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Citation: Chin. Phys. B . 2019, 28(12): 120401 . **doi:** 10.1088/1674-1056/ab4e80
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Thermal properties of regular black hole with electric charge in Einstein gravity coupled to nonlinear electrodynamics*

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(Received 31 July 2019; revised manuscript received 23 September 2019; published online 22 November 2019)

We propose a regular spherically symmetric spacetime solution with three parameters in Einstein gravity coupled to nonlinear electrodynamics (NED), which describes the NED black hole with electric charge. It is found that the system enclosed by the horizon of NED spacetime satisfies the first law of thermodynamics. In order to obtain the NED spacetime with only electric charge, the case of two parameters taking the same value is considered. In this case, we express the mass of the NED spacetime as a function of the entropy and electric charge of the NED black hole, give the Smarr-like formula and the approximate Smarr formula for the mass of NED spacetime.

Keywords: regular nonlinear electrodynamics (NED) spacetime, NED black hole with electric charge, first law of thermodynamics, Smarr formula

PACS: 04.70.Dy, 98.80.–k, 98.80.Es **DOI:** 10.1088/1674-1056/ab4e80

1. Introduction

The center of some spherically symmetric spacetimes, such as Reissner–Nordström (RN) spacetime, is singular. The spacetime without any geometric singularity can be said to be regular. A black hole in regular spacetime is called regular black hole. The Bardeen regular black hole is the first black hole with magnetic monopole to be found in Einstein gravity coupled to nonlinear electrodynamics (NED).[1,2] In the NED gravity, some regular black hole solutions have been proposed,[3–26] the electromagnetic field of which behaves as the Maxwell field in the weak field limit.[3–5] The NED solutions with multiple parameters given in Refs. [19]–[22] describe the regular NED black holes with electric charge. It is of great interest to generalize this class of the NED solutions.

The Smarr formula for the black hole with a central singularity, such as RN black hole, plays an important role. However, the usual Smarr formula is not applicable to the case of the NED black hole.[27–29] Starting from the Komar integral, the Smarr-type formula may be derived, where the boundary of Komar integral is taken on the infinite surface.[30] On the other hand, one can assume that the Smarr formula for the NED black hole is valid.[31] From the presupposed Smarr formula for the NED black hole, the quantities similar to the temperature and electric potential of NED black hole can be obtained. The latter study does not seem to yield physically meaningful results. The thermodynamic properties of the regular NED black hole need to be further studied.

In this paper, we propose the NED solution with electric charge and study its thermodynamic properties. In Section 2, we give the NED solution with three parameters, determine the radius of the NED black hole and calculate its electric potential. In Section 3, we obtain the first law of thermodynamics of the NED black hole. In a special case, we show the Smarr-like formula for the mass of the NED spacetime, and obtain the approximate Smarr formula associated with the outer horizon of NED black hole. Lastly, we discuss some problems on the regularity parameter and the Smarr formula of NED spacetime.

2. Regular NED black hole with electric charge

The action for Einstein gravity coupling to nonlinear electrodynamics may be taken as

$$S=\int\sqrt{-g}\,\mathrm{d}^4x\left(\frac{1}{16\pi}R-\mathcal{L}\right), \tag{1}$$

where $\mathcal{L}=\mathcal{L}(F)$ is Lagrangian density of the electromagnetic field in the NED gravity, $F=\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, $F_{\mu\nu}=\nabla_\mu A_\nu-\nabla_\nu A_\mu$, and A_μ is the gauge potential of electromagnetic field. The energy–momentum tensor of electromagnetic field in the NED gravity is[23]

$$T_{\mu\nu}=g_{\mu\nu}\mathcal{L}-\mathcal{L}_F F^{\lambda}_{\mu}F_{\lambda\nu}, \tag{2}$$

where $\mathcal{L}_F=\mathrm{d}\mathcal{L}/\mathrm{d}F$. The electromagnetic tensor $F^{\mu\nu}$ satisfies the modified Maxwell equation

$$\nabla_\mu\left(\mathcal{L}_F F^{\mu\nu}\right)=0. \tag{3}$$

For the spherically symmetric spacetime with electric charge q, we have

$$\mathrm{d}s^2=-f\,\mathrm{d}t^2+f^{-1}\,\mathrm{d}t^2+r^2(\mathrm{d}\theta^2+\sin^2\varphi\,\mathrm{d}\varphi^2) \tag{4}$$

with $f=1-2m/r$ the metric function of the NED spacetime, the electric field takes

$$\mathcal{E}=\mathcal{L}_F^{-1}\frac{q}{r^2}, \tag{5}$$

*Project supported by the National Natural Science Foundation of China (Grant Nos. 11504027 and 11847011).

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the Lagrangian density and its derivative to F are respectively[23] read

$$\mathcal{L} = \frac{1}{8\pi r} m'', \tag{6}$$

$$\mathcal{L}_F = \frac{8\pi q^2}{r^2(2m' - rm'')} \tag{7}$$

with $m = m(r)$ the mass function of the NED spacetime, where a prime represents the derivative to r.

The metric function of the NED spacetime in Ref. [20] takes

$$f = 1 - \frac{2M}{r} - \frac{2q^3 r^\sigma}{\alpha(r^\beta + \lambda\beta)^{\sigma/\beta}},$$

where M, q, α, β, σ, and λ are some constants. By considering regularity, the metric function can be simplified to

$$f = 1 - \frac{2M}{r} \frac{r^\sigma}{(r^\beta + \lambda\beta)^{\sigma/\beta}}$$

with four parameters $M > 0$, $\beta > 0$, $\sigma > 0$, and $\lambda > 0$.[20,21] For $\beta = 1$ and $\sigma = 3$, it yields the NED spacetime solution with electric charge. In order to describe the regular NED spacetime with electric charge, we consider the metric function with three parameters: the mass of spacetime, the electric charge parameter and the regularity parameter of spacetime. We propose the following metric function

$$f = 1 - \frac{2Mr^2}{r^3 + \chi r^2 + \chi^2 r + \lambda^3}, \tag{8}$$

or the mass function

$$m = \frac{Mr^3}{r^3 + \chi r^2 + \chi^2 r + \lambda^3} \tag{9}$$

with the mass $M > 0$, the electric charge parameter $\chi \geq 0$, and the regularity parameter $\lambda > 0$. The regular NED spacetime (8) with a nonzero χ is not included in the ones given in Refs. [20] and [21]. For $\chi = 0$, the metric function (8) describes the Hayward spacetime with magnetic monopole.[8] For $\chi \neq 0$, it describes the NED spacetime with electric charge. In what follows, we will consider the regular NED spacetime with a nonzero electric charge. In the central region, the NED spacetime behaves as a de-Sitter spacetime, $f \approx 1 - r^2/l_{\text{eff}}^2$ with $l_{\text{eff}} = \sqrt{\lambda^3/2M}$ an effective cosmological constant. For $r \gg \chi$ and $r \gg \lambda$, the metric function (8) is expanded as

$$f = 1 - \frac{2M}{r} + \frac{q^2}{r^2} + \frac{2M(\lambda^3 - \chi^3)}{r^4} + O(r^{-5}), \tag{10}$$

where the term r^{-3} is absent, and $q = \sqrt{2M\chi}$ is the electric charge of the NED spacetime.

The spacetime described by the metric function (8) can have two black hole horizons. Three solutions (r_1, r_2, and r_3) of the equation $f = 0$ satisfy the following relationships:

$$2M - \chi = r_1 + r_2 + r_3, \tag{11}$$

$$\chi^2 = r_1 r_2 + r_2 r_3 + r_3 r_1, \tag{12}$$

$$\lambda^3 = -r_1 r_2 r_3. \tag{13}$$

From Eq. (13), it can be seen that at least one of the three solutions is negative. Letting $r_3 < 0$, $r_1 = r_+$, $r_2 = r_-$ with $r_+ \geq r_-$, then two positive solutions of $f = 0$ are given as

$$r_+ = \frac{1}{3}(2M - \chi) + \frac{2\sqrt{2}}{3}\sqrt{2M^2 - 2M\chi - \chi^2}\cos\psi, \tag{14}$$

$$r_- = \frac{1}{3}(2M - \chi) - \frac{2\sqrt{2}}{3}\sqrt{2M^2 - 2M\chi - \chi^2}\cos\left(\psi + \frac{\pi}{3}\right) \tag{15}$$

with

$$\cos 3\psi = \frac{16M^3 - 27\lambda^3 - 24M^2\chi - 6M\chi^2 + 7\chi^3}{4\sqrt{2}\sqrt{(2M^2 - 2M\chi - \chi^2)^3}}, \tag{16}$$

where r_+ and r_- denote radii of the outer and inner horizon of the NED black hole. As $\chi \to 0$ and $\lambda \to 0$, radii of the outer and inner horizons of the NED black hole approach $2M$ and 0, respectively.

From $\kappa_{\text{H}} = \frac{1}{2}f'(r_{\text{H}})$, the surface gravity on the horizon of the NED black hole is obtained as

$$\kappa_{\text{H}} = \frac{Mr_{\text{H}}(r_{\text{H}}^3 - 2\lambda^3 - r_{\text{H}}\chi^2)}{(r_{\text{H}}^3 + \lambda^3 + r_{\text{H}}^2\chi + r_{\text{H}}\chi^2)^2} \tag{17}$$

with $r_{\text{H}} = r_\pm$. From $\kappa_{\text{H}} = 0$, one obtains the equation satisfied by the extremal NED black hole

$$r_{\text{E}}^3 - 2\lambda^3 - r_{\text{E}}\chi^2 = 0, \tag{18}$$

where $r_{\text{E}} = r_+ = r_-$ denotes radius of the extremal NED black hole. As $\lambda \to 0$, $r_{\text{E}} \to \chi$; as $\chi \to 0$, $r_{\text{E}} \to 2^{1/3}\lambda$. For the four cases of $\lambda/M = 0.1$, 0.5, 0.75, 1.0, the allowable ranges of parameter χ are determined to be $0 < \chi/M \leq 0.666$, 0.582, 0.41, 0.102, the corresponding radii of extremal black hole are given as $r_{\text{E}}/M = 0.668$, 0.805, 1.004, 1.263, respectively (See Fig. 1). For $\lambda \approx 0$, radius of the extremal black hole is $r_{\text{E}} \approx \chi \approx 0.667M$. The maximum of parameter λ takes $\lambda_{\max} = 1.05827M$ corresponding to $\chi \approx 0$. In this case, $r_{\text{E}} \approx 1.333M$ is the largest radius of the extremal NED black hole.

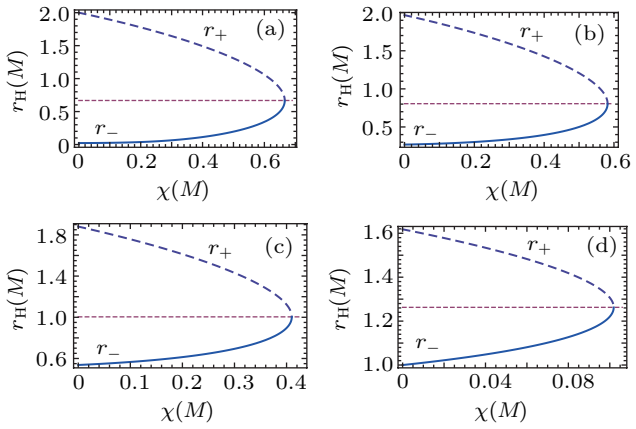


Fig. 1. The relationship between radii of horizons and parameter χ with some fixed values of λ. Dashed horizontal line represents radius of extremal black hole r_{E} corresponding to $\chi = \chi_{\text{E}}$. Panel (a): $r_{\text{E}} \approx 0.668M$ and $\chi_{\text{E}} \approx 0.666M$ for $\lambda = 0.1M$. Panel (b): $r_{\text{E}} \approx 0.805M$ and $\chi_{\text{E}} \approx 0.582M$ for $\lambda = 0.5M$. Panel (c): $r_{\text{E}} \approx 1.004M$ and $\chi_{\text{E}} \approx 0.41M$ for $\lambda = 0.75M$. Panel (d): $r_{\text{E}} \approx 1.263M$ and $\chi_{\text{E}} \approx 0.102M$ for $\lambda = M$.

Substituting Eq. (9) into Eqs. (6) and (7) gives the Lagrangian of the NED spacetime and its derivative as

$$\mathcal{L} = -\frac{M\left[6r^3\lambda^3 - 3\lambda^6 + r^5\chi + 3r^4\chi^2 - 3r\lambda^3\chi^2 + r^2\chi(\lambda^3 - \chi^3)\right]}{4\pi(r^3+\lambda^3+r^2\chi+r\chi^2)^3}, \tag{19}$$

$$\mathcal{L}_F = \frac{4\pi q^2(r^3+\lambda^3+r^2\chi+r\chi^2)^3}{Mr^5[2r^4\chi+6r^3\chi^2+2\lambda^3\chi^2+3r^2(3\lambda^3+\chi^3)+r(5\lambda^3\chi+\chi^4)]}. \tag{20}$$

As $r\to\infty$, $\mathcal{L}$ approaches $-M\chi/4\pi r^4$ and $\mathcal{L}_F$ is close to 4π. From Eq. (20), the strength of the electromagnetic tensor is given as

$$F = -\frac{M^2r^6[2r^4\chi+6r^3\chi^2+2\lambda^3\chi^2+3r^2(3\lambda^3+\chi^3)+r(5\lambda^3\chi+\chi^4)]^2}{32\pi^2q^2(r^3+\lambda^3+r^2\chi+r\chi^2)^6}. \tag{21}$$

As $r\to\infty$, F approaches $-q^2/16\pi^2r^4$. From Eq. (5) (or $E=\sqrt{2F}$), the electrostatic field is obtained as

$$\mathcal{E} = \frac{qr^3[2r^4+6r^3\chi+3r^2(3\lambda^3/\chi+\chi^2)+r(5\lambda^3+\chi^3)+2\lambda^3\chi]}{8\pi(r^3+\lambda^3+r^2\chi+r\chi^2)^3}. \tag{22}$$

On the horizon of the NED black hole, the electrostatic field takes

$$\mathcal{E}_{\rm H} = \frac{q[2x_{\rm H}^4+6x_{\rm H}^3\tilde{\chi}+3x_{\rm H}^2(3\tilde{\lambda}^3/\tilde{\chi}+\tilde{\chi}^2)+x_{\rm H}(5\tilde{\lambda}^3+\tilde{\chi}^3)+2\tilde{\lambda}^3\tilde{\chi}]}{64\pi M^2x_{\rm H}^3} \tag{23}$$

with $x_{\rm H}=r_{\rm H}/M$, $\tilde{\lambda}=\lambda/M$, and $\tilde{\chi}=\chi/M$. As $\tilde{\chi}\to0$ and $\tilde{\lambda}\to0$, equation (23) reduces approximately to

$$\mathcal{E}_{\rm H}^{(0)} = \frac{q(2x_{\rm H}^4+9x_{\rm H}^2\tilde{\lambda}^3/\tilde{\chi})}{64\pi M^2x_{\rm H}^3}. \tag{24}$$

For the case of $\tilde{\lambda}^3/\tilde{\chi}\approx0$, the electrostatic field takes approximately $\mathcal{E}_{\rm H}^{(0)}=q/4\pi r_{\rm H}^2$ with $r_{\rm H}=x_{\rm H}M$ and $x_{\rm H}\approx2$. Assuming that $\tilde{\lambda}^3/\tilde{\chi}$ is a nonzero constant c, then $\mathcal{E}_{\rm H}^{(0)}\approx(1+9c/8)q/4\pi r_{\rm H}^2$.

By integrating $\Phi=-\int\mathcal{E}\,{\rm d}r$, the electrostatic potential of the NED spacetime is given as

$$\Phi = \frac{q[4r^5+8r^4\chi+6r^3(\lambda^3/\chi+\chi^2)+3r^2(2\lambda^3+\chi^3)+6r\lambda^3\chi+3\lambda^6/\chi]}{16\pi(r^3+\lambda^3+r^2\chi+r\chi^2)^2} \tag{25}$$

with $\Phi(\infty)=0$. On the horizon of the NED black hole, the electrostatic potential takes

$$\Phi_{\rm H} = \frac{q[4x_{\rm H}^5+8x_{\rm H}^4\tilde{\chi}+6x_{\rm H}^3(\tilde{\lambda}^3/\tilde{\chi}+\tilde{\chi}^2)+3x_{\rm H}^2(2\tilde{\lambda}^3+\tilde{\chi}^3)+6x_{\rm H}\tilde{\lambda}^3\tilde{\chi}+3\tilde{\lambda}^6/\tilde{\chi}]}{64\pi Mx_{\rm H}^4}. \tag{26}$$

As $\tilde{\chi}\to0$ and $\tilde{\lambda}\to0$, equation (26) approximately takes

$$\Phi_{\rm H}^{(0)} = \frac{q}{4\pi r_{\rm H}}\left(1+\frac{3c}{8}+\frac{3c\tilde{\lambda}^3}{16x_{\rm H}^3}\right). \tag{27}$$

For $c\approx0$, $\Phi_{\rm H}^{(0)}$ is approximately the electrostatic potential of the RN black hole. In order to be consistent with the usual definition of electrostatic potential of black hole, Φ should be rescaled as $\Phi\to4\pi\Phi$.

3. Thermal properties of regular NED black hole

For a spherically symmetric spacetime with multiple horizons, one can define multiple temperatures associated with the surface gravities of the black hole horizons. From $T_{\rm H}=(1/2\pi)\kappa_{\rm H}$ with $\kappa_{\rm H}=(1/2)f'(r_{\rm H})$, the horizon temperature of the NED black hole is obtained as

$$T_{\rm H} = \frac{x_{\rm H}^3-2\tilde{\lambda}^3-x_{\rm H}\tilde{\chi}^2}{8\pi Mx_{\rm H}^3}. \tag{28}$$

As $\chi\to0$ and $\lambda\to0$, $T_{\rm H}$ approaches $1/8\pi M$. In $f(R)$ gravity with the action $\mathcal{S}=\int{\rm d}^4x\sqrt{-g}f(R)$, the black hole entropy takes $S_{\rm H}=f'A_{\rm H}/4$ with $A_{\rm H}$ the horizon area of black hole and $f'={\rm d}f/{\rm d}R$.[32,33] Clearly, the entropy of the NED regular black hole is still the Bekenstein–Hawking entropy.[34]

Einstein quasi-local energy enclosed by the sphere of radius r in the spacetime (4) takes $E=(1/2)r(1-f)$,[35] which is consistent with the mass function (9). Letting $f=0$, one obtains the energy enclosed by the horizon of the NED black hole as $E_{\rm H}=(1/2)r_{\rm H}$. The first law of thermodynamics for the NED black hole reads[36–38]

$${\rm d}E_{\rm H} = {\rm d}Q_{\rm H}+{\rm d}W_{\rm H} = T_{\rm H}{\rm d}S_{\rm H}-p_{\rm H}{\rm d}V_{\rm H}, \tag{29}$$

where $p_{\rm H}=p_{\rm r}(r_{\rm H})$ denotes the radial pressure on the horizon of the NED black hole, $S_{\rm H}=\pi r_{\rm H}^2$ is the entropy of the NED black hole, and $V_{\rm H}=(4/3)\pi r_{\rm H}^3$ is the volume enclosed by the horizon of the NED black hole. For the system enclosed by the horizon of the NED spacetime described by Eq. (8), the energy flux through the horizon and the work done by the horizon are

$$\mathrm{d}Q_{\rm H}=\frac{r_{\rm H}^3-r_{\rm H}\chi^2-2\lambda^3}{4Mr_{\rm H}^2}\,\mathrm{d}r_{\rm H}, \tag{30}$$

$$\mathrm{d}W_{\rm H}=\frac{3\lambda^3+r_{\rm H}\chi(r_{\rm H}+2\chi^2)}{4Mr_{\rm H}^2}\,\mathrm{d}r_{\rm H}. \tag{31}$$

From Eqs. (30) and (31), one can check the first law of thermodynamics Eq. (29) with $\mathrm{d}E_{\rm H}=(1/2)\mathrm{d}r_{\rm H}$. The first law of thermodynamics Eq. (29) can be applied to the system enclosed by the outer horizon as well as to the one enclosed by the inner horizon with a negative temperature. This is similar to the case of the RN black hole.

From Eq. (9), one can see that the mass function (or energy) of the NED spacetime m monotonously increases with the increase of r. The mass function tends to M, as r tends to be infinite. So, the total energy of the NED spacetime is its mass M. From $f(r_{\rm H})=0$, the mass of the NED spacetime is given as

$$M=\frac{1}{2}r_{\rm H}\left(1+\frac{\lambda^3+r_{\rm H}^2\chi+r_{\rm H}\chi^2}{r_{\rm H}^3}\right) \tag{32}$$

with $\chi=q^2/2M$. From Eq. (10), one can see that there is an asymptotic term $2M(\lambda^3-\chi^3)/r^4$ in the expansion of the metric function f. This means that the NED spacetime described by Eq. (8) with $\lambda\neq\chi$ probably contains a nonzero magnetic charge. For $\lambda=\chi$, the mass of the NED spacetime is obtained as

$$\begin{aligned}M=&\frac{r_{\rm H}}{8}+\frac{\sqrt{3}r_{\rm H}}{24}\sqrt{\frac{8q^2}{r_{\rm H}^2}+3+\mathcal{D}}\\&+\frac{\sqrt{6}r_{\rm H}}{24}\sqrt{\frac{8q^2}{r_{\rm H}^2}+3-\frac{\mathcal{D}}{2}+\frac{3\sqrt{3}(r_{\rm H}^4+4r_{\rm H}^2q^2+8q^4)}{r_{\rm H}^3\sqrt{3r_{\rm H}^2+8q^2+r_{\rm H}^2\mathcal{D}}}},\end{aligned} \tag{33}$$

where

$$\mathcal{D}=\frac{4\mathcal{C}}{r_{\rm H}^4}-\frac{8q^4(1+6q^2/r_{\rm H}^2)}{\mathcal{C}}, \tag{34}$$

$$\begin{aligned}\mathcal{C}=&\left(-10q^6r_{\rm H}^6-\frac{45}{2}q^8r_{\rm H}^4+\frac{3\sqrt{3}}{2}q^6r_{\rm H}^3\right.\\&\left.\times\sqrt{256q^6+203q^4r_{\rm H}^2+88q^2r_{\rm H}^4+16r_{\rm H}^6}\right)^{1/3}.\end{aligned} \tag{35}$$

Equations (33)–(35) show that the mass of the NED spacetime may be expressed as $M=M(S_{\rm H},q)$. Mathematically, one can always define the quantities $\tilde{T}_{\rm H}=(\frac{\partial M}{\partial S_{\rm H}})|_q$ and $\tilde{\Phi}_{\rm H}=(\frac{\partial M}{\partial q})|_{S_{\rm H}}$. For the NED black hole, the two quantities are different from $T_{\rm H}$ and $\Phi_{\rm H}$ (see Fig. 2). This means that there is not the exact Smarr formula for the NED black hole with a nonzero electric charge.

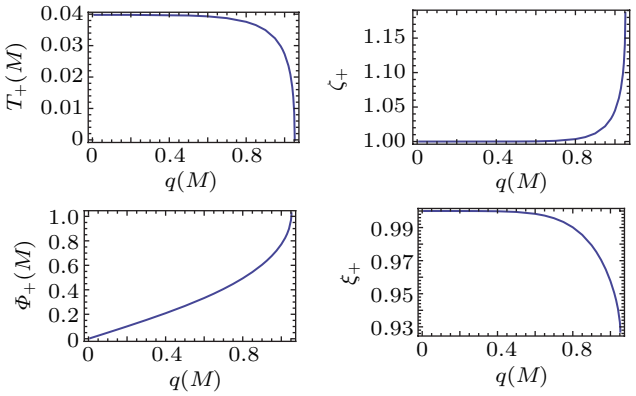


Fig. 2. The figure shows that the temperature of the NED black hole T_+ and the ratio $\zeta_+=\tilde{T}_+/T_+$, with minimum $\zeta_{+\min}\approx 1$ and maximum $\zeta_{+\max}\approx 1.1866$ corresponding to $q=1.0525$. It shows that the electrostatic potential of the NED black hole Φ_+ and the ratio $\xi_+=\tilde{\Phi}_+/\Phi_+$, with maximum $\xi_{+\max}\approx 1$ and minimum $\xi_{+\min}\approx 0.9276$ corresponding to $q=1.0525$.

In terms of entropy S_+, temperature T_+ and electric potential Φ_+ of the outer horizon of the NED black hole, the mass of the NED spacetime may be expressed as

$$M=2T_+S_++\Phi_+q+\Delta M_+, \tag{36}$$

where $\Delta M_+/M$ is a small quantity. We call Eq. (36) the Smarr-like formula associated with the outer horizon of the NED black hole. For $q=0.1M$, the value of ΔM_+ takes $-5.9\times10^{-11}M$. For a larger value of q, it is still a very small quantity. Letting $q=0.5M$, the value is $\Delta M_+\approx-0.000028M$. In the near extremal case ($q\approx q_m$ with $q_m=1.052526825$), it is $\Delta M_+\approx-0.079M$. As a result, there is the approximate formula expressed in terms of the quantities defined on the outer horizon of the NED black hole

$$M\approx 2T_+S_++\Phi_+q. \tag{37}$$

The above formula shows how is the total energy in the NED spacetime made up. The smaller the electric charge q in Eq. (37), the more accurate the approximate Smarr formula is. As $q\to 0$, it gradually approaches an exact Smarr formula.

Assuming a non-zero cosmological constant, then the metric function of the NED spacetime takes

$$\tilde{f}=f-\frac{r^2}{l^2}=1-\frac{2Mr^2}{r^3+\chi r^2+\chi^2r+\lambda^3}-\frac{r^2}{l^2} \tag{38}$$

with $l\gg M$ being a constant and f given by Eq. (8). The mass of the NED spacetime may be expressed as

$$M=\frac{1}{2}r_{\rm C}+\frac{\lambda^3+r_{\rm C}^2\chi+r_{\rm C}\chi^2}{2r_{\rm C}^2}-\frac{r_{\rm C}^3+r_{\rm C}^2\chi+r_{\rm C}\chi^2+\lambda^3}{2l^2}$$

with $r_{\rm C}\gg M$ being the radius of the cosmological horizon. It can be checked that there exists an approximate Smarr formula with an extremely high accuracy

$$M\simeq 2T_{\rm C}S_{\rm C}+\Phi_{\rm C}q+\Theta_{\rm C}l, \tag{39}$$

where $\Phi_C = \Phi(r_C)$ with Φ given by Eq. (25) is the electric potential on the cosmological horizon, $\Theta_C = -(\frac{\partial M}{\partial l})|_{r_C,q}$ is the potential associated with l, $S_C = \pi r_C^2$ is the entropy of the cosmological horizon, and $T_C = (1/4\pi)\tilde{f}'(r_C)$ is its temperature.

For the topological dilaton black hole with power-Maxwell field and a nonzero cosmological constant, there is the exact Smarr formula.[39] However, the mass of the NED spacetime described by Eq. (38) cannot be exactly expressed as the form of the Smarr formula. This should be related to the regularity of the NED spacetime.

4. Discussion

In the asymptotic expansion of the metric function Eq. (10), there is a term proportional to r^{-4}. As is well known, there is also such a term in the asymptotic expansion of the metric function of the NED spacetime with magnetic monopole.[8] So, the meaning of this term is not completely clear since the magnetic monopole can have a contribution of the same magnitude in the far region of the NED spacetime. As an NED spacetime with only electric charge, we have considered the situation of $\lambda = \chi$. From Eq. (8), one can see that $f \to 1-(2M/\lambda^3)r^2$ with $r \to 0$. In the case of $\lambda \neq 0$, there is a regular core in the center of the NED spacetime. In this sense, we call λ the regularity parameter of the NED spacetime.

The systems enclosed by the horizons of the NED spacetime described by Eq. (8) satisfy the first law of thermodynamics. The mass function (9) shows that M is not an integration constant of the energy of the NED spacetime. So, one can only define the mass of spacetime, but not the mass of black hole. For $\lambda = \chi$, we give the Smarr-like formula for the mass of the NED spacetime and the approximate Smarr formula associated with the outer horizon of the NED black hole. For a nonzero cosmological constant, there is also the approximate Smarr formula associated with the cosmological horizon. The approximate Smarr formula is an important thermodynamic relation for the NED spacetime with electric charge.

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