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Quantum search for unknown number of target items by hybridizing fixed-point method with trail-and-error method*

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For the unsorted database quantum search with the unknown fraction λ of target items, there are mainly two kinds of methods, *i.e.*, fixed-point and trail-and-error. (i) In terms of the fixed-point method, Yoder *et al*. [*Phys. Rev. Lett.* **113** 210501 (2014)] claimed that the quadratic speedup over classical algorithms has been achieved. However, in this paper, we point out that this is not the case, because the query complexity of Yoder's algorithm is actually in $O(1/\sqrt{\lambda_0})$ rather than $O(1/\sqrt{\lambda})$, where λ_0 is a known lower bound of λ. (ii) In terms of the trail-and-error method, currently the algorithm without randomness has to take more than 1 times queries or iterations than the algorithm with randomly selected parameters. For the above problems, we provide the first hybrid quantum search algorithm based on the fixed-point and trail-and-error methods, where the matched multiphase Grover operations are trialed multiple times and the number of iterations increases exponentially along with the number of trials. The upper bound of expected queries as well as the optimal parameters are derived. Compared with Yoder's algorithm, the query complexity of our algorithm indeed achieves the optimal scaling in λ for quantum search, which reconfirms the practicality of the fixed-point method. In addition, our algorithm also does not contain randomness, and compared with the existing deterministic algorithm, the query complexity can be reduced by about 1/3. Our work provides a new idea for the research on fixed-point and trial-and-error quantum search.

Keywords: quantum search, fixed-point, trail-and-error, unknown number of target items

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1. Introduction

For the unsorted database search, the famous Grover algorithm[1,2] achieves a quadratic speedup over classical algorithms. Afterwards many analyses, generalizations and variants have been investigated.[3–25] Grover's algorithm has been proven optimal.[26–30] However, the Grover algorithm cannot apply to the case where the fraction λ of target items is completely unknown expect for a lower bound, because iterating too much will pass by the target states, that is the so-called soufflé problem.[31]

For the case of unknown λ, there are mainly two quantum search methods. One is the fixed-point method.[32,33] In Ref. [32], the final state gradually converges to the target states along with the iterations, avoiding "overcooking" the state. However, the algorithm loses the valuable quantum speedup of quantum search,[33] which has also been proven asymptotically optimal.[34,35]

Fortunately, Yoder *et al*.[33] creatively reduces the "fixed point" to a bounded region of the target states, and claimed that the quadratic speedup over classical search is maintained consequently, which has gained many recognitions.[36–38] For example, in Ref. [38], Dalzell *et al*. pointed out that: "it (Yoder's algorithm) produces $|E\rangle$ (the uniform superposition over all the target states) with fidelity at least $\sqrt{1-\delta^2}$ in only $O(\ln(1/\delta)/\sqrt{\omega})$ time, thus exhibiting the quadratic speedup", where ω is a known lower bound of the unknown λ. Note that, for consistency, in the following, λ_0 instead of ω will be used to denote the lower bound of λ.

However, as stated in Ref. [38], the complexity in the quantum query model[39,40] of Yoder's algorithm is in $O(1/\sqrt{\lambda_0})$ rather than $O(1/\sqrt{\lambda})$, which is dependent on the lower bound λ_0 rather than λ. For example, if $\lambda_0 = 1/N$, while $\lambda = 1/\sqrt{N}$, then $O(1/\sqrt{\lambda_0}) = O(\sqrt{N})$, while $O(1/\sqrt{\lambda}) = O(\sqrt[4]{N})$. As another example, if $\lambda_0 = 1/N$, while $\lambda = (N-1)/N$, then $O(1/\sqrt{\lambda_0}) = O(\sqrt{N})$, while $O(1/\sqrt{\lambda}) = O(1)$. Therefore, we confirm that the order of query complexity of Yoder's algorithm is in fact not really optimal (for a detailed complexity analysis of Yoder's algorithm see Section 2).

Another method for the case of unknown λ is the trial-and-error method.[27,41–43] In this regard, the original work was proposed by Boyer *et al*.,[27] which trials Grover's algorithm multiple times, where the number of iterations is

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randomly selected from an exponentially increasing interval along with the number of trials. This randomness enables the average success probability at each trial is *always* at least 1/4 after the algorithm reaches the critical stage, and a target item can thus be found with an expected number of Grover iterations no more than $4/\sqrt{\lambda}$, but the algorithm is therefore referred to as a randomized application of Grover's algorithm. Based on this randomized trial-and-error method, replacing the internal Grover's algorithm[1] by the partial diffusion algorithm[44] or fixed-phase algorithm,[42] Younes *et al.* proposed two different variants,[41,42] but the number of iterations was not reduced (for details see Table 1 in Section 4).

In order to remove the randomness of trial-and-error algorithms, Okamoto *et al.*[43] directly set the number of Grover iterations exponentially increasing along with the trials and thus obtained a simpler deterministic trial-and-error algorithm, where the success probability at each trial can *often* (but not always) be no less than 3/4. This allows the algorithm to find a target state also in the optimal scaling of quantum search. But this also results in more than 1 times iterations or queries than Boyer's randomized algorithm.

In this paper, we expect to design the first quantum search algorithm that hybridizes the fixed-point method and the trial-and-error method for the case of unknown λ, to confirm that the fixed-point method can also actually achieve the real optimal query complexity by selecting the number of iterations under the trial-and-error method, and confirm that the number of queries of the deterministic trial-and-error quantum search algorithm can be closer to (and even the number of iterations can be fewer than) the randomized versions.

The paper is organized as follows. Section 2 provides an introduction of Yoder's fixed-point quantum search algorithm as well as an analysis of the query complexity. Section 3 describes our hybrid quantum search algorithm of the fixed-point method and the trial-and-error method. Section 4 discusses the comparisons between the existing algorithms and the algorithm in this paper, which also gives a brief conclusion.

2. Yoder's algorithm and its query complexity

Yoder's algorithm[33] aims to overcome the loss of quadratic speedup in the original fixed-point quantum search[32] and avoid the soufflé problem in the original Grover algorithm.[1]

The initial state of Yoder's algorithm is prepared by applying operator A to the state $|0\rangle$, which can be written as

$$|\psi\rangle = A|0\rangle = \sqrt{\lambda}\,|\alpha\rangle + \sqrt{1-\lambda}\,|\beta\rangle, \tag{1}$$

where $A = H$ is the Hadamard transform, $|\alpha\rangle$ ($|\beta\rangle$) represents the equal superposition of all target (nontarget) states, *i.e.*,

$$|\alpha\rangle = \frac{1}{\sqrt{M}} \sum_{x\in f^{-1}(1)} |x\rangle, \tag{2}$$

$$|\beta\rangle = \frac{1}{\sqrt{N-M}} \sum_{x\in f^{-1}(0)} |x\rangle, \tag{3}$$

where $\lambda = M/N$ is the fraction of target items, and M is the number of target items in the database of size N.

Yoder's algorithm performs on $|\psi\rangle$ the following sequence of matched-multiphase Grover operations (referred to as Yoder's sequence)

$$G(\phi_l, \varphi_l)\, G(\phi_{l-1}, \varphi_{l-1}) \cdots G(\phi_1, \varphi_1). \tag{4}$$

Here l denotes the number of iterations, $G(\phi,\varphi) = -AS_0^{\phi}A^{\dagger}S_f^{\varphi}$ is the generalized Grover iteration,[45] S_f^{φ} (S_0^{ϕ}) is the selective phase shift, conditionally changing the phase of target states (zero state) by a factor of φ (ϕ), expressed as $\left(\mathrm{i} = \sqrt{-1}\right)$

$$S_f^{\varphi} = I - \left(-\mathrm{e}^{\mathrm{i}\varphi} + 1\right) \sum_{x\in f^{-1}(1)} |x\rangle\langle x|, \tag{5}$$

$$S_0^{\phi} = I - \left(-\mathrm{e}^{\mathrm{i}\phi} + 1\right) |0\rangle\langle 0|, \tag{6}$$

and the phases $\{\phi_j, \varphi_j : 1 \le j \le l\}$ satisfy the following multiphase matching condition[33]

$$\phi_j = \varphi_{l-j+1} = -2\,\mathrm{arccot}\left(\sqrt{1-\gamma^2}\tan(2\pi j/L)\right), \tag{7}$$

where $L = 2l+1$, $\gamma = T_{1/L}^{-1}(1/\delta)$, $\delta \in (0,1)$, and $T_L(x)$ is the L^{th} Chebyshev polynomial of the first kind,[46] defined as

$$T_L(x) = \begin{cases} \cos(L\arccos(x)), & \text{if } |x| \le 1, \\ \cosh(L\,\mathrm{arcosh}(x)), & \text{if } x \ge 1, \\ (-1)^L \cosh(L\,\mathrm{arcosh}(-x)), & \text{if } x \le -1. \end{cases} \tag{8}$$

The final state of Yoder's algorithm can be expressed as

$$|C_L\rangle = \sqrt{P_L}\,|\alpha\rangle + \sqrt{1-P_L}\,|\beta\rangle, \tag{9}$$

where P_L denotes the success probability, satisfying

$$P_L = 1 - \delta^2 T_L^2\left[T_{1/L}(1/\delta)\sqrt{1-\lambda}\right], \tag{10}$$

and for a given L, $P_L \ge 1-\delta^2$ as long as

$$\lambda \ge \left(\frac{\ln(2/\delta)}{L}\right)^2 \equiv \omega. \tag{11}$$

Then, for a given λ, to ensure the success probability no less than $1-\delta^2$, based on Eq. (11), letting $\omega \le \lambda$, the following condition of L can be obtained,[33] *i.e.*,

$$L \ge \ln(2/\delta)/\sqrt{\lambda} \equiv L_{\min}, \tag{12}$$

which demonstrates the fixed-point property. However, in the case of unknown λ, $L_{\min}$ is unknown. For this, it is assumed

that there exists a known lower bound λ_0 of λ.[33] Then, from Eq. (11), letting $\omega \leq \lambda_0$, it follows that

$$L \geq \ln(2/\delta)/\sqrt{\lambda_0} \equiv L_0. \tag{13}$$

Note that L_0 is known, because λ_0 is known. Therefore, the query complexity of Yoder's algorithm is actually in $O(1/\sqrt{\lambda_0})$, rather than $O(1/\sqrt{\lambda})$, which is independent on λ and not really optimal. For example, if the lower bound $\lambda_0 = 1/N$, while $\lambda = M/N = 1/\sqrt{N}$ (*i.e.*, $M = \sqrt{N}$), then the order of query complexity of Yoder's algorithm is $O(1/\sqrt{\lambda_0}) = O(\sqrt{N})$, which is actually the same as that of classical search, *i.e.*, $O(1/\lambda) = O(\sqrt{N})$, while the algorithm that achieves the real quadratic speedup over classical algorithms should be in $O(1/\sqrt{\lambda}) = O(\sqrt[4]{N})$.

3. Hybrid fixed-point and trail-and-error quantum search

As described in Section 2, choosing the number of iterations relying on the lower bound λ_0 of λ, results in the problem that query complexity of Yoder's algorithm[33] is not really optimal. In addition, as described in Ref. [43], the existing deterministic trial-and-error algorithm requires more than 1 times iterations than the randomized version,[27] due to the success probability of the original Grover algorithm[1] oscillates intensively about λ. We expect to handle these problems by hybridizing the fixed-point method with the trial-and-error method, that is, we trials Yoder's sequence (defined by Eq. (4)) multiple times and exponentially increase the number of iterations. At this time, the condition Eq. (12) of λ, rather than Eq. (13) of λ_0, could be satisfied rapidly, and after that the success probability at each trial would *always* (not just often) be no less than a given value, due to the fixed-point property. In this way, we could expect that a target item would be found with the expected Oracle queries in the real optimal order, and fewer number of iterations than the existing trial-and-error algorithms could be cost. First, we give the following lemma.

Lemma 1 For any real number $\lambda \in (0,1)$, $\delta \in (0,1)$, and integer $L > 0$, there exists a lower bound of the success probability P_L of Eq. (10), denoted by $P_L^{\rm lb}$, which is given as

$$P_L^{\rm lb} \equiv \begin{cases} 1-\delta^2 T^2_{\sqrt{1-L^2/L_{\rm cri}^2}}(1/\delta), & \text{if } L \leq L_{\rm cri}, \\ 1-\delta^2, & \text{if } L > L_{\rm cri}, \end{cases} \tag{14}$$

where $T_L(x)$ is defined by Eq. (8), and

$$L_{\rm cri} = \frac{\operatorname{arcosh}(1/\delta)}{\operatorname{arcosh}\left(1/\sqrt{1-\lambda}\right)}. \tag{15}$$

Proof First of all, based on Eq. (15), for any $\lambda \in (0,1)$, $\delta \in (0,1)$, and $L > 0$, P_L of Eq. (10) can be rewritten as

$$P_L = 1-\delta^2 T_L^2\left[T_{1/L}(1/\delta)\, T^{-1}_{1/L_{\rm cri}}(1/\delta)\right]. \tag{16}$$

Note that, for any $x > 0$ and $\theta \in [0, \pi/2]$, we can derive the following inequality about the hyperbolic cosine function, *i.e.*,

$$\begin{aligned} \cosh(x) &= \sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!}\left(\cos^2(\theta)+\sin^2(\theta)\right)^n \\ &\leq \frac{1}{2}\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}\left[(\cos\theta+\sin\theta)^{2n}+(\cos\theta-\sin\theta)^{2n}\right] \\ &= \cosh(x\sin\theta)\cosh(x\cos\theta). \end{aligned} \tag{17}$$

Then, in the case of $L \leq L_{\rm cri}$, substituting $x = L^{-1}\operatorname{arcosh}(1/\delta)$ and $\theta = \arcsin\left(L/L_{\rm cri}\right)$ into Eq. (17), we obtain

$$1 \leq T_L\left[T_{1/L}(1/\delta)\, T^{-1}_{1/L_{\rm cri}}(1/\delta)\right] \leq T_{\sqrt{1-L^2/L_{\rm cri}^2}}(1/\delta). \tag{18}$$

Furthermore, from Eqs. (16) and (18) it follows that

$$P_L \geq 1-\delta^2 T^2_{\sqrt{1-L^2/L_{\rm cri}^2}}(1/\delta). \tag{19}$$

Finally, in the case of $L > L_{\rm cri}$, $P_L \geq 1-\delta^2$ can be easily obtained due to the fact that $|T_L(x)| \leq 1$ for $|x| \leq 1$. Therefore, the conclusion is proved. ■

Now, we are ready to describe the hybrid quantum search algorithm for the unknown λ (the corresponding flow diagram is shown in Fig. 1).

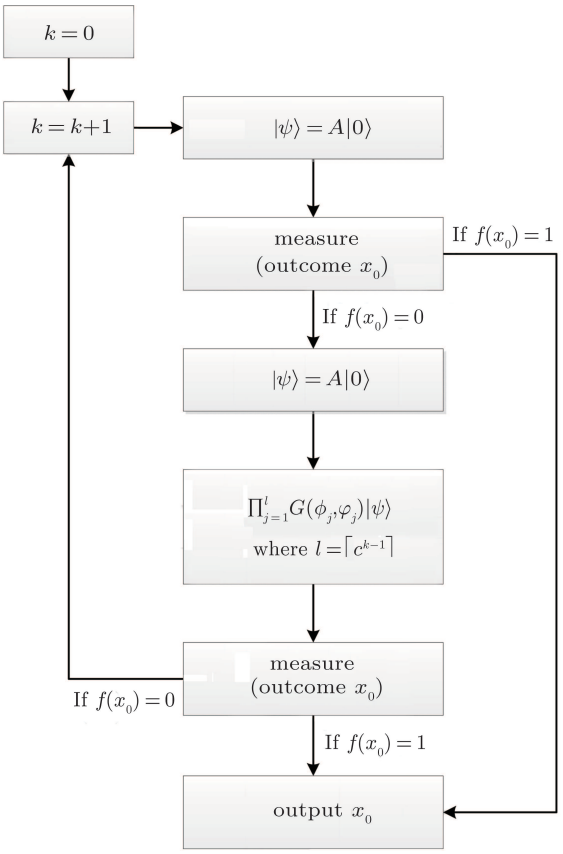


Fig. 1. Flow diagram of the hybrid fixed-point and trail-and-error quantum search algorithm.

Step 1 Initialize $k=0$, and let $\delta=0.5659$ and $c=1.523$. The reason for such selection of δ and c is given in the proof of Theorem 1 below.

Step 2 Increase k by 1.

Step 3 Apply H to the state $|0\rangle$, and measure the system. If the outcome $|x_0\rangle$ is a target item, *i.e.*, $f(x_0)=1$, then output x_0 and stop the procedure.

Step 4 Prepare the initial state $|\psi\rangle=H|0\rangle$, and perform on $|\psi\rangle$ the Yoder's sequence $\prod_{j=1}^{l}G(\phi_j,\varphi_j)$, where $l=\lceil c^{k-1}\rceil$ is the number of iterations and the phases $\{\phi_j,\varphi_j\}$ satisfy the multiphase matching condition of Eq. (7).

Step 5 Measure the system. If the outcome $|x_0\rangle$ is a target item, then output x_0 and stop the procedure. Otherwise, go to Step 2.

Theorem 1 This algorithm outputs a target item in expected number of Oracle queries in $O(1/\sqrt{\lambda})$.

Proof On the one hand, if $\lambda\geq 1-\delta^2$, then Step 3 ensures that a target item can be measured soon. On the other hand, if $0<\lambda<1-\delta^2$, we let

$$l_{\rm cri}=\left\lceil\frac{\operatorname{arcosh}(1/\delta)}{2\operatorname{arcosh}\left(1/\sqrt{1-\lambda}\right)}-\frac{1}{2}\right\rceil, \tag{20}$$

and set

$$s_0=\lfloor\log_c l_{\rm cri}\rfloor+1. \tag{21}$$

Then, using l_s, Q_s, and P_{L_s} to denote the value of the number of iterations, the probability of occurrence and the probability of success on the s-th round of the main loop (Steps 2–5), the expected number of Oracle queries of the whole algorithm can be expressed by

$$E(T_{\rm all})=E(T_1)+E(T_2), \tag{22}$$

where $E(T_1)$ and $E(T_2)$ denote the expected queries the algorithm takes while $1\leq s\leq s_0$ and $s\geq s_0+1$, respectively,

$$E(T_1)=\sum_{s=1}^{s_0}Q_s(2l_s+2), \tag{23}$$

$$E(T_2)=\sum_{s=s_0+1}^{\infty}Q_s(2l_s+2), \tag{24}$$

with the l_s and Q_s satisfying

$$l_s=\lceil c^{s-1}\rceil, \tag{25}$$

$$Q_s=\prod_{j=1}^{s-1}\left(1-P_{L_j}\right). \tag{26}$$

By virtue of Lemma 1, we can obtain the lower bound of the success probability P_{L_s}, *i.e.*,

$$\begin{aligned}P_{L_s}&\geq\begin{cases}1-\delta^2T^2_{\sqrt{1-c^{-2(s_0+1-s)}}}(1/\delta), & \text{if } 1\leq s\leq s_0,\\ 1-\delta^2, & \text{if } s\geq s_0+1,\end{cases}\\ &\equiv P_{L_s}^{\rm lb},\end{aligned} \tag{27}$$

and further obtain the upper bound of the occurrence probability Q_s, *i.e.*,

$$\begin{aligned}&Q_{s_0+1}\leq\delta^2T^2_{\sqrt{1-c^{-2}}}(1/\delta)\equiv Q_{s_0+1}^{\rm ub},\\ &Q_{s_0+1+u}\leq Q_{s_0+1}^{\rm ub}\delta^{2u},\ \ u\geq 0.\end{aligned} \tag{28}$$

Then, based on Eqs. (21), (23)–(28) we can obtain

$$\begin{aligned}E(T_1)&\leq\sum_{s=1}^{s_0}\left(2c^{s-1}+4\right)\\ &\leq\frac{2c}{c-1}l_{\rm cri}+4\log_c l_{\rm cri}+4,\end{aligned} \tag{29}$$

$$\begin{aligned}E(T_2)&=\sum_{u=0}^{\infty}Q_{s_0+1+u}\left(2l_{s_0+1+u}+2\right)\\ &\leq Q_{s_0+1}^{\rm ub}\left(\frac{2c}{1-c\delta^2}l_{\rm cri}+\frac{4}{1-\delta^2}\right),\end{aligned} \tag{30}$$

where we have assumed that $c<\delta^{-2}$. Finally, from Eqs. (22), (28)–(30), and $l_{\rm cri}\gg\log_c l_{\rm cri}\gg 1$ for $\lambda\ll 1-\delta^2$, it follows that

$$\begin{aligned}E(T_{\rm all})&\leq\left(\frac{2c}{c-1}+\frac{2Q_{s_0+1}^{\rm ub}c}{1-c\delta^2}\right)l_{\rm cri}+4\left(\log_c l_{\rm cri}+\frac{Q_{s_0+1}^{\rm ub}}{1-\delta^2}+1\right)\\ &\approx\left(\frac{c}{c-1}+\frac{c\delta^2T^2_{\sqrt{1-c^{-2}}}(1/\delta)}{1-c\delta^2}\right)\frac{\operatorname{arcosh}(1/\delta)}{\sqrt{\lambda}}\\ &\equiv g(\delta,c)/\sqrt{\lambda},\end{aligned} \tag{31}$$

and numerical calculation shows that

$$\begin{aligned}&\min\left\{g(\delta,c):0<\delta<1,1<c<\delta^{-2}\right\}\\ &=g(\delta\approx 0.5659,c\approx 1.523)\approx 5.643.\end{aligned} \tag{32}$$

Therefore, the query complexity of our algorithm is in $O(1/\sqrt{\lambda})$. In addition, we can see that $\delta=0.5659$ and $c=1.523$ are just the optimal parameters minimizing the upper bound of the expected query complexity.

4. Discussion and conclusion

In this section, we first discuss the comparisons between our algorithm and the existing quantum search algorithms[27,32,33,41–43] in the case of unknown λ and then give a brief conclusion. Table 1 lists the method, query complexity, and phase(s) of our algorithm and other algorithms. The main advantages of our algorithm are discussed in detail as follows.

Compared with the fixed-point quantum search algorithms,[32,33] as shown in Eq. (31) our algorithm indeed reaches the optimal scaling for quantum search, *i.e.*, $O(1/\sqrt{\lambda})$, while, references [32] and [33] do not. Reasons are in the following: First, for the original fixed-point quantum search algorithm,[32] to achieve a success probability no less than than $1-\delta^2$ ($\delta\in(0,1)$) for an unknown λ with a known lower

Table 1. Detailed comparisons between our algorithm and other algorithms.

Algorithm	Method	Query complexity	Phase(s)
Grover[32]	fixed-point	$O\left(\ln(1/\delta)/\lambda_0\right)$ [a,b]	$\pi/3$
Yoder *et al.*[33]	fixed-point	$O\left(\ln(2/\delta)/\sqrt{\lambda_0}\right)$ [a,b]	$\{\phi_j,\varphi_j\}$ of Eq. (7)
Boyer *et al.*[27]	trail-and-error (randomized)	$\leq 4/\sqrt{\lambda}$	π
Younes *et al.*[41]	trail-and-error (randomized)[c]	$\leq 4\sqrt{2}/\sqrt{\lambda}$	π
Younes[42]	trail-and-error (randomized)[d]	$\leq 61.42/\sqrt{\lambda}$ [b]	1.91684π
Okamoto *et al.*[43]	trail-and-error (deterministic)	$\leq 8.378/\sqrt{\lambda}$	π
Pro. Alg.	fixed-point + trail-and-error (deterministic)	$\leq 5.643/\sqrt{\lambda}$ [b]	$\{\phi_j,\varphi_j\}$ of Eq. (7)

[a] λ_0 is the assumed known lower bound of the unknown fraction λ of target items in Refs. [32] and [33], and the obtained success probability is no less than $1-\delta^2$, where $\delta\in(0,1)$.
[b] The number of Grover iterations is half of it. Note that in the derivation of query complexity, the viewpoint of Ref. [33] has been adopted, that is, one Grover iteration with arbitrary phases other than π contains two queries of the standard quantum Oracle with phase-π, defined by Eq. (33).
[c] Different from Boyer's algorithm, the internal Grover's algorithm is replaced by the partial diffusion algorithm.[44]
[d] The internal Grover's algorithm is replaced by the (1.91684π) fixed-phase algorithm.[42]

bound λ_0, the number of required Oracle queries[35] is in

$$O\left(\frac{\ln(1/\delta)}{\lambda_0}\right)\geq O\left(\frac{\ln(1/\delta)}{\lambda}\right),$$

yielding the loss of quadratic speedup over classical exhaustive search, which is in $O(1/\lambda)$. Second, as shown in Eq. (13), the query complexity of Yoder's algorithm[33] is in $O\left(\ln(2/\delta)/\sqrt{\lambda_0}\right)$, which indeed achieves a quadratic speedup over the original fixed-point algorithm. However, for any $\lambda\geq\lambda_0$, the same as Ref. [32], the least number of iterations of Yoder's algorithm is fixed. For example, if $\lambda_0=1/N$, while $\lambda=(N-1)/N\approx 1$, then a target item could almost be found by performing the classical search once, but the complexity of Yoder's algorithm is still in $O(1/\sqrt{\lambda_0})=O(\sqrt{N})$. Therefore, Yoder's algorithm does not achieve the quadratic speedup over classical search. Note that, this advantage of our algorithm comes from the use of trial-and-error method.

Compared with the trial-and-error quantum search algorithms,[27,41–43] first, with respect to the method, our algorithm does not contain randomness (*i.e.*, random selection of the number of iterations), and enables the success probability *always* no less than an arbitrary given lower bound between 0 and 1 after the number of iterations exceeds the critical value (*i.e.*, $l_{\rm cri}$ defined by Eq. (20)). While, the existing randomized trial-and-error algorithms[27,41,42] rely on the randomness, and the existing deterministic trial-and-error algorithm[43] can only makes the success probability *often* (but not always) no less than a lower bound, fixed at 3/4. Second, with respect to the query/iteration complexity, compared with the existing deterministic trial-and-error algorithm,[43] our query complexity can be reduced by $(8.378-5.643)/8.378\approx 1/3$. Although, the upper bound of query complexity of our algorithm displayed in Table 1 is slightly higher than Boyer's randomized algorithm[27] (about $5.643/4\approx 1.4$ times), however, in the derivation of query complexity, we have adopted the viewpoint in Ref. [33] that one Grover iteration with arbitrary phases queries two standard quantum Oracle (denoted by $S_{\rm f}^{\pi}$) with phase-π, which flips the ancilla qubit when the input is the target state, *i.e.*,

$$S_f^{\pi}|x\rangle|y\rangle=|x\rangle|y\oplus f(x)\rangle. \tag{33}$$

Therefore, in fact, our algorithm has fewer number of iterations, about $1-(5.643/2)/4\approx 30\%$ can be saved than Ref. [27]. This advantage comes from the use of fixed-point method in our algorithm.

Finally, it is worth mentioning that, for the quantum Oracle with arbitrary phases (*i.e.*, S_f^{φ}, defined by Eq. (5)), Yoder *et al.* designed a feasible quantum circuit, including two standard quantum Oracle S_f^{π} with phase-π, which appears to show that the complexity of S_f^{φ} is twice that of S_f^{π}. However, we have noticed that in many practical applications of quantum search, *e.g.*, key search in classical cryptography,[47–49] the satisfiability problem,[50,51] quantum machine learning,[52] *etc.*, the design of the standard quantum Oracle S_f^{π} contains a quantization (denoted by Q_f) of the classical Oracle and the corresponding un-computation step (denoted by $Q_f^{\dagger}$) which restores the ancillary qubits to their initial state,[53] as shown in Fig. 2. Therefore, the complexity of S_f^{π} is twice that of Q_f, also as pointed in Ref. [54]. What is more, based on this, we can find out that follow the circuit of S_f^{φ} in Ref. [33], as depicted in Fig. 2, the $Q_f^{\dagger}$ in the first S_f^{π} and the Q_f in the second S_f^{π} could be omitted because they offset each other ($Q_f^{\dagger}Q_f=I$). Therefore, the complexity of quantum Oracle S_f^{φ} with arbitrary phases would also be twice that of Q_f, which is the same as that of the standard Oracle S_f^{π}. In this case, our algorithm will has the least query complexity among the current algorithms for the unknown λ.[27,32,33,41–43] While, this conclusion may be related to the specific application scenario of the algorithms, and more rigorous proofs can be studied further in the future.

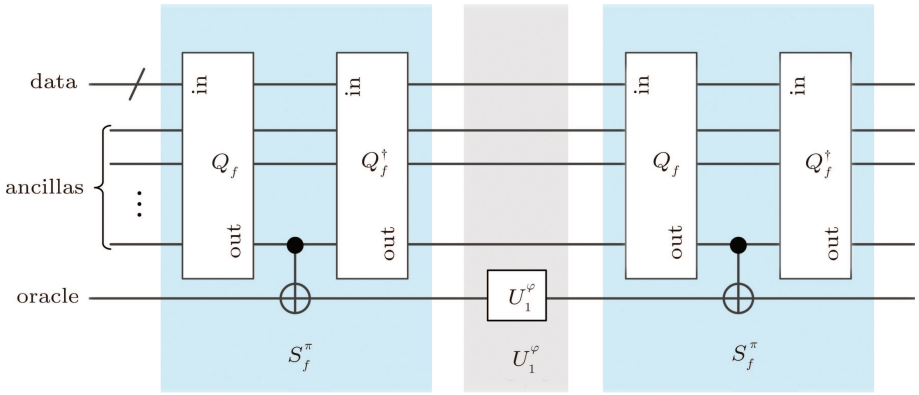


Fig. 2. Circuit for the quantum Oracle S_f^φ with arbitrary phases, defined by Eq. (5), where S_f^π denotes the quantum Oracle with phase-π (defined by Eq. (33)), Q_f is the quantization of the classical Oracle, $Q_f^\dagger$ corresponds to the un-computation step to restore the ancillary qubits, and U_1^φ is the quantum gate adding a phase of $e^{i\varphi}$ to $|1\rangle$ state.

In summary, we have presented the first quantum search algorithm hybridizing the fixed-point method with the trial-and-error method for the unknown λ, which integrates the advantages of both methods and solves the non-optimal problem of the "optimal" fixed-point algorithm[33] and the problem that the deterministic trial-and-error algorithm[43] requires more than 1 times queries or iterations than the randomized versions. In our algorithm, the Yoder's sequence[33] is conducted multiple trials, and the number of iterations increases exponentially along with the number of trials. The upper bound of expected queries of our algorithm to find a target item was derived, as well as the optimal parameters, which minimize the query complexity. Our study reconfirms the practicality of the fixed-point method, and shows that it is very advantageous to hybridize the trial-and-error method and the fixed-point method, which provides a different approach for the research on quantum search algorithms and can be applied to a variety of scenarios where Grover's search is used.[47–52]

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