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Resonant multiple wave solutions to some integrable soliton equations*

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To transform the exponential traveling wave solutions to bilinear differential equations, a sufficient and necessary condition is proposed. Motivated by the condition, we extend the results to the (2+1)-dimensional Kadomtsev–Petviashvili (KP) equation, the (3+1)-dimensional generalized Kadomtsev–Petviashvili (g-KP) equation, and the B-type Kadomtsev–Petviashvili (BKP) equation. Aa a result, we obtain some new resonant multiple wave solutions through the parameterization for wave numbers and frequencies via some linear combinations of exponential traveling waves. Finally, these new resonant type solutions can be displayed in graphs to illustrate the resonant behaviors of multiple wave solutions.

Keywords: linear superposition principle, resonant multiple wave solutions, (2+1)-dimensional Kadomtsev–Petviashvili (KP) equation, (3+1)-dimensional g-KP and BKP equations

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1. Introduction

It is well known that the Hirota bilinear method is an efficient tool to construct exact solutions of nonlinear differential equations.[1,2] Based on Hirota bilinear forms, rational solutions, solitons solutions, rogue wave solutions, and periodic solutions could be obtained, e.g. in Refs. [3]–[7]. Meanwhile, the exact solutions play an essential role in proper understanding of qualitative features of the concerned phenomena and processes in various fields of science and engineering.[8–11] Recently, the lump solutions, lump type solutions and its various interaction solutions between lump solutions and other kinds of dispersive wave solutions are explored.[12–18] Meanwhile, using the Riemann–Hilbert method to construct N-soliton solutions has also attracted people's attention in the field of mathematical physics.[19,20] More recently, the linear superposition principle has been applied to exponential traveling waves of Hirota bilinear equations, and formed a specific sub-class of solutions from linear combinations of exponential wave solutions.[21,22] Some useful and extended results have been discussed in Refs. [23]–[27].

Let Υ be a polynomial in M variables, and then we consider a Hirota bilinear equation

$$\Upsilon(D_{x_1}, D_{x_2}, \ldots, D_{x_M}) f \cdot f = 0, \tag{1}$$

where D_{x_i} $(1 \le i \le M)$ is the Hirota bilinear derivatives defined by[28]

$$D_x^m D_t^n f \cdot g = (\frac{\partial}{\partial x} - \frac{\partial}{\partial x'})^m (\frac{\partial}{\partial t} - \frac{\partial}{\partial t'})^n f(x,t) \cdot g(x',t')|_{x=x',t=t'}. \tag{2}$$

The polynomial Υ needs to satisfy

$$\Upsilon(\overbrace{0,0,\ldots,0}^{M}) = 0, \tag{3}$$

and

$$\Upsilon(-x_1, -x_2, \ldots, -x_M) = \Upsilon(x_1, x_2, \ldots, x_M). \tag{4}$$

Next, we introduce N-wave variables

$$\varpi_i = k_{1,i}x_1 + k_{2,i}x_2 + \ldots + k_{M,i}x_M, \quad 1 \le i \le N, \tag{5}$$

and exponential wave functions

$$f_i = \mathrm{e}^{\varpi_i}, \quad 1 \le i \le N, \tag{6}$$

where $k_{i,j}$, $1 \le i \le N$, $1 \le j \le M$ are all real constants. Based on Eq. (3), we can see that each of the exponential wave functions f_i $(1 \le i \le N)$ is a solution to the Hirota bilinear Eq. (1) with the following bilinear identity

$$\begin{aligned} &\Upsilon(D_{x_1}, D_{x_2}, \ldots, D_{x_M}) \mathrm{e}^{\varpi_i} \cdot \mathrm{e}^{\varpi_j} \\ &= \Upsilon(k_{1,i} - k_{1,j}, k_{2,i} - k_{2,j}, \ldots, k_{M,i} - k_{M,j}) \mathrm{e}^{\varpi_i + \varpi_j}. \end{aligned} \tag{7}$$

Substitute the linear combination function

$$f = \mu_1 f_1 + \mu_2 f_2 + \ldots + \mu_N f_N, \tag{8}$$

into Eq. (1), with μ_i, $1 \le i \le N$, being arbitrary real constants, which needs to satisfy the following bilinear identity

$$\Upsilon(D_{x_1}, D_{x_2}, \ldots, D_{x_M}) f \cdot f$$

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$$= \sum_{1\le i<j\le N} 2\mu_i\mu_j \Upsilon(k_{1,i}-k_{1,j}, k_{2,i}-k_{2,j}, \ldots, k_{M,i}-k_{M,j}) e^{\varpi_i+\varpi_j}, \quad (9)$$

which indicates that a linear combination function f defined by Eq. (8) solves Eq. (1) if and only if the following condition

$$\Upsilon(k_{1,i}-k_{1,j}, k_{2,i}-k_{2,j}, \ldots, k_{M,i}-k_{M,j}) e^{\varpi_i+\varpi_j} = 0, \quad (10)$$
$$1 \le i < j \le N,$$

is satisfied.

For the above processes (2)–(9), the details can also refer to Refs. [21]–[27]. We now summarize it by two theorems as follows:

Theorem 1[23] Let $\Upsilon(x)$ be a polynomial in $x \in R^M$ and let ϖ_i, $1 \le i \le N$, be the N-wave variables $\varpi_i = \sum_{j=1}^{M} b_j k_i^{\alpha_j} x_j$, for $1 \le i \le N$, where b_j's are all real constants. Then any linear combination of the exponential waves $f_i = e^{\varpi_i}$, $1 \le i \le N$, solves linear differential equation $\Upsilon(D_{x_1}, D_{x_2}, \ldots, D_{x_M}) f \cdot f = 0$ if and only if

$$\Upsilon(k_{1,i}-k_{1,j}, k_{2,i}-k_{2,j}, \ldots, k_{M,i}-k_{M,j}) e^{\varpi_i+\varpi_j} = 0 \quad (11)$$

for $1 \le j \le i \le N$.

Moreover, Eq. (11) is true for arbitrary $k_i, k_j \in R$ if and only if it is also true for arbitrary constants $k_i, k_j \in \mathscr{C}$. Then, we have the following Theorem 2.

Theorem 2[23] Let $\Upsilon(x)$ be a polynomial in $x \in R^M$. Suppose that $N_1, N_2 \in N$, $N = N_1 + N_2$, $b_j \in R$, $\alpha_j \in Z$ for $j = 1, 2, \ldots, M$ are all fixed. The N-wave variables $\varpi_i = \sum_{j=1}^{M} b_j k_i^{\alpha_j} x_j$, $1 \le i \le N$ with parameters

$$k_i \in R, \quad 1 \le i \le N_1, \quad k_i = \gamma_i + I\delta_i,$$
$$\gamma_i, \delta_i \in R, \quad \delta_i \ne 0, N_1 \le i \le N, \quad I = \sqrt{-1}. \quad (12)$$

If equation (11) is true for arbitrary $k_j, k_i \in R$, $1 \le j, i \le N$, then any linear combination of the wave f_i defined by

$$f_i = \begin{cases} e^{\varpi_i}, & 1 \le i \le N_1, \\ e^{\mathrm{Re}(\varpi_i)} \cos(\mathrm{Im}(\varpi_i) + \theta_i), & N_1 \le i \le N, \end{cases} \quad (13)$$

with parameters θ_i, $N_1 \le i \le N$, being arbitrary real constants, solves the bilinear differential equation

$$\Upsilon(D_{x_1}, D_{x_2}, \ldots, D_{x_M}) f \cdot f = 0.$$

In what follows, we apply the linear superposition principle to extend the results for the (2+1)-dimensional KP equation, the (3+1)-dimensional g-KP equation and the BKP equation to obtain resonant multiple wave solutions in this paper.[29] To the best of our knowledge, they have not been obtained before.

2. Applications

2.1. The (2+1)-dimensional KP equation

In this section, we apply the linear superposition principle to solve the (2+1)-dimensional KP equation,[29] which is important because it serves as a kernel model in the universe Sato's theory,[30–32] that is,

$$(u_{xxx} + 6uu_x + u_t)_x + 3\delta^2 u_{yy} = 0, \quad (14)$$

where $\delta = \pm 1$.

Equation (14) can be transformed into the following Hirota bilinear equation

$$(D_x D_t + D_x^4 + 3\delta^2 D_y^2) F \cdot F = 0, \quad (15)$$

under the dependent variable transformation $u = 2(\ln F)_{xx}$.

By Eq. (5) and applying a kind of the parameterization for wave numbers and frequencies in Ref. [33], we have

$$\varpi_i = a_1 k_i x + a_2 k_i^2 y + a_3 k_i^3 t + \vartheta_i, 1 \le i \le N, \quad (16)$$

where a_1, a_2, a_3 are constants to be determined later, and ϑ_i is arbitrary.

Applying Theorem 1 and Eq. (15), one can derive

$$\begin{aligned} &\Upsilon(k_{1,i}-k_{1,j}, k_{2,i}-k_{2,j}, k_{3,i}-k_{3,j}) \\ &= 3\delta^2 a_2^2 k_i^4 - 6\delta^2 a_2^2 k_i^2 k_j^2 + 3\delta^2 a_2^2 k_j^4 \\ &\quad + a_1^4 k_i^4 - 4a_1^4 k_i^3 k_j + 6a_1^4 k_i^2 k_j^2 - 4a_1^4 k_i k_j^3 + a_1^4 k_j^4 \\ &\quad + a_1 a_3 k_i^4 - a_1 a_3 k_i^3 k_j - a_1 a_3 k_i k_j^3 + a_1 a_3 k_j^4 \\ &= 0. \end{aligned} \quad (17)$$

Solving Eq. (17) we can obtain a set of solutions as follows:

$$\left\{ a_2 = \pm \frac{a_1^2}{\delta}, \ a_3 = -4a_1^3 \right\}, \quad (18)$$

where a_1 is a free parameter.

Suppose that $N_1, N_2 \in N$, $N = N_1 + N_2$. The N-wave variables $\varpi_m = a_1 k_m x + a_2 k_m^2 y + a_3 k_m^3 t + \phi_m, 1 \le m \le N$ with parameters $k_m, 1 \le m \le N$, defined by (5). Moreover, for $N_1 < m \le N$, we have

$$\begin{aligned} \varpi_m =& a_1 k_m x + a_2 k_m^2 y + a_3 k_m^3 t + \phi_m \\ &- 4a_1{}^4 \gamma_m{}^3 + 12 a_1{}^4 \gamma_m \sigma_m{}^2 + \frac{a_1{}^2 \gamma_m{}^2}{\delta} - \frac{a_1{}^2 \sigma_m{}^2}{\delta} + a_1 \gamma_m \\ &+ i\left(-12 a_1{}^4 \gamma_m{}^2 \sigma_m + 4 a_1{}^4 \sigma_m{}^3 + 2 \frac{a_1{}^2 \gamma_m \sigma_m}{\delta} + a_1 \sigma_m \right). \end{aligned} \quad (19)$$

Using Theorem 2, any linear combination of the exponential

and trigonometric waves is taken as follows:

$$
\begin{aligned}
&u=2(\ln F)_{xx},\\
&F=\sum_{m=1}^{N_1}\varepsilon_m \mathrm{e}^{a_1k_mx\pm\frac{a_1^2}{\delta}k_m^2y-4a_1k_m^3t+\phi_m}\\
&+\sum_{m=N_1+1}^{N}\xi_m \mathrm{e}^{-4ta_1{}^4\gamma_m{}^3+12ta_1{}^4\gamma_m\sigma_m{}^2\pm(\frac{a_1{}^2\gamma_m{}^2}{\delta}-\frac{a_1{}^2\sigma_m{}^2}{\delta})y+xa_1\gamma_m+\phi_m}\\
&\times\cos\left(\left((-12a_1^4\gamma_m^2\sigma_m+4a_1^4\sigma_m^3)t\pm\frac{2a_1^2\gamma_m\sigma_m}{\delta}y+xa_1\sigma_m\right)+\theta_m\right).
\end{aligned}
\tag{20}
$$

Taking $\theta_m=0$, we have

$$
\begin{aligned}
&u=2(\ln F)_{xx},\\
&F=\sum_{m=1}^{N_1}\varepsilon_m \mathrm{e}^{a_1k_mx\pm\frac{a_1^2}{\delta}k_m^2y-4a_1k_m^3t+\phi_m}\\
&+\sum_{m=N_1+1}^{N}\xi_m \mathrm{e}^{-4ta_1{}^4\gamma_m{}^3+12ta_1{}^4\gamma_m\sigma_m{}^2\pm(\frac{a_1{}^2\gamma_m{}^2}{\delta}-\frac{a_1{}^2\sigma_m{}^2}{\delta})y+xa_1\gamma_m+\phi_m}\\
&\times\cos\left((-12a_1^4\gamma_m^2\sigma_m+4a_1^4\sigma_m^3)t\pm\frac{2a_1^2\gamma_m\sigma_m}{\delta}y+xa_1\sigma_m\right).
\end{aligned}
\tag{21}
$$

Taking $\theta_m=2k\pi+\pi/2,\ k\in K$, then we have

$$
\begin{aligned}
&u=2(\ln F)_{xx},\\
&F=\sum_{m=1}^{N_1}\varepsilon_m \mathrm{e}^{a_1k_mx\pm\frac{a_1^2}{\delta}k_m^2y-4a_1k_m^3t+\phi_m}\\
&+\sum_{m=N_1+1}^{N}\xi_m \mathrm{e}^{-4ta_1{}^4\gamma_m{}^3+12ta_1{}^4\gamma_m\sigma_m{}^2\pm(\frac{a_1{}^2\gamma_m{}^2}{\delta}-\frac{a_1{}^2\sigma_m{}^2}{\delta})y+xa_1\gamma_m+\phi_m}\\
&\times\sin\left((-12a_1^4\gamma_m^2\sigma_m+4a_1^4\sigma_m^3)t\pm\frac{2a_1^2\gamma_m\sigma_m}{\delta}y+xa_1\sigma_m\right).
\end{aligned}
\tag{22}
$$

Furthermore, we can obtain the following mixed type function solutions like complexiton solutions

$$
\begin{aligned}
&u=2(\ln F)_{xx},\\
&F=\sum_{m=1}^{N_1}\varepsilon_i \mathrm{e}^{a_1k_mx\pm\frac{a_1^2}{\delta}k_m^2y-4a_1k_m^3t+\phi_m}\\
&+\sum_{m=N_1+1}^{N}\xi_m \mathrm{e}^{-4ta_1{}^4\gamma_m{}^3+12ta_1{}^4\gamma_m\sigma_m{}^2\pm(\frac{a_1{}^2\gamma_m{}^2}{\delta}-\frac{a_1{}^2\sigma_m{}^2}{\delta})y+xa_1\gamma_m+\phi_m}\\
&\times\left[\xi_{1m}\cos\left((-12a_1^4\gamma_m^2\sigma_m+4a_1^4\sigma_m^3)t\pm\frac{2a_1^2\gamma_m\sigma_m}{\delta}y+xa_1\sigma_m\right)\right.\\
&\left.+\xi_{2m}\sin\left((-12a_1^4\gamma_m^2\sigma_m+4a_1^4\sigma_m^3)t\pm\frac{2a_1^2\gamma_m\sigma_m}{\delta}y+xa_1\sigma_m\right)\right],
\end{aligned}
\tag{23}
$$

where ξ_{1m} and ξ_{2m} are constants.

2.2. The (3+1)-dimensional g-KP equation

In this section, we apply the linear superposition principle to solve the (3+1)-dimensional g-KP equation[29]

$$u_{xxxy}+3(u_yu_x)_x+u_{tx}+u_{ty}-u_{zz}=0. \tag{24}$$

Equation (24) can be changed into the following Hirota bilinear equation

$$(D_x^3D_y+D_xD_t+D_yD_t-D_z^2)F\cdot F=0, \tag{25}$$

under the dependent variable transformation $u=2(\ln F)_x$.

Using Eq. (5) and applying a kind of the parameterization for wave numbers and frequencies in Ref. [33], we have

$$\varpi_i=a_1k_ix+a_2k_iy+a_3k_i^2z+a_4k_i^3t+\vartheta_i,\quad 1\le i\le N, \tag{26}$$

where a_1,a_2,a_3,a_4 are constants to be determined later, and ϑ_i is arbitrary.

Applying Theorem 1 and Eq. (25), we have

$$
\begin{aligned}
&\Upsilon(k_{1,i}-k_{1,j},k_{2,i}-k_{2,j},k_{3,i}-k_{3,j},k_{4,i}-k_{4,j})\\
&=a_1{}^3a_2k_i{}^4-4a_1{}^3a_2k_i{}^3k_j+6a_1{}^3a_2k_i{}^2k_j{}^2-4a_1{}^3a_2k_ik_j{}^3\\
&\quad+a_1{}^3a_2k_j{}^4+a_1a_4k_i{}^4-a_1a_4k_i{}^3k_j-a_1a_4k_ik_j{}^3\\
&\quad+a_1a_4k_j{}^4+a_2a_4k_i{}^4-a_2a_4k_i{}^3k_j-a_2a_4k_ik_j{}^3\\
&\quad+a_2a_4k_j{}^4-a_3{}^2k_i{}^4+2a_3{}^2k_i{}^2k_j{}^2-a_3{}^2k_j{}^4\\
&=0.
\end{aligned}
\tag{27}
$$

Solving Eq. (27), we can obtain a set of solutions in the form of

$$\left\{a_3^2=-3a_1^3a_2,\quad a_4=-\frac{4a_1^3}{(a_1+a_2)}\right\}, \tag{28}$$

where a_1 and a_2 are arbitrary parameters.

Suppose that $N_1,N_2\in N$, $N=N_1+N_2$. We have the N-wave variables $\varpi_m=a_1k_mx+a_2k_my+a_3k_m^2z+a_4k_m^3t+\phi_m$, $1\le m\le N$ with parameters k_m, $1\le m\le N$, defined by (5). Moreover, for $N_1<m\le N$, we have

$$
\begin{aligned}
\varpi_m&=a_1k_mx+a_2k_my+a_3k_m^2z+a_4k_m^3t+\phi_m\\
&=a_1\gamma_mx+a_2\gamma_my\pm(a_1\sqrt{-3a_1a_2}\gamma_m^2-a_1\sqrt{-3a_1a_2}\sigma_m^2)z\\
&\quad+\left(\frac{12a_1^4\gamma_m\sigma_m^2}{a_1+a_2}-\frac{4a_1^4\gamma_m^3}{a_1+a_2}\right)t+\phi_m\\
&\quad+i\left[a_1\sigma_mx+a_2\sigma_my\pm2a_1\sqrt{-3a_1a_2}\gamma_m\sigma_mz\right.\\
&\quad\left.+\left(\frac{4a_1^4\sigma_m^3}{a_1+a_2}-\frac{12a_1^4\gamma_m^2\sigma_m}{a_1+a_2}\right)t\right],\quad 1\le m\le N.
\end{aligned}
\tag{29}
$$

Using Theorem 2, any linear combination of exponential and trigonometric waves can be taken as follows:

$$
\begin{aligned}
u&=2(\ln F)_x,\\
F&=\sum_{m=1}^{N_1}\varepsilon_m \mathrm{e}^{a_1k_mx+a_2k_my+a_3k_m^2z+a_4k_m^3t+\phi_m}\\
&\quad+\sum_{m=1+N_1}^{N}\xi_m \mathrm{e}^{a_1\gamma_mx+a_2\gamma_my+(a_1\sqrt{-3a_1a_2}\gamma_m^2-a_1\sqrt{-3a_1a_2}\sigma_m^2)z}\\
&\quad\times\mathrm{e}^{(\frac{12a_1^4\gamma_m\sigma_m^2}{a_1+a_2}-\frac{4a_1^4\gamma_m^3}{a_1+a_2})t+\phi_m}\cos\left[\left(a_1\sigma_mx+a_2\sigma_my\right.\right.\\
&\quad\left.\left.+2a_1\sqrt{-3a_1a_2}\gamma_m\sigma_mz+\left(\frac{4a_1^4\sigma_m^3}{a_1+a_2}-\frac{12a_1^4\gamma_m^2\sigma_m}{a_1+a_2}\right)t\right)+\theta_m\right].
\end{aligned}
\tag{30}
$$

Taking $\theta_m = 0$, we have

$$\begin{aligned} u &= 2(\ln F)_x, \\ F &= \sum_{m=1}^{N_1} \varepsilon_m \mathrm{e}^{a_1 k_m x + a_2 k_m y + a_3 k_m^2 z + a_4 k_m^3 t + \phi_m} \\ &\quad + \sum_{m=1+N_1}^{N} \xi_m \mathrm{e}^{a_1\gamma_m x + a_2\gamma_m y + (a_1\sqrt{-3a_1a_2}\gamma_m^2 - a_1\sqrt{-3a_1a_2}\sigma_m^2)z} \\ &\quad \times \mathrm{e}^{(\frac{12a_1^4\gamma_m\sigma_m^2}{a_1+a_2} - \frac{4a_1^4\gamma_m^3}{a_1+a_2})t + \phi_m} \cdot \cos\left[\left(a_1\sigma_m x + a_2\sigma_m y \right.\right. \\ &\quad \left.\left. + 2a_1\sqrt{-3a_1a_2}\gamma_m\sigma_m z + \left(\frac{4a_1^4\sigma_m^3}{a_1+a_2} - \frac{12a_1^4\gamma_m^2\sigma_m}{a_1+a_2}\right)t\right)\right]. \end{aligned} \tag{31}$$

Taking $\theta_m = 2k\pi + \pi/2,\ k \in K$, we have

$$\begin{aligned} u &= 2(\ln F)_x, \\ F &= \sum_{m=1}^{N_1} \varepsilon_m \mathrm{e}^{a_1 k_m x + a_2 k_m y + a_3 k_m^2 z + a_4 k_m^3 t + \phi_m} \\ &\quad + \sum_{m=1+N_1}^{N} \varepsilon_m \mathrm{e}^{a_1\gamma_m x + a_2\gamma_m y + (a_1\sqrt{-3a_1a_2}\gamma_m^2 - a_1\sqrt{-3a_1a_2}\sigma_m^2)z} \\ &\quad \times \mathrm{e}^{(\frac{12a_1^4\gamma_m\sigma_m^2}{a_1+a_2} - \frac{4a_1^4\gamma_m^3}{a_1+a_2})t + \phi_m} \sin\left[\left(a_1\sigma_m x + a_2\sigma_m y \right.\right. \\ &\quad \left.\left. + 2a_1\sqrt{-3a_1a_2}\gamma_m\sigma_m z + \left(\frac{4a_1^4\sigma_m^3}{a_1+a_2} - \frac{12a_1^4\gamma_m^2\sigma_m}{a_1+a_2}\right)t\right)\right]. \end{aligned} \tag{32}$$

Furthermore, we can obtain the following mixed-type function solutions like complexiton solutions

$$\begin{aligned} u &= 2(\ln F)_x, \\ F &= \sum_{m=1}^{N_1} \varepsilon_m \mathrm{e}^{a_1 k_m x + a_2 k_m y + a_3 k_m^2 z + a_4 k_m^3 t + \phi_m} \\ &\quad + \sum_{m=1+N_1}^{N} \xi_m \mathrm{e}^{a_1\gamma_m x + a_2\gamma_m y + (a_1\sqrt{-3a_1a_2}\gamma_m^2 - a_1\sqrt{-3a_1a_2}\sigma_m^2)z} \\ &\quad \times \mathrm{e}^{(\frac{12a_1^4\gamma_m\sigma_m^2}{a_1+a_2} - \frac{4a_1^4\gamma_m^3}{a_1+a_2})t + \phi_m} \left[\xi_{1m}\cos\left(a_1\sigma_m x + a_2\sigma_m y \right.\right. \\ &\quad \left.\left. + 2a_1\sqrt{-3a_1a_2}\gamma_m\sigma_m z + \left(\frac{4a_1^4\sigma_m^3}{a_1+a_2} - \frac{12a_1^4\gamma_m^2\sigma_m}{a_1+a_2}\right)t\right)\right. \\ &\quad \left. + \xi_{2m}\sin\left(a_1\sigma_m x + a_2\sigma_m y + 2a_1\sqrt{-3a_1a_2}\gamma_m\sigma_m z \right.\right. \\ &\quad \left.\left. + \left(\frac{4a_1^4\sigma_m^3}{a_1+a_2} - \frac{12a_1^4\gamma_m^2\sigma_m}{a_1+a_2}\right)t\right)\right], \end{aligned} \tag{33}$$

where ξ_{1m} and ξ_{2m} are constants.

2.3. The (3+1)-dimensional BKP equation

In this section, we apply the linear superposition principle to solve the (3+1)-dimensional BKP equation[29]

$$u_{zt} - u_{xxxy} - 3(u_y u_x)_x + 3u_{xx} = 0. \tag{34}$$

Equation (34) can be transformed into the following Hirota bilinear equation

$$(D_z D_t - D_x^3 D_y + 3D_x^2)F \cdot F = 0 \tag{35}$$

under the dependent variable transformation $u = 2(\ln F)_x$.

Along with Eq. (5) and a kind of the parameterization for wave numbers and frequencies in Ref. [33], we have

$$\varpi_i = a_1 k_i^{\alpha_1} x + a_2 k_i^{\alpha_2} y + a_3 k_i^{\alpha_3} z + a_4 k_i^{\alpha_4} t + \vartheta_i, 1 \le i \le N, \tag{36}$$

where a_1, a_2, a_3, a_4 are constants to be determined later, and $\alpha_i, (i = 1,2,3,4)$ are integers which can be positive or negative, and ϑ_i is arbitrary.

Applying Theorem 1 and Eq. (35), we obtain

$$\begin{aligned} &\Upsilon(k_{1,i} - k_{1,j}, k_{2,i} - k_{2,j}, k_{3,i} - k_{3,j}, k_{4,i} - k_{4,j}) \\ &= a_3 a_4 k_i{}^{\alpha_3+\alpha_4} - a_3 a_4 k_i{}^{\alpha_3} k_j{}^{\alpha_4} - a_3 a_4 k_j{}^{\alpha_3} k_i{}^{\alpha_4} \\ &\quad + a_3 a_4 k_j{}^{\alpha_3+\alpha_4} - a_1^3 a_2 k_i{}^{3\alpha_1+\alpha_2} + a_1^3 a_2 k_i{}^{3\alpha_1} k_j{}^{\alpha_2} \\ &\quad + 3a_1^3 a_2 k_j{}^{\alpha_1} k_i{}^{2\alpha_1+\alpha_2} - 3a_1^3 a_2 k_i{}^{2\alpha_1} k_j{}^{\alpha_1+\alpha_2} \\ &\quad - 3a_1^3 a_2 k_i{}^{\alpha_1+\alpha_2} k_j{}^{2\alpha_1} + 3a_1^3 a_2 k_i{}^{\alpha_1} k_j{}^{2\alpha_1+\alpha_2} \\ &\quad + a_1^3 a_2 k_j{}^{3\alpha_1} k_i{}^{\alpha_2} - a_1^3 a_2 k_j{}^{3\alpha_1+\alpha_2} + 3a_1^2 k_i{}^{2\alpha_3} \\ &\quad - 6a_1^2 k_i{}^{\alpha_3} k_j{}^{\alpha_3} + 3a_1^2 k_j{}^{2\alpha_3} = 0. \end{aligned} \tag{37}$$

Solving Eq. (37), we can obtain a set of solutions to Eq. (37) in the following two cases:

Case (i)

$$\{a_2 = \frac{1}{a_1}, \quad a_4 = \frac{a_1^2}{a_3}\}, \tag{38}$$

where a_1 and a_4 are free parameters.

Suppose that $N_1, N_2 \in N,\ N = N_1 + N_2$. We have the N-wave variables $\varpi_m = a_1 k_m x + a_2 k_m^{-1} y + a_3 k_m^{-1} z + a_4 k_m^3 t + \phi_m, 1 \le m \le N$ with parameters $k_m,\ 1 \le m \le N$, defined by (5). Moreover, for $N_1 < m \le N$, we have

$$\begin{aligned} \varpi_m &= a_1 k_m x + a_2 k_m^{-1} y + a_3 k_m^{-1} z + a_4 k_m^3 t + \phi_m \\ &= a_1 x\gamma_m + \frac{\gamma_m y}{a_1(\gamma_m{}^2 + \sigma_m{}^2)} + \frac{a_3\gamma_m z}{\gamma_m{}^2 + \sigma_m{}^2} + \frac{a_1{}^2 t\gamma_m{}^3}{a_3} \\ &\quad - 3\frac{a_1{}^2 t\gamma_m\sigma_m{}^2}{a_3} + \phi_m + i\left(a_1 x\sigma_m - \frac{\sigma_m y}{a_1(\gamma_m{}^2 + \sigma_m{}^2)}\right. \\ &\quad \left. - \frac{a_3\sigma_m z}{\gamma_m{}^2 + \sigma_m{}^2} + 3\frac{a_1{}^2 t\gamma_m{}^2\sigma_m}{a_3} - \frac{a_1{}^2 t\sigma_m{}^3}{a_3}\right). \end{aligned} \tag{39}$$

Using Theorem 2, any linear combination of exponential and trigonometric waves gives

$$\begin{aligned} &u = 2(\ln F)_x, \\ &F = \sum_{m=1}^{N_1} \varepsilon_m \mathrm{e}^{a_1 k_m x + a_2 k_m^{-1} y + a_3 k_m^{-1} z + a_4 k_m^3 t + \phi_m} \\ &\quad + \sum_{m=N_1+1}^{N} \xi_m \mathrm{e}^{a_1\gamma_m x + \frac{\gamma_m}{a_1(\gamma_m{}^2+\sigma_m{}^2)} y + \frac{a_3\gamma_m}{\gamma_m{}^2+\sigma_m{}^2} z + (\frac{a_1{}^2\gamma_m{}^3}{a_3} - \frac{3a_1{}^2\gamma_m\sigma_m{}^2}{a_3})t + \phi_m} \\ &\quad \times \cos\left(\left(a_1 x\sigma_m - \frac{\sigma_m}{a_1(\gamma_m{}^2 + \sigma_m{}^2)} y - \frac{a_3\sigma_m}{\gamma_m^2 + \sigma_m{}^2} z \right.\right. \\ &\quad \left.\left. + \left(\frac{3a_1^2\gamma_m^2\sigma_m}{a_3} - \frac{a_1^2\sigma_m^3}{a_3}\right)t\right) + \theta_m\right). \end{aligned} \tag{40}$$

Taking $\theta_m = 0$, we have

$$
\begin{aligned}
&u = 2(\ln F)_x,\\
&F = \sum_{m=1}^{N_1} \varepsilon_m \mathrm{e}^{a_1 k_m x + a_2 k_m^{-1} y + a_3 k_m^{-1} z + a_4 k_m^3 t + \phi_m}\\
&+ \sum_{m=N_1+1}^{N} \xi_m \mathrm{e}^{a_1 \gamma_m x + \frac{\gamma_m}{a_1\left({\gamma_m}^2+{\sigma_m}^2\right)} y + \frac{a_3 \gamma_m}{{\gamma_m}^2+{\sigma_m}^2} z + \left(\frac{{a_1}^2 {\gamma_m}^3}{a_3} - \frac{3{a_1}^2 \gamma_m {\sigma_m}^2}{a_3}\right) t + \phi_m}\\
&\times \cos\left(a_1 x \sigma_m - \frac{\sigma_m}{a_1\left({\gamma_m}^2+{\sigma_m}^2\right)} y - \frac{a_3 \sigma_m}{{\gamma_m}^2+{\sigma_m}^2} z\right.\\
&\left.+ \left(\frac{3a_1^2 \gamma_m^2 \sigma_m}{a_3} - \frac{a_1^2 \sigma_m^3}{a_3}\right) t\right).
\end{aligned}
\tag{41}
$$

Taking $\theta_m = 2k\pi + \pi/2,\ k \in K$, we have

$$
\begin{aligned}
&u = 2(\ln F)_x,\\
&F = \sum_{m=1}^{N_1} \varepsilon_m \mathrm{e}^{a_1 k_m x + a_2 k_m^{-1} y + a_3 k_m^{-1} z + a_4 k_m^3 t + \phi_m}\\
&+ \sum_{m=N_1+1}^{N} \xi_m \mathrm{e}^{a_1 \gamma_m x + \frac{\gamma_m}{a_1\left({\gamma_m}^2+{\sigma_m}^2\right)} y + \frac{a_3 \gamma_m}{{\gamma_m}^2+{\sigma_m}^2} z + \left(\frac{{a_1}^2 {\gamma_m}^3}{a_3} - \frac{3{a_1}^2 \gamma_m {\sigma_m}^2}{a_3}\right) t + \phi_m}\\
&\times \sin\left(a_1 x \sigma_m - \frac{\sigma_m}{a_1\left({\gamma_m}^2+{\sigma_m}^2\right)} y - \frac{a_3 \sigma_m}{{\gamma_m}^2+{\sigma_m}^2} z\right.\\
&\left.+ \left(\frac{3a_1^2 \gamma_m^2 \sigma_m}{a_3} - \frac{a_1^2 \sigma_m^3}{a_3}\right) t\right).
\end{aligned}
\tag{42}
$$

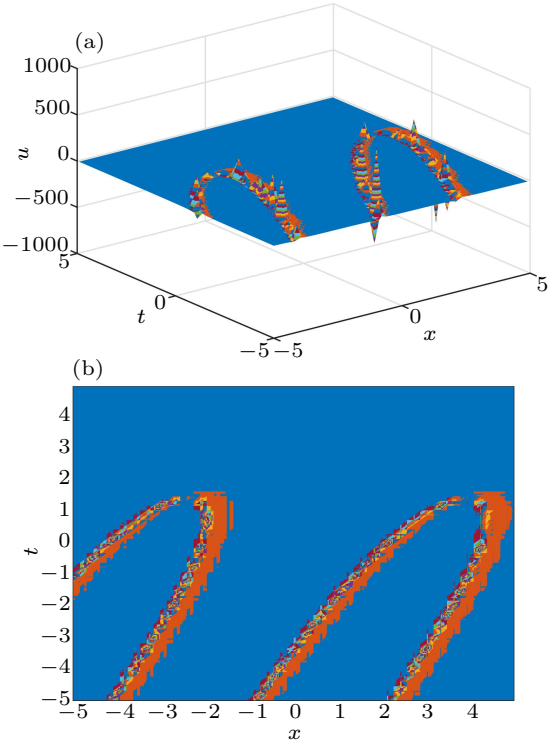


Fig. 1. Characteristics of the resonant multiple wave solution (47) with $N_1 = 1,\ N = 3,\ a_1 = a_4 = 1,\ \phi_1 = \phi_2 = \phi_3 = 0,\ k_1 = 1,\ \varepsilon_1 = 1,\ \xi_2 = \xi_3 = 1,\ \gamma_2 = \gamma_3 = 1,\ \sigma_2 = \sigma_3 = 1,\ y = 0,\ z = 0$. (a) 3D-plot (b) 2D-density plot.

Furthermore, we can obtain the following mixed-type function solutions like complexiton solutions

$$
\begin{aligned}
&u = 2(\ln F)_x,\\
&F = \sum_{m=1}^{N_1} \varepsilon_m \mathrm{e}^{a_1 k_m x + a_2 k_m^{-1} y + a_3 k_m^{-1} z + a_4 k_m^3 t + \phi_m}\\
&+ \sum_{m=N_1+1}^{N} \xi_m \mathrm{e}^{a_1 \gamma_m x + \frac{\gamma_m}{a_1\left({\gamma_m}^2+{\sigma_m}^2\right)} y + \frac{a_3 \gamma_m}{{\gamma_m}^2+{\sigma_m}^2} z + \left(\frac{{a_1}^2 {\gamma_m}^3}{a_3} - \frac{3{a_1}^2 \gamma_m {\sigma_m}^2}{a_3}\right) t + \phi_m}\\
&\times \left[\xi_{1m} \cos\left(a_1 x \sigma_m - \frac{\sigma_m}{a_1\left({\gamma_m}^2+{\sigma_m}^2\right)} y - \frac{a_3 \sigma_m}{{\gamma_m}^2+{\sigma_m}^2} z\right.\right.\\
&\left.+ \left(\frac{3a_1^2 \gamma_m^2 \sigma_m}{a_3} - \frac{a_1^2 \sigma_m^3}{a_3}\right) t\right) + \xi_{2m} \sin\left(a_1 x \sigma_m - \frac{\sigma_m}{a_1\left({\gamma_m}^2+{\sigma_m}^2\right)} y\right.\\
&\left.\left.- \frac{a_3 \sigma_m}{{\gamma_m}^2+{\sigma_m}^2} z + \left(\frac{3a_1^2 \gamma_m^2 \sigma_m}{a_3} - \frac{a_1^2 \sigma_m^3}{a_3}\right) t\right)\right],
\end{aligned}
\tag{43}
$$

where ξ_{1m} and ξ_{2m} are constants.

Case (ii)

$$
\{a_2 = \frac{1}{a_1},\quad a_3 = \frac{a_1^2}{a_4}\},
\tag{44}
$$

where a_1 and a_3 are free parameters.

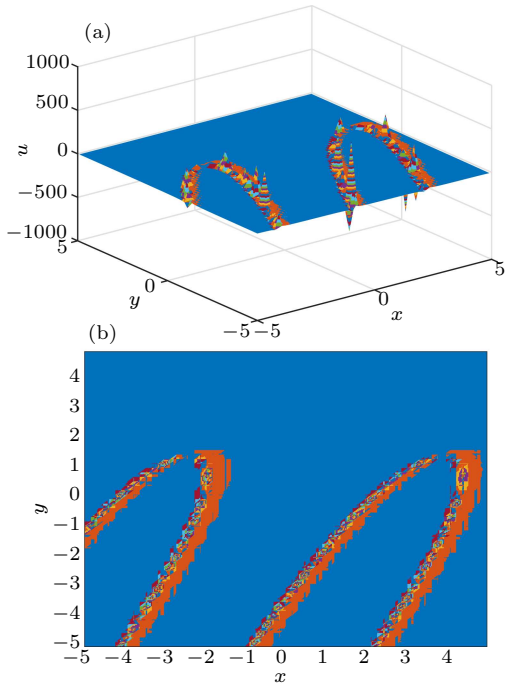


Fig. 2. Characteristics of the resonant multiple wave solution (47) with $N_1 = 1,\ N = 3,\ a_1 = a_4 = 1,\ \phi_1 = \phi_2 = \phi_3 = 0,\ k_1 = 1,\ \varepsilon_1 = 1,\ \xi_2 = \xi_3 = 1,\ \gamma_2 = \gamma_3 = 1,\ \sigma_2 = \sigma_3 = 1,\ t = 0,\ z = 0$. (a) 3D-plot (b) 2D-density plot.

Suppose that $N_1, N_2 \in N,\ N = N_1 + N_2$. We have the N-wave variables $\varpi_m = a_1 k_m x + a_2 k_m^{-1} y + a_3 k_m^3 z + a_4 k_m^{-1} t + \phi_m, 1 \le m \le N$ with parameters $k_m,\ 1 \le m \le N$, defined by Eq. (5). Moreover, for $N_1 < m \le N$, we have

$$
\begin{aligned}
\varpi_m &= a_1 k_m x + a_2 k_m^{-1} y + a_3 k_m^3 z + a_4 k_m^{-1} t + \phi_m\\
&= a_1 \gamma_m x + \frac{\gamma_m}{a_1\left(\gamma_m^2+\sigma_m^2\right)} y + \left(\frac{a_1^2 \gamma_m^3}{a_4} - \frac{3a_1^2 \gamma_m \sigma_m^2}{a_4}\right) z\\
&\quad + \frac{a_4 \gamma_m}{\gamma_m^2+\sigma_m^2} t + \phi_m + i\left(a_1 \sigma_m x - \frac{\sigma_m}{a_1\left(\gamma_m^2+\sigma_m^2\right)} y\right.
\end{aligned}
$$

$$+\Big(\frac{3a_1^2\gamma_m^2\sigma_m}{a_4}-\frac{a_1^2\sigma_m^3}{a_4}\Big)z-\frac{a_4\sigma_m}{\gamma_m^2+\sigma_m^2}t\Big). \tag{45}$$

Using Theorem 2, any linear combination of exponential and trigonometric waves reads

$$\begin{aligned}
&u=2(\ln F)_x,\\
&F=\sum_{m=1}^{N_1}\varepsilon_m \mathrm{e}^{a_1k_mx+a_2k_m^{-1}y+a_3k_m^3z+a_4k_m^{-1}t+\phi_m}\\
&+\sum_{m=N_1+1}^{N}\xi_m \mathrm{e}^{a_1\gamma_mx+\frac{\gamma_m}{a_1(\gamma_m^2+\sigma_m^2)}y+(\frac{a_1^2\gamma_m^3}{a_4}-\frac{3a_1^2\gamma_m\sigma_m^2}{a_4})z+\frac{a_4\gamma_m}{\gamma_m^2+\sigma_m^2}t+\phi_m}\\
&\times\cos\Big(\Big(a_1\sigma_mx-\frac{\sigma_m}{a_1(\gamma_m^2+\sigma_m^2)}y+\Big(\frac{3a_1^2\gamma_m^2\sigma_m}{a_4}-\frac{a_1^2\sigma_m^3}{a_4}\Big)z\\
&-\frac{a_4\sigma_m}{\gamma_m^2+\sigma_m^2}t\Big)+\theta_m\Big).
\end{aligned} \tag{46}$$

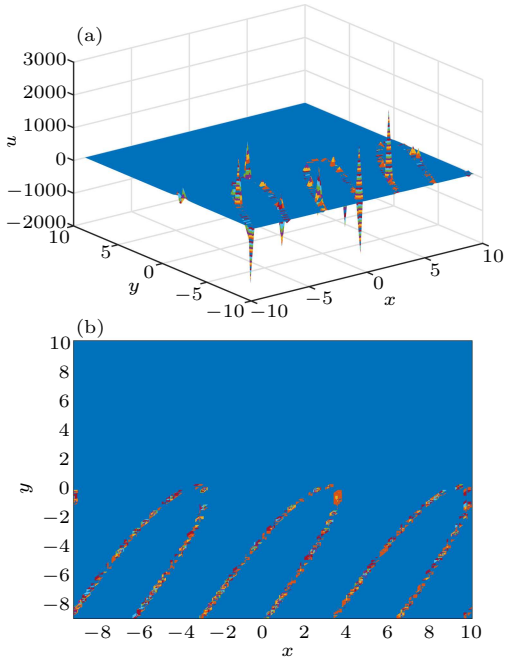


Fig. 3. Characteristics of the resonant multiple wave solution (47) with $N_1=2$, $N=4$, $a_1=a_4=1$, $\phi_1=\phi_2=\phi_3=\phi_4=0$, $k_1=k_2=1$, $\varepsilon_1=\varepsilon_2=1$, $\xi_2=\xi_3=\xi_4=1$, $\gamma_2=\gamma_3=\gamma_4=1$, $\sigma_2=\sigma_3=\sigma=1$, $y=0$, $z=0$. (a) 3D-plot (b) 2D-density plot.

Taking $\theta_m=0$, we have

$$\begin{aligned}
&u=2(\ln F)_x,\\
&F=\sum_{m=1}^{N_1}\varepsilon_m \mathrm{e}^{a_1k_mx+a_2k_m^{-1}y+a_3k_m^3z+a_4k_m^{-1}t+\phi_m}\\
&+\sum_{m=N_1+1}^{N}\xi_m \mathrm{e}^{a_1\gamma_mx+\frac{\gamma_m}{a_1(\gamma_m^2+\sigma_m^2)}y+(\frac{a_1^2\gamma_m^3}{a_4}-\frac{3a_1^2\gamma_m\sigma_m^2}{a_4})z+\frac{a_4\gamma_m}{\gamma_m^2+\sigma_m^2}t+\phi_m}\\
&\cdot\cos\Big(a_1\sigma_mx-\frac{\sigma_m}{a_1(\gamma_m^2+\sigma_m^2)}y+\Big(\frac{3a_1^2\gamma_m^2\sigma_m}{a_4}-\frac{a_1^2\sigma_m^3}{a_4}\Big)z\\
&-\frac{a_4\sigma_m}{\gamma_m^2+\sigma_m^2}t\Big).
\end{aligned} \tag{47}$$

Taking $\theta_m=2k\pi+\pi/2,\ k\in K$, we have

$$\begin{aligned}
&u=2(\ln F)_x,\\
&F=\sum_{m=1}^{N_1}\varepsilon_m \mathrm{e}^{a_1k_mx+a_2k_m^{-1}y+a_3k_m^3z+a_4k_m^{-1}t+\phi_m}\\
&+\sum_{m=N_1+1}^{N}\xi_m \mathrm{e}^{a_1\gamma_mx+\frac{\gamma_m}{a_1(\gamma_m^2+\sigma_m^2)}y+(\frac{a_1^2\gamma_m^3}{a_4}-\frac{3a_1^2\gamma_m\sigma_m^2}{a_4})z+\frac{a_4\gamma_m}{\gamma_m^2+\sigma_m^2}t+\phi_m}\\
&\times\sin\Big(a_1\sigma_mx-\frac{\sigma_m}{a_1(\gamma_m^2+\sigma_m^2)}y+\Big(\frac{3a_1^2\gamma_m^2\sigma_m}{a_4}-\frac{a_1^2\sigma_m^3}{a_4}\Big)z\\
&-\frac{a_4\sigma_m}{\gamma_m^2+\sigma_m^2}t\Big).
\end{aligned} \tag{48}$$

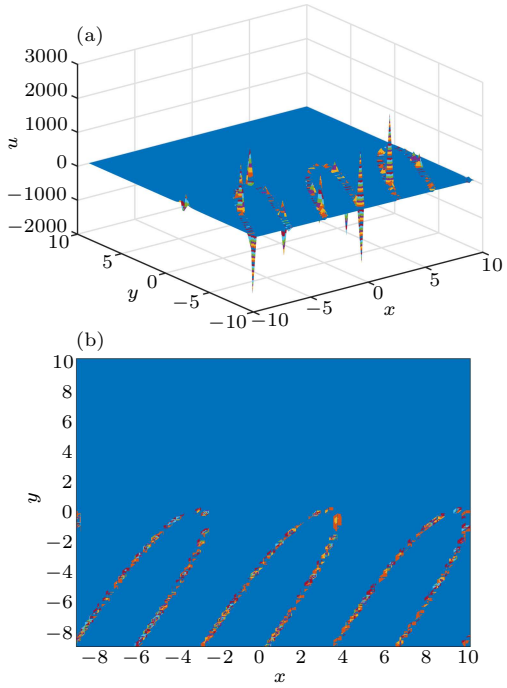


Fig. 4. Characteristics of the resonant multiple wave solution (47) with $N_1=2$, $N=4$, $a_1=a_4=1$, $\phi_1=\phi_2=\phi_3=\phi_4=0$, $k_1=k_2=1$, $\varepsilon_1=\varepsilon_2=1$, $\xi_2=\xi_3=\xi_4=1$, $\gamma_2=\gamma_3=\gamma_4=1$, $\sigma_2=\sigma_3=\sigma=1$, $t=0$, $z=0$. (a) 3D-plot (b) 2D-density plot.

Furthermore, we can obtain the following mixed type function solutions like complexiton solutions

$$\begin{aligned}
&u=2(\ln F)_x,\\
&F=\sum_{m=1}^{N_1}\varepsilon_m \mathrm{e}^{a_1k_mx+a_2k_m^{-1}y+a_3k_m^3z+a_4k_m^{-1}t+\phi_m}\\
&+\sum_{m=N_1+1}^{N}\xi_m \mathrm{e}^{a_1\gamma_mx+\frac{\gamma_m}{a_1(\gamma_m^2+\sigma_m^2)}y+(\frac{a_1^2\gamma_m^3}{a_4}-\frac{3a_1^2\gamma_m\sigma_m^2}{a_4})z+\frac{a_4\gamma_m}{\gamma_m^2+\sigma_m^2}t+\phi_m}\\
&\times\Big[\xi_{1m}\cos\Big(a_1\sigma_mx-\frac{\sigma_m}{a_1(\gamma_m^2+\sigma_m^2)}y+\Big(\frac{3a_1^2\gamma_m^2\sigma_m}{a_4}-\frac{a_1^2\sigma_m^3}{a_4}\Big)z\\
&-\frac{a_4\sigma_m}{\gamma_m^2+\sigma_m^2}t\Big)+\xi_{2m}\sin\Big(a_1\sigma_mx-\frac{\sigma_m}{a_1(\gamma_m^2+\sigma_m^2)}y\\
&+\Big(\frac{3a_1^2\gamma_m^2\sigma_m}{a_4}-\frac{a_1^2\sigma_m^3}{a_4}\Big)z-\frac{a_4\sigma_m}{\gamma_m^2+\sigma_m^2}t\Big)\Big],
\end{aligned} \tag{49}$$

where ξ_{1m} and ξ_{2m} are constants.

In order to better understand the resonant multiple wave solution, we display them by 3D-plot in Figs. 1–4.

3. Conclusions

In summary, resonant multiple-wave solutions including soliton and complexion type solutions have been generated for a generalized (3+1)-dimensional KP equation with the aid of the linear superposition principle in complex field sense. These type solutions could help us better understand resonant nonlinear phenomena. Meanwhile, we would like to point that this paper can be regarded as an extension to the article.[29] Furthermore, for other types of special solutions of soliton equations appearing in this paper, we would further consider them in the future work.

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