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Analytic solutions for generalized $\mathcal{PT}$-symmetric Rabi models*

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We theoretically investigate the exact solutions for generalized parity–time($\mathcal{PT}$)-reversal-symmetric Rabi models driven by external fields with monochromatic periodic, linear, and parabolic forms, respectively. The corresponding exact solutions are presented in terms of the confluent Heun equations without any approximation. In principle, the analytic solutions derived here are valid in the whole parameter space. Such a kind of study may offer potential coherent control schemes of the $\mathcal{PT}$-symmetric two-level systems.

Keywords: Rabi model, $\mathcal{PT}$ symmetry, population dynamics

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1. Introduction

In the past twenty years, the special non-Hermitian systems with $\mathcal{PT}$ symmetry have received widespread attention in several research fields since they were proposed firstly by Bender.[1,2] Such systems may exhibit exotic properties in certain conditions, for example, the non-Hermitian symmetric system has real eigenvalues. In order to carry out experimental research, one has to utilize other platforms to simulate $\mathcal{PT}$-symmetric systems because they do not exist in nature. The platforms based on optical waveguide or optical waveguide arrays with balanced gain and loss have become an important alternative candidate. As we know, under the paraxial approximation the equation of a laser beam propagating in a waveguide can be approximated as the Schrödinger equation which is the same as the equation of quantum mechanics. This induces that the optical platforms take a natural advantage in quantum simulations. If the laser beam propagates along the z axis and the complex refractive index $n(x,z)$ of the media satisfies the following $\mathcal{PT}$-symmetry distribution: $n^*(x,z) = n(-x,-z)$,[3–8] a $\mathcal{PT}$-symmetric Schrödinger equation is then achieved. A series of optical experiments to investigate the $\mathcal{PT}$-symmetric quantum systems have been performed.[9,10]

The two-level system driven by an external field[11] is one of basic models in quantum optics, which is usually used to describe the coherent interaction between classical radiation and matter. For a weak monochromatic driving field and in the resonance case, the population on these two levels is to transfer with sinusoidal periodicity, which is termed as Rabi oscillations. Such the driven two-level system is well known as a Rabi model, which has been applied in different fields of physics. It should be pointed out that the two-level model was not only used to describe the Rabi oscillation, but also to depict other physical quantities equivalent to population, such as the energy oscillation in two coupled optical waveguides.[12] As the field strength increases, periodic level avoided crossings are associated with the process of population transfer.[13] That phenomenon happens only in the strong coupling regime. Although the Rabi model is very simple in form, it is very difficult to obtain its analytical solution. Many efforts are devoted to obtaining the analytic solutions for the Hermitian two-level models.[14–23]

The two-level model mentioned above is Hermitian. With the in-depth study of non-Hermitian quantum systems, more and more researchers are turning their attention to non-Hermitian two-level systems. The characteristics of the time-periodic-driven two-level system with $\mathcal{PT}$ symmetry were studied and the corresponding phase diagram as a function of the perturbation strength and frequency was obtained.[24–29] The time evolution of the two-level system with a time-dependent non-Hermitian Hamiltonian without $\mathcal{PT}$ symmetry was investigated and showed that the system was in general unstable with exponential growth or decay.[30] Recently, the analytical results for the parity–time-symmetric two-level system under synchronous combined modulations was studied.[31] To the best of our knowledge, an exact analytic solution for the $\mathcal{PT}$-symmetric Rabi model is still lacking up to now. In this article, we will focus on how to obtain the analytic solutions for $\mathcal{PT}$-symmetric Rabi models. These

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analytic solutions are beneficial to mastering the dynamic behaviors of non-Hermitian two-level systems.

The rest of the paper is organized as follows. At first, the mathematical model of a generalized Rabi model is formulated. The following three parts present three different driving-field cases, such as monochromatic periodic, linear, and parabolic forms, respectively. Here the corresponding analytic solutions are shown in terms of the confluent Heun functions[17–20,32,33] as well. Finally, a conclusion is given at the last section.

2. Generalized $\mathcal{PT}$-symmetric Rabi model

Our discussion starts from the following Hamiltonian

$$H = -\mathrm{i}2\lambda\cos(\omega z)\sigma_z + \frac{\omega_0}{2}\sigma_x. \tag{1}$$

Here σ_x, σ_y, and σ_z are the Pauli spin matrices. In principle, such a system can be constructed by parallel dimer waveguides where the distribution of gain (or dissipation) is modulated with the amplitude 2λ and period ω and satisfies $\mathcal{PT}$ symmetry. The laser beam is set to propagate in this dimer waveguide and along the z direction. The $\omega_0/2$ denotes the coupling strength (corresponding to the tunneling coefficient) and depends on the distance between the parallel waveguides.

Introducing the unitary rotating matrix

$$U = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

this unitary transformation yields[24]

$$H' = UHU^{\dagger} = \frac{\omega_0}{2}\sigma_z + 2\,\mathrm{i}\lambda\cos(\omega z)\sigma_x. \tag{2}$$

This equation might be named as the standard $\mathcal{PT}$-symmetric Rabi model and its form is similar to the standard Hermitian Rabi model.[34] In the above form, the periodic 'driving field' $[2\mathrm{i}\lambda\cos(\omega z)]$ is considered. The corresponding instantaneous eigenvalues are given by the diagonal form of Eq. (2)

$$\varepsilon_{\pm}(z) = \pm\sqrt{\left(\frac{\omega_0}{2}\right)^2 - [2\lambda\cos(\omega z)]^2}. \tag{3}$$

Inspecting Eq. (3), we can find that the eigenvalues with the respect to z are always real when $\lambda \leq \omega_0/4$, which is corresponding to the unbroken $\mathcal{PT}$ symmetric phase. On the contrary, when $\lambda > \omega_0/4$, the system has complex eigenvalues and is in the broken $\mathcal{PT}$ symmetric phase. As an illustration, figure 1 shows the instantaneous eigenvalues expressed as the above formula under different parameter conditions. All the parameters of the simulations in Fig. 1 have been given out in its caption. The solid and dash lines correspond to the cases for the real and imaginary parts of the instantaneous eigenvalues $\varepsilon(z)$, respectively. It is easy to get that the imaginary parts of the instantaneous eigenvalues are not equal to 0 when $\lambda > \omega_0/4$ (see Fig. 1(b)). For such a case, the dynamical behaviors with exponential growth (or decay) appear. The cases for different $\mathcal{PT}$ symmetric phases will be investigated in the following parts.

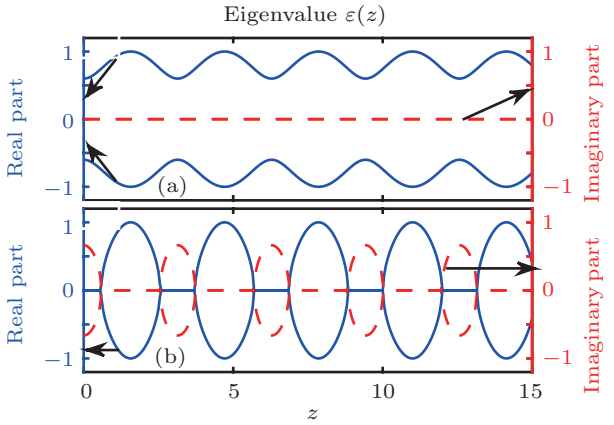


Fig. 1. The real (solid line) and imaginary (dash line) parts of the instantaneous eigenvalues $\varepsilon(z)$ with respect to z for different cases. (a) $\omega_0 = 2.0$; $\omega = 1.0$, $\lambda = 0.4$, and $z \in [0.0, 15.0]$; (b) the parameters are the same to the former except for $\lambda = 0.6$. Panels (a) and (b) belong to the unbroken and broken $\mathcal{PT}$ symmetric phases, respectively.

In fact, the criterion of $\mathcal{PT}$ symmetric phase is dependent on the carrier wave frequency ω of the driving field. When $\lambda \ll \omega_0/4$ (the perturbation theory is valid here) and $\omega_0 \simeq \omega$ (corresponding to single-photon resonance), the Hamiltonian (2) can be approximated and read as

$$H'_{\mathrm{app}} = \begin{pmatrix} \omega_0/2 & \mathrm{i}\lambda \\ \mathrm{i}\lambda & -\omega_0/2+\omega \end{pmatrix}, \tag{4}$$

where the Salwen's perturbation theory[24,35] and rotating-wave approximation are used. At the same time, the above Hamiltonian is derived on account of the lowest order approximation. It is easy to get the corresponding eigenvalues:

$$\varepsilon_{\pm} = \frac{\omega \pm \sqrt{(\omega-\omega_0)^2 - 4\lambda^2}}{2}.$$

Inspecting the eigenvalue expression, one sees that the above two-level system is in the broken $\mathcal{PT}$ symmetric phase when $|\lambda| > |\omega - \omega_0|/2$. This indicates that the above system driven by an external field with arbitrary strength is always in the broken $\mathcal{PT}$ symmetric phase when $\omega = \omega_0$. Our numerical simulation results also show that the criterion depends on ω. Whether or not the instantaneous eigenvalues are real cannot be used to arrive this point. In order to get the more exact criterion, we derive an explicit analytical approximated Hamiltonian for the Bloch–Siegert shift[36–38] of the Rabi model which is valid over the entire driving-strength range.[39]

Utilizing the transformation $|\Psi'(z)\rangle = \exp[S(z)]|\Psi'(z)\rangle$ with $S(z) = -(2\lambda/\omega)\xi\cos(\omega z)\sigma_x$, one obtains a new Hamiltonian as follows:

$$H_s' = \mathrm{e}^{S(z)} H' \mathrm{e}^{-S(z)} - \mathrm{i}\,\mathrm{e}^{S(z)} \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{-S(z)}$$
$$= \frac{\omega_0}{2} \cos\left[\frac{4\lambda\,\mathrm{i}}{\omega}\xi\cos(\omega z)\right]\sigma_z + \frac{\omega_0}{2}\sin\left[\frac{4\lambda\,\mathrm{i}}{\omega}\xi\cos(\omega z)\right]\sigma_y$$
$$+ 2\mathrm{i}\lambda(1-\xi)\cos(\omega z)\sigma_x. \tag{5}$$

Here parameter ξ is to be determined. Taking advantage of the identities

$$\cos(\zeta\sin\theta) = \mathrm{J}_0(\zeta) + 2\sum_{n=1}^{\infty}\mathrm{J}_{2n}(\zeta)\cos(2n\theta),$$
$$\sin(\zeta\sin\theta) = 2\sum_{n=1}^{\infty}\mathrm{J}_{2n-1}(\zeta)\sin[(2n-1)\theta]$$

with the n-th-order Bessel function of the first kind $\mathrm{J}_n(\cdot)$ and $\zeta = (4\lambda\,\mathrm{i}/\omega)\xi$ and neglecting the higher-order Bessel functions, we then have the following approximated expression:

$$H_s' \simeq \frac{\omega_0}{2}\mathrm{I}_0\left(\frac{4\lambda}{\omega}\xi\right)\sigma_z + \mathrm{i}\omega_0\mathrm{I}_1\left(\frac{4\lambda}{\omega}\xi\right)\sin(\omega z)\sigma_y$$
$$+ 2\,\mathrm{i}\lambda(1-\xi)\cos(\omega z)\sigma_x. \tag{6}$$

Here we have used $\mathrm{J}_n(\mathrm{i}\zeta) = \mathrm{i}^n\mathrm{I}_n(\zeta)$ with the n-th-order Bessel function of the second kind. The above approximation is reasonably since the higher-order harmonic terms introduced by the higher-order Bessel functions can be negligible over a wide range of parameter space.[40] If setting $\omega_0\mathrm{I}_1[(4\lambda/\omega)\xi] = 2\lambda(1-\xi)$, the above Hamiltonian is simplified as

$$H_s' \simeq \frac{\omega_0}{2}\mathrm{I}_0\left(\frac{4\lambda}{\omega}\xi\right)\sigma_z + 2\,\mathrm{i}\lambda(1-\xi)\exp(\mathrm{i}\omega z)\sigma_x.$$

So we may obtain the following criterion of the unbroken $\mathcal{PT}$ symmetric phase:

$$\left|\frac{\omega_0}{2}\mathrm{I}_0\left(\frac{4\lambda}{\omega}\xi\right) - \frac{\omega}{2}\right| > |2\lambda(1-\xi)|. \tag{7}$$

It needs to be emphasized that the constraint condition $\omega_0\mathrm{I}_1[(4\lambda/\omega)\xi] = 2\lambda(1-\xi)$ should be considered at the same time. Hereto we have obtained the criterion of the unbroken $\mathcal{PT}$ symmetric phase.

The model (1) may be generalized as the following expressions

$$H_0' = \mathrm{i}f(z)\sigma_x + \frac{\omega_0}{2}\sigma_z, \tag{8a}$$
$$H_0 = U^{\dagger}H_0'U = -\mathrm{i}f(z)\sigma_z + \frac{\omega_0}{2}\sigma_x. \tag{8b}$$

The above equations may be called as generalized $\mathcal{PT}$-symmetric Rabi models. These generalized models may be constructed in experiment with current optical technologies. The parameter $\omega_0/2$ can be controlled by setting the distance between adjacent waveguides. The $f(z)$ corresponds to the gain (or loss) coefficients of each waveguide, which can be achieved by proper doping.[10] If such special waveguids are well designed and arranged, one may achieve the generalized $\mathcal{PT}$-symmetric Rabi model. In this article, we have not considered the effects of optical nonlinearity. With the increase of laser intensity in optical waveguids, nonlinear optical effects should be included, such as Kerr effect. The $\mathcal{PT}$ symmetric phase transition for such cases will rely on nonlinear optical effects.

With the help of Hamiltonian (8a), the equations of motion for the amplitudes c_1 and c_2 can be derived from the Shrödinger equation $\mathrm{i}|\dot{\Psi}\rangle = H_0|\Psi\rangle$ where $|\Psi\rangle = (c_2, c_1)^{\mathrm{T}}$:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z}c_2 = -\mathrm{i}f(z)c_2 + \frac{\omega_0}{2}c_1, \tag{9a}$$
$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z}c_1 = \frac{\omega_0}{2}c_2 + \mathrm{i}f(z)c_1. \tag{9b}$$

Introducing the transformation $C_2 = \exp[\delta/2]c_2$ and $C_1 = \exp[-\delta/2]c_1$ with $\delta(z) = \int_0^z 2f(z')\mathrm{d}z'$, the time evolution of the population amplitudes $C_{1,2}$ can be obtained ($g = \omega_0/2$)

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z}C_2 = g\exp[\delta]C_1, \tag{10a}$$
$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z}C_1 = g\exp[-\delta]C_2. \tag{10b}$$

By means of elimination of C_1 (or C_2), the second-order differential equation for C_2 (or C_1) can be derived

$$\frac{\mathrm{d}^2C_2}{\mathrm{d}z^2} + \left(-\delta_z - \frac{g_z}{g}\right)\frac{\mathrm{d}C_2}{\mathrm{d}z} + g^2C_2 = 0, \tag{11a}$$
$$\frac{\mathrm{d}^2C_1}{\mathrm{d}z^2} + \left(\delta_z - \frac{g_z}{g}\right)\frac{\mathrm{d}C_1}{\mathrm{d}z} + g^2C_1 = 0, \tag{11b}$$

where and hereafter the lowercase Latin index denotes the differentiation with respect to the corresponding variable.

By assuming that $C_2(z)$ has the following form $C_2^*(z)$ and introducing a set of auxiliary variables t for $g^*(t)$ and $\delta^*(t)$, one can solve Eq. (11a) for the field-configuration defined as

$$g(z) = g^*(t)\frac{\mathrm{d}t}{\mathrm{d}z}, \quad \delta_z(z) = \delta_t^*(t)\frac{\mathrm{d}t}{\mathrm{d}z} \tag{12}$$

for arbitrary complex-valued function $t(z)$.[41] By using the transformation of the independent variable $C_2 = \varphi(t)u(t)$ together with expression (12), one may see that equation (11a) is reduced to the following equation for the new dependent variable $u(t)$:

$$u_{tt} + \left(2\frac{\varphi_t}{\varphi} - \delta_t^* - \frac{g_t^*}{g^*}\right)u_t + \left[\frac{\varphi_{tt}}{\varphi} + \left(-\delta_t^* - \frac{g_t^*}{g^*}\right)\frac{\varphi_t}{\varphi} + g^{*2}\right]u = 0. \tag{13}$$

The analytic solutions of the above equation can be achieved by recognizing whether a given equation is of the confluent Heun function type.[32,33] The confluent Heun equations under consideration can be written in the following form

$$P(t)u_{tt}+\left(a_2+a_1t+a_0t^2\right)u_t+(b_1t-b_0)u=0, \tag{14}$$

where $P(t)=t^2$, t, and 1 for the double-, bi-, and tri-confluent Heun equations, respectively. It is easy to obtain that equation (13) has the one type of the above confluent Heun equations if

$$2\frac{\varphi_t}{\varphi}-\delta_t^*-\frac{g_t^*}{g^*}=\frac{a_2+a_1t+a_0t^2}{P(t)}, \tag{15a}$$

$$\frac{\varphi_{tt}}{\varphi}+\left(-\delta_t^*-\frac{g_t^*}{g^*}\right)\frac{\varphi_t}{\varphi}+g^{*2}=\frac{b_1t-b_0}{P(t)}. \tag{15b}$$

Though these are a set of over-determined equations for $g^*(t)$, $\delta^*(t)$, and $\varphi(t)$, the corresponding solutions are not known. In principle, a serial of special particular solutions can be obtained by starting from some guessed ansatz. Our main aim is to formulate a general theory which would overcome the deficiencies of the analytic solutions for the generalized $\mathcal{PT}$-symmetric model. It should be emphasized that C_1 has a similar analytic solution for C_2. According to Eqs. (11a) and (11b), one can find that the solution for C_1 can be achieved by changing δ_z to be $-\delta_z$. Because of this reason, only the analytic solutions for C_2 is expressed. Hereto we have derived the general equations of motion for the population amplitudes $C_{1,2}$. In the following sections, we will present several exact solutions with the help of the above method.

3. Three different driving-field cases

In this section, we will present several exact solutions with the help of the above method.

3.1. The case of periodic driving fields

Firstly we consider the dynamics of the $\mathcal{PT}$-symmetric Rabi model driven by periodic external fields. For this case, the driving field is set as $\mathrm{i}\,2\lambda\cos(\omega z)$. In this section, we will derive the corresponding analytic solutions.

By setting $P(t)=t^2$ and assuming that

$$2\frac{\varphi_t}{\varphi}-\delta_t^*-\frac{g_t^*}{g^*}=\frac{a_2}{t^2}+\frac{a_1}{t}+a_0, \tag{16a}$$

$$\frac{\varphi_{tt}}{\varphi}+\left(-\delta_t^*-\frac{g_t^*}{g^*}\right)\frac{\varphi_t}{\varphi}+g^{*2}=\frac{b_1}{t}-\frac{b_0}{t^2}, \tag{16b}$$

one can obtain the double-confluent Heun equation as follows [see Eqs. (15a) and (15b)]:

$$u_{tt}+\left(\frac{a_2}{t^2}+\frac{a_1}{t}+a_0\right)u_z+\frac{b_1t-b_0}{t^2}u=0. \tag{17}$$

Here we suggest their particular solutions in the following form: $\varphi=t^{c_1}\,\mathrm{e}^{c_0t-c_2/t}$, $g^*=g_0^*t^k$, and $\delta_t^*=\delta_2/t^2+\delta_1/t+\delta_0$. It is easy to check that k is an integer or half-integer obeying the inequalities $0\leq 2k+4\leq 4$.[41,42] This yields that $k=-2$, $-3/2$, -1, $-1/2$, 0. According to Eqs. (12), the actual field configurations for the periodic-driven two-level system can be written in terms of the double-confluent Heun function:

$$g(z)=g_0^*t^k\frac{\mathrm{d}t}{\mathrm{d}z},$$

$$\delta_t(z)=\left(\frac{\delta_2}{t^2}+\frac{\delta_1}{t}+\delta_0\right)\frac{\mathrm{d}t}{\mathrm{d}z},$$

where the parameters g_0^* and $\delta_{0,1,2}$ are complex constants and should be chosen so that the functions $g(z)$ and $\delta(z)$ are real for the chosen complex-valued $t(z)$. The solution of such a two-level system (11a) is explicitly written as

$$C_2=t^{c_1}\,\mathrm{e}^{c_0t-c_2/t}H_{\mathrm{D}}(a_2,a_1,a_0;b_1,b_0;t), \tag{18}$$

where the parameters a_2, a_1, a_0, b_1, b_0 are given as

$$\begin{aligned}
&a_0=2c_0-\delta_0,\quad a_1=2c_1-k+\delta_1,\\
&a_2=2c_2-\delta_2,\quad c_0^2-c_0\delta_0+Q^{(4)}(0)/4!=0,\\
&0=-c_1\delta_2-c_2(2+k+\delta_1-2c_1)+Q'(0),\\
&0=c_2^2-c_2\delta_2+Q(0),\\
&b_0=c_1(1+k+\delta_1-c_1)-c_0a_2+c_2\delta_0-\frac{Q''(0)}{2},\\
&b_1=c_1a_0-c_0(k+\delta_1)+Q^{(3)}(0)/3!.
\end{aligned}$$

Here $Q(t)=g_0^{*2}t^{2k+4}$.

Setting $f(z)=2\lambda\cos(\omega z)$ and according to the coefficient expressions in Eqs. (10a), one gets the following expression:

$$g=\frac{\omega_0}{2}=g_0^*t^k\frac{\mathrm{d}t}{\mathrm{d}z},$$

$$\delta(z)=\int_0^z\left[4\lambda\cos(\omega z')\right]\mathrm{d}z',$$

$$\delta_z=4\lambda\cos(\omega z)=\left(\frac{\delta_2}{t^2}+\frac{\delta_1}{t}+\delta_0\right)\frac{\mathrm{d}t}{\mathrm{d}z}.$$

One may identify that equation (11a) is equal to the double-confluent Heun equation if setting $t(z)=\mathrm{e}^{\mathrm{i}\omega z}$, $k=-1$, $a_2=-\delta_2$, $a_1=1-\delta_1$, $a_0=-\delta_0$, $b_1=0$, $b_0=-g_0^{*2}$, $c_{0,1,2}=0$, $g_0^*=-\mathrm{i}(\omega_0/2\omega)$, $\delta_1=0$, and $\delta_0=\delta_2=-\mathrm{i}\,2(\lambda/\omega)$. The analytic solutions for the case of periodic driving fields then are read as

$$C_2=H_{\mathrm{D}}\left(\mathrm{i}\frac{2\lambda}{\omega},1,\mathrm{i}\frac{2\lambda}{\omega};0,\frac{\omega_0^2}{4\omega^2};\mathrm{e}^{\mathrm{i}\omega z}\right). \tag{19}$$

In the same way, C_1 can be obtained as follows:

$$C_1=H_{\mathrm{D}}\left(-\mathrm{i}\frac{2\lambda}{\omega},1,\mathrm{i}\frac{2\lambda}{\omega};0,\frac{\omega_0^2}{4\omega^2};\mathrm{e}^{\mathrm{i}\omega z}\right). \tag{20}$$

It is worth noticing that there is no approximation used. Our analytic solutions are valid for the population dynamics of the general periodic-driven system with arbitrary strength. As an example, the population dynamics driven by a periodic field is shown in Fig. 2. In the caption, we have shown the parameters in our simulations. For the case of Fig. 2(a), the two-level system is in the unbroken $\mathcal{PT}$ symmetric phase because of $\lambda < \omega_0/4$. One obtains that the population is oscillating as expected. The system would exhibit the characteristics of the broken $\mathcal{PT}$ symmetric phase when $\lambda > \omega_0/4$, which is illustrated in Fig. 2(b). Inspecting that figure, one sees that the population is exponential increasing. Until now, the former discussion focuses on the case of periodic driving fields.

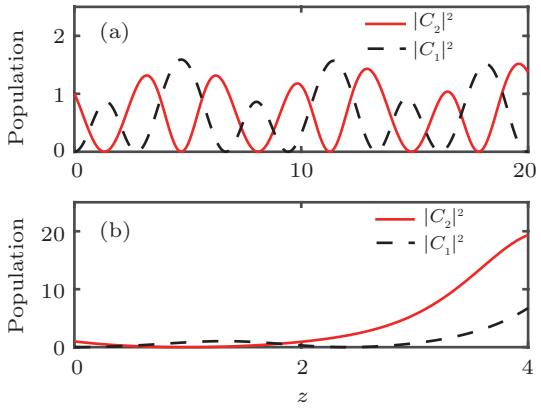


Fig. 2. The evolution of the population $|C_i|^2$ for the $\mathcal{PT}$-symmetric Rabi model driven by a periodic field. (a) $\omega_0 = 2.0$, $\omega = 1.0$, $\lambda = 0.2$, $z_0 = 0$, and $z \in [0.0, 20.0]$; (b) the parameters are the same to the former except for $\lambda = 0.6$ and $z \in [0.0, 4.0]$. The solid and dash lines correspond to $|C_2|^2$ and $|C_1|^2$, respectively. The cases in panels (a) and (b) belong to the unbroken and broken $\mathcal{PT}$ symmetric phases, respectively.

3.2. The case of linear driving fields

The common two-level model with a time-dependent Hamiltonian where the energy separation of the two states is a linear function of time is used to investigate the population transition dynamics near an avoided crossing. This is the well-known Landau–Zener transition.[43,44] It should be mentioned that the avoided crossing could not happen in the generalized $\mathcal{PT}$-symmetric Rabi model [see the eigenvalue expression (3)]. The aim of this section is to derive the analytic solution of the non-Hermitian two-level system with the driving field [$f(z) = 2\lambda z$] and try to study the dynamical behavior. Such a driving field can be considered as the first-order Taylor series expansion of $\cos(\zeta)$ around $\zeta = \pi/2 + n\pi$ with integer n.

By setting $P(t) = t$, and

$$2\frac{\varphi_t}{\varphi} - \delta_t^* - \frac{g_t^*}{g^*} = \frac{a_2}{t} + a_1 + a_0 t,$$
$$\frac{\varphi_{tt}}{\varphi} + \left(-\delta_t^* - \frac{g_t^*}{g^*}\right)\frac{\varphi_t}{\varphi} + g^{*2} = b_1 - \frac{b_0}{t},$$

one can obtain the double-confluent Heun equation as follows:

$$u_{tt} + \left(\frac{a_2}{t} + a_1 + a_0 t\right)u_t + \frac{b_1 t - b_0}{t}u = 0. \tag{21}$$

Using the same procedure as in the former section, one may obtain the the solutions of these equations in the following form: $\varphi = t^{c_2}\,\mathrm{e}^{c_1 t + c_0 t^2/2}$, $g^* = g_0^* t^k$, and $\delta_t^* = \delta_2/t + \delta_1 + \delta_0 t$. In the same way, one can find that k is an integer or half-integer obeying the inequalities $0 \le 2k+2 \le 4$. This leads to $k = -1$, $-1/2$, 0, $1/2$, and 1. In the light of Eqs. (12), the actual field configurations for the linear two-level system can be written in terms of the bi-confluent Heun function:

$$g(z) = g_0^* t^k \frac{\mathrm{d}t}{\mathrm{d}z},$$
$$\delta_z(z) = \left(\frac{\delta_2}{t} + \delta_1 + \delta_0 t\right)\frac{\mathrm{d}t}{\mathrm{d}z}.$$

Here the parameters g_0^* and $\delta_{0,1,2}$ are complex constants.

The corresponding solution of the two-state systems (11a) is explicitly written as

$$C_2 = t^{c_2}\,\mathrm{e}^{c_1 t + c_0 t^2/2} H_{\mathrm{B}}(a_2, a_1, a_0; b_1, b_0; t), \tag{22}$$

where the parameters a_2, a_1, a_0, b_1, and b_0 are given as

$$a_0 = 2c_0 - \delta_0, \quad a_1 = 2c_1 - \delta_1, \quad a_2 = 2c_2 - k - \delta_2,$$
$$c_0^2 - c_0\delta_0 + Q^{(4)}(0)/4! = 0,$$
$$c_1 a_0 - c_0\delta_1 + Q^{(3)}(0)/3! = 0,$$
$$c_2^2 - c_2\delta_2 - c_2 - c_2 k + Q(0) = 0,$$
$$b_0 = c_2\delta_1 + c_1(k + \delta_2 - 2c_2) - Q'(0),$$
$$b_1 = c_0(1 - k - \delta_2 + 2c_2) + c_1(c_1 - \delta_1) - c_2\delta_0 + Q''(0)/2.$$

Here $Q(t) = g_0^{*2} t^{2k+2}$. According to the coefficient expressions in Eqs. (12), one achieves

$$g = \frac{\omega_0}{2} = g_0^* t^k \frac{\mathrm{d}t}{\mathrm{d}z},$$
$$\delta(z) = \int_0^t \left[4\lambda z'\right]\mathrm{d}z',$$
$$\delta_z = 4\lambda z = \left(\frac{\delta_2}{t} + \delta_1 + \delta_0 t\right)\frac{\mathrm{d}t}{\mathrm{d}z}.$$

One may find that equation (11a) is equal to the bi-confluent Heun equation if setting $t(z) = z$, $k = 0$, $a_2 = -\delta_2$, $a_1 = -\delta_1$, $a_0 = -\delta_0$, $b_1 = g_0^{*2}$, $b_0 = 0$, $c_{0,1,2} = 0$, $g_0^* = \omega_0/2$, $\delta_2 = 0$, $\delta_1 = 0$, and $\delta_0 = 4\lambda$. The analytic solutions for the case of linear driving fields then are read as

$$C_2 = H_{\mathrm{B}}(0, 0, -4\lambda; \omega_0^2/4, 0; z). \tag{23}$$

With the help of the same procedure, C_1 can be obtained as follows:

$$C_1 = H_{\mathrm{B}}(0, 0, 4\lambda; \omega_0^2/4, 0; z). \tag{24}$$

As an illustration, the population dynamics driven by a linear field is shown in Fig. 3. In the caption, the parameters of our simulation have been shown. For the case of Fig. 3, the two-level system is in the unbroken $\mathcal{PT}$ symmetric phase because of $\lambda|z| > \omega_0/4$. According to the parameters used here, the two-level system approaches in the broken $\mathcal{PT}$ symmetric phase when $|z| > 2.5$. At the same time, one can obtain that the driving field is zero when $z = 0$. In the Hermitian two-level system, the level avoiding crossing happens at $z = 0$ and the population varies fast due to the Landau–Zener tunneling. But we cannot observe a similar phenomenon in the case of the non-Hermitian two-level system (see the curve near $z = 0$ in Fig. 3). The level avoiding crossing is the fundamental reason of dynamical tunneling in quantum systems. The rapid change of population does not exist in our $\mathcal{PT}$-symmetric Rabi models because such a phenomenon on level avoiding crossing does not appear.

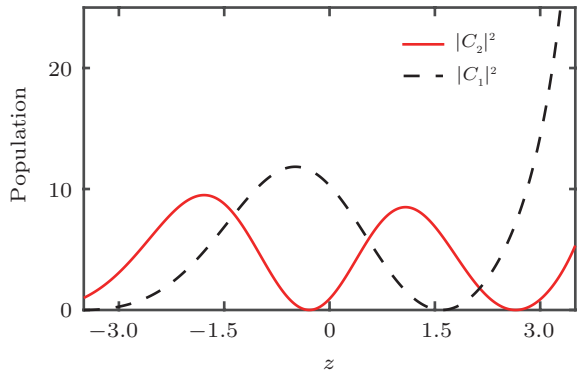


Fig. 3. The evolution of the population $|C_i|^2$ for for the $\mathcal{PT}$-symmetric Rabi model driven by a linear field $[f(z) = 2\lambda z]$. Here the parameters are chosen as the following: $\omega_0 = 2.0$, $\lambda = 0.2$, and $z \in [-3.5, 3.5]$.

3.3. The case of quadratic nonlinear driving fields

In this part, we try to search the analytical solutions for the non-Hermitian two-level systems driven by external fields with a quadratic form $[f(z) = 2\lambda z^2]$. Such a kind of field can be thought of as the second-order Taylor series expansion of the periodic field $\cos(\zeta)$ around $\zeta = n\pi$ with integer n. As in the former case of linear driving fields, there is no avoid crossing.

By setting $P(t) = 1$, the tri-confluent Heun equation can be obtained according to Eq. (14) and written as

$$u_{tt} + \left(a_2 + a_1 t + a_0 t^2\right) u_t + (b_1 t - b_0) u = 0. \tag{25}$$

It is obviously that one can obtain analytic solutions with tri-confluent Heun function forms if the following two equations can be established

$$2\frac{\varphi_t}{\varphi} - \delta_t^* - \frac{g_t^*}{g^*} = a_2 + a_1 t + a_0 t^2, \tag{26a}$$

$$\frac{\varphi_{tt}}{\varphi} + \left(-\delta_t^* - \frac{g_t^*}{g^*}\right)\frac{\varphi_t}{\varphi} + g^{*2} = b_1 t - b_0. \tag{26b}$$

Using the same procedure as that used in the former section, one may find the solutions of these equations in the following form

$$\varphi = \mathrm{e}^{c_2 t + c_1 t^2/2 + c_0 t^3/3}, \quad g^* = g_0^*,$$
$$\delta_t^* = \delta_2 + \delta_1 t + \delta_0 t^2.$$

The field configurations for the solution written in terms of the tri-confluent Heun functions are given as

$$g = g_0^* \frac{\mathrm{d}t}{\mathrm{d}z},$$
$$\delta_z(z) = \left(\delta_2 + \delta_1 t + \delta_0 t^2\right)\frac{\mathrm{d}t}{\mathrm{d}z},$$

where the parameters g_0^* and $\delta_{0,1,2}$ are complex constants.

The analytical solution of the two-state system (11a) is explicitly written as $(c_{2,1,0} = \{0, 0, 0\})$

$$C_2 = H_{\mathrm{T}}(a_2, a_1, a_0; b_1, b_0; t), \tag{27}$$

where the parameters a_2, a_1, a_0, b_1, and b_0 are given as

$$\{a_2, a_1, a_0, b_1, b_0\} = \{-\delta_2, -\delta_1, -\delta_0, 0, -g_0^{*2}\}.$$

In this case, the coefficients in Eqs. (11a) may be expressed as

$$g = \frac{\omega_0}{2} = g_0^* \frac{\mathrm{d}z}{\mathrm{d}t},$$
$$\delta(z) = \int_0^z \left[4\lambda z'^2\right] \mathrm{d}z',$$
$$\delta_z = 4\lambda z^2 = \left(\delta_2 + \delta_1 t + \delta_0 t^2\right)\frac{\mathrm{d}t}{\mathrm{d}z}.$$

One may find that equation (11a) is equal to the tri-confluent Heun equation if setting $t(z) = z$, $k = 0$, $a_2 = -\delta_2$, $a_1 = -\delta_1$, $a_0 = -\delta_0$, $b_1 = g_0^{*2}$, $b_0 = 0$, $c_{0,1,2} = 0$, $g_0^* = \omega_0/2$, $\delta_2 = 0$, $\delta_1 = 0$, and $\delta_0 = 4\lambda$. The solution can be explicitly written as follows:

$$C_2 = H_{\mathrm{T}}\left(0, 0, -4\lambda; 0, -\omega_0^2/4; z\right). \tag{28}$$

By the same procedure, C_1 can be obtained in the following

$$C_1 = H_{\mathrm{T}}\left(0, 0, 4\lambda; 0, -\omega_0^2/4; z\right). \tag{29}$$

The evolution of the population for the $\mathcal{PT}$-symmetric Rabi model driven by a quadratic nonlinear external field is depicted in Fig. 4. As the case in Fig. 3, we cannot observe the avoiding crossing or level-glancing phenomena which exist in the Hermitian two-level systems driven by external fields with parabolic forms.[20,41] So far we have discussed the exact analytic solution of the generalized $\mathcal{PT}$-symmetric Rabi model with different driving fields.

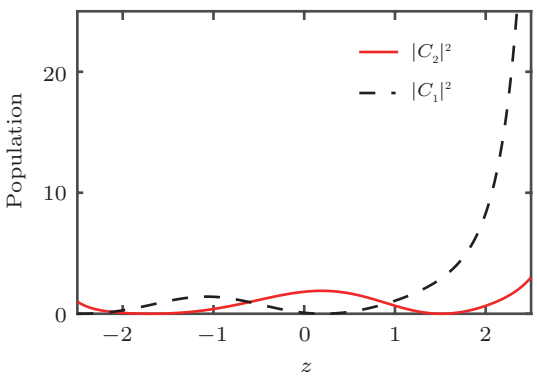


Fig. 4. The evolution of the population $|C_i|^2$ for the $\mathcal{PT}$-symmetric Rabi model driven by a quadratic nonlinear external field $[f(z) = 2\lambda z^2]$. Here the parameters are chosen as the same to those in Fig. 3, except for $z \in [-2.5, 2.5]$.

4. Conclusion

In conclusion, the exact analytic solutions of the generalized $\mathcal{PT}$-symmetric Rabi model in different cases were given. By comparing the second-order differential equation for the population and the form of the confluent Henu equations, we obtained three kinds of analytically solvable models and provided the corresponding explicit analytic formulas. According to our knowledge this is the first report on the exact analytic solutions for the generalized $\mathcal{PT}$-symmetric Rabi model. We have provided a powerful tool to study the population dynamics of the generalized $\mathcal{PT}$-symmetric two-level model and coherent control in non-Hermitian quantum systems with two states involved. Finally, it needs to be emphasized that the three types of confluent Henu functions used in this article can be calculated with the help of the well-defined functions Heun I $(I = D, B, T)$ in Maple.[20]

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