on the network's average degree K.

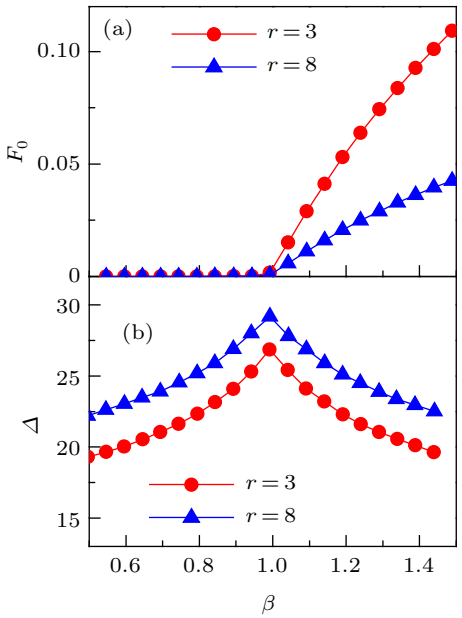


Fig. 4. Response F_0 versus β. The network size is $N = 10^5$. The average degree is $K = 10$. The length of refractory period is $r = 3$ and 8, respectively.

The shift of the critical point with the refractory period can be understood through the approach proposed in Refs. [11] and [17]. The probability that the ith node at t is in the excited state is denoted by P_i^t. It was obtained in Ref. [17] that $P_i^t = \sum_j^N P_j^t A_{ij}$, and the critical point is $\lambda = 1$, where λ is the largest eigenvalue of the adjacent matrix $\{A_{ij}\}$. Here we consider that the neighborhood of excited node is different from the network, and use the following dynamical equation

$$P_i^t = z' \sum_j^N P_j^t A_{ij}, \tag{8}$$

where $\sum_j^N P_j^t A_{ij}$ is the excitations created by the neighborhood of node i. Assuming $P_i^t = \beta^t u_i$ produces

$$\beta u_i = z' \sum_j^N u_j A_{ij}. \tag{9}$$

The stability of the solution $P = 0$ is governed by β, with $\beta < 1$ being stable and $\beta > 1$ being unstable. According to Eq. (9), β is the largest eigenvalue of $z'A$.

Using the largest eigenvalue of the adjacency matrix $\boldsymbol{A}$, the critical condition is

$$\lambda = 1/z'. \tag{10}$$

Therefore, the critical point can be expressed as

$$\lambda = \begin{cases} \frac{1}{1-F_0}, & r = 0, \\ \frac{1}{1-1/K-(r+1)F_0}, & r > 0. \end{cases} \tag{11}$$

By ignoring F_0, one can see that the critical point shifts from $\lambda = 1$ to $\lambda = \frac{1}{1-1/K}$. The analytic result presents the interpretation for the shift of the critical point.

To show that the analytic result is reasonable, we plot the simulation result of the relation between F_0 and β in Fig. 4(a). In computation of β, the F_0 is ignored, that is, $\beta = \lambda(1-1/K)$. One can see that the transition occurs at the same point $\beta = 1$ for networks with $r = 3$ and 8. The simulation result is identical with the analysis. As the same as the transition of response, the dynamical range reaches the maximal value at $\beta = 1$ for $r = 3$ and 8, as shown in Fig. 4(b).

Furthermore, the analysis shows that when the average degree of the network is large, the shift of critical point can be ignored. We show the simulation results of the relation between F_0 and λ for the network with $K = 200$ in Fig. 5. The simulation results agree with the critical condition $\lambda = 1$ for $r = 0$, 3, and 8.

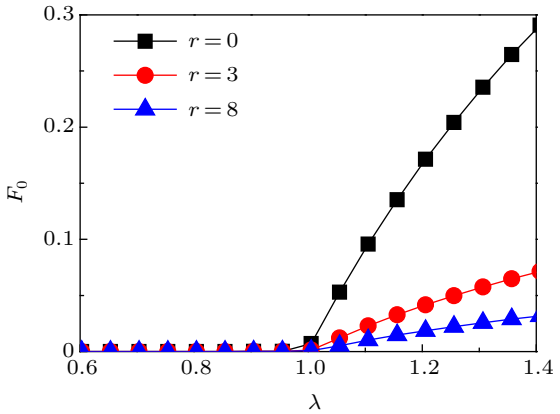


Fig. 5. Response F_0 versus the largest eigenvalue of the adjacency matrix. The network size is $N = 10^5$. The average degree is $K = 200$. The length of refractory period is $r = 0$, 3, and 8, respectively.

The effect of the refractory period on the dynamical range is independent of the average degree. We show the relation between the dynamical range and λ for networks with $K = 200$ in Fig. 6. In the network, the maximal dynamical range increases with the refractory period.

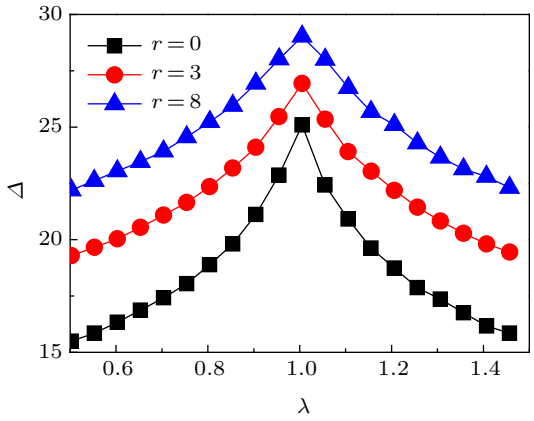


Fig. 6. The dynamical range Δ versus the largest eigenvalue λ of adjacency matrix. Network parameters are $N = 10^5$ and $K = 200$. The length of refractory period is $r = 0$, 3, and 8, respectively.

We show the response F versus the input signal intensity η in Fig. 7. As the refractory period increases, the maximal response $F_{\max}$ decreases. The value of maximal response is $F_{\max} = 1/(r+2)$. As the decreases of $F_{\max}$, the value of $F_{0.1}$ decreases. We illustrate the value of $F_{0.1}$ by the horizontal dashed lines in Fig. 3. Therefore the stimulus strength $\eta_{0.1}$, corresponding to the vertical lines in Fig. 7, decreases with the refractory period r. According to Eq. (4), the change of the value of $\eta_{0.1}$ induces the increases of the dynamical range.

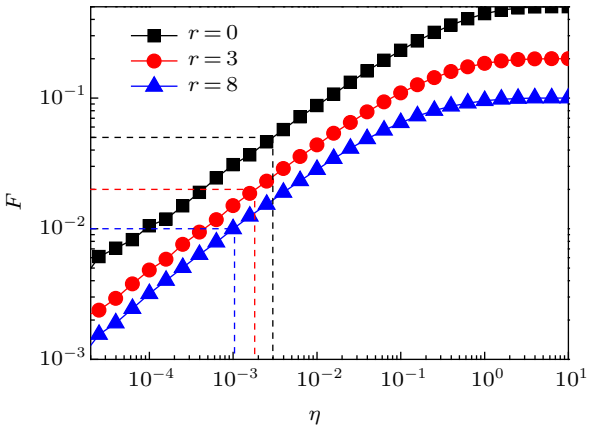


Fig. 7. Response F versus η in networks with $\lambda = 1$. The network size is $N = 10^5$. The average degree is $K = 200$. The length of refractory is $r = 0$, 3, and 8, respectively.

4. Conclusion

We have studied the dynamical range of networks consisting of excitable elements with the refractory period, and obtained that in networks with small average degree, the maximal dynamical range appears when the value of the largest eigenvalue of adjacent matrix is larger than 1, and the dynamical range is enhanced by the refractory period.

We present the mechanism of the effect of refractory period. The refractory period brings the correlation between the excited nodes and its neighbors. As a result, the fraction of resting state node in the neighborhood of excited nodes is less than that in the whole network. Considering the correlation, we obtain that the critical point of the network with refractory period appears when the largest eigenvalue of adjacency matrix is larger than one. The critical point is also the condition that the dynamical range reaches the maximal value. In networks with refractory period, the critical value of the largest eigenvalue decreases monotonically with the average degree. As the average degree is large, the critical condition becomes such that the largest eigenvalue of adjacency matrix is one, which was proposed in Ref. [17].

The effect that refractory period enhances the dynamical range is independent of the average degree. We present the mechanism of the effect. The saturated response decreases as the refractory period becomes longer. As a result, the low boundary of stimulus intensity becomes smaller. The decrease of low boundary induces the increase of the dynamical range.

We obtain the effects of the refractory period on enhancing the dynamical range and the shift of the critical point. The results provide a new way of enhancing dynamical range. The mechanism of these effects is helpful for understanding the critical behaviors of neural networks.

References

[1] Bak P, Tang C and Wiesenfeld K 1987 *Phys. Rev. Lett.* **59** 381
[2] Li B Q and Wang S J 2018 *Commun. Theor. Phys.* **69** 280
[3] Sun B A and Wang L F 2017 *Acta Phys. Sin.* **66** 178103 (in Chinese)
[4] Zheng Q, Wang Z and Shen J 2017 *Chin. Phys. B* **26** 020501
[5] Qu L H, Du L and Deng Z C 2018 *Chin. Phys. B* **27** 118707
[6] Gu C G, Wang P and Yang H J 2019 *Chin. Phys. B* **28** 018701
[7] Li G F and Sun X J 2017 *Acta Phys. Sin.* **66** 240501 (in Chinese)
[8] Xue X D, Wang M L 2019 *Acta Phys. Sin.* **68** 078701 (in Chinese)
[9] Beggs J M, Plenz D 2003 *J. Neurosci.* **23** 11167
[10] Haldeman C and Beggs J M 2005 *Phys. Rev. Lett.* **94** 058101
[11] Kinouchi O and Copelli M 2006 *Nat. Phys.* **2** 348
[12] Wang S J, Ouyang G, Guang J, Zhang M, Wong K M and Zhou C 2016 *Phys. Rev. Lett.* **116** 018101
[13] Shew W L, Yang H, Petermann T, Roy R and Plenz D 2009 *J. Neurosci.* **29** 15595
[14] Gautam S H, Hoang T T, McClanahan K, Grady S K and Shew W L 2015 *PLoS Comput. Biol.* **11** e1004576
[15] Wu A C, Xu X J and Wang Y H 2007 *Phys. Rev. E* **75** 032901
[16] Wang C Y, Wu Z X and Chen M Z Q 2017 *Phys. Rev. E* **95** 012310
[17] Larremore D B, Shew W L and Restrepo J G 2011 *Phys. Rev. Lett.* **106** 058101
[18] Larremore D B, Shew W L, Ott E and Restrepo J G 2011 *Chaos* **21** 025117
[19] Wang F and Wang S J 2019 *Commun. Theor. Phys.* **71** 746
[20] Pei S, Tang S, Yan S, Jiang S, Zhang X and Zheng Z 2012 *Phys. Rev. E* **86** 021909
[21] Gollo L L, Mirasso C and Eguíluz V M 2012 *Phys. Rev. E* **85** 040902
[22] Moosavi S A, Montakhab A and Valizadeh A 2017 *Sci. Rep.* **7** 7107