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RAPID COMMUNICATION

Atom-pair tunneling and quantum phase transition in asymmetry double-well trap in strong-interaction regime*

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The quantum effect of nonlinear co-tunnelling process, which is dependent on atom-pair tunneling and asymmetry of an double-well trap, is studied by using an asymmetrical extended Bose–Hubbard model. Due to the existence of atom-pair tunneling that describes quantum phenomena of ultracold atom-gas clouds in an asymmetrical double-well trap, the asymmetrical extended Bose–Hubbard model is better than the previous Bose–Hubbard model model by comparing with the experimental data cited from the literature. The dependence of dynamics and quantum phase transition on atom-pair tunneling and asymmetry are investigated. Importantly, it shows that the asymmetry of the extended Bose–Hubbard model, corresponding to the bias between double wells, leads to a number of resonance tunneling processes, which tunneling is renamed conditional resonance tunneling, and corrects the atom-number parity effect by controlling the bias between double wells.

Keywords: atom-pair tunneling, quantum phase transition

PACS: 03.75.Lm, 05.30.–d, 67.85.Hj **DOI:** 10.1088/1674-1056/ab44b6

1. Introduction

Recently the study of quantum phase transition and quantum tunneling has shown a broad prospect, with the rapid development of experimental techniques in optical traps and magnetic traps,such as from a superfluid to a Mott insulator, where the tunnel coupling through the inter well barriers and the atom–atom interaction play a crucial role.[1–8] Up to date the quantum phase transition has been studied only based on the well-known Bose–Hubbard (BH) model, which can describe the Josephson oscillation as well as self-trapping of Bose–Einstein condensates qualitatively.[9–11]

For the many-body case in strong interaction regime, the strong interaction between particles may change the tunnel configuration fundamentally, so the superexchange interaction between atoms on neighboring lattice sites cannot be ignored. In 2007, Sacha Zölner *et al.* performed an *ab initio* investigation from uncorrelated to fermionized tunneling of two atoms in a double well, and found that the correlated pair tunneling increases with the interaction strength increasing.[12] In this case, the BH Hamiltonian cannot describe the dynamical effect of quantum tunneling very well. It is very necessary to be free from an onsite approximation. In 2009, Liang *et al.* extended the well-known Bose–Hubbard model, where a peculiar atom-pair hopping term appears naturally in the new Hamiltonian in Ref. [13]. According to the second quantization theory, the atom-pair hopping term leads to a new dynamic process where strongly interacting between atoms can tunnel as a fragmented pair. The extended Bose–Hubbard model can very well explain the recently reported experimental observation of correlated tunneling. In some problems a double-well trap is often asymmetrical, due to the bias between double wells, whether in theory or in experiment. Even to a certain degree the bias between double wells can be arbitrarily tuned by using the advanced experimental technique.[14,15] It is possible and significative to investigate the quantum effect of nonlinear quantum tunneling in an asymmetry double-well trap. In 2008, Cheinet *et al.* reported on the observation of an interaction blockade effect for ultracold atoms in optical lattices, and they detected a discrete set of steps in the well population for increasing bias potentials, which may be used to count and control the number of atoms within a given well.[15] In 2016, Rubeni *et al.* studied the extended Bose–Hubbard model by a classical analysis and a quantum analysis. One determined the existence of three different quantum phases: self-trapping, phase-locking, and Josephson states in Ref. [16].

In this paper the quantum effect dependent on asymmetry of a double-well trap is analytically investigated beyond the onsite approximation. The atom-pair tunneling gives rise to a new dynamics of atom occupation-number oscillation and the insulator state. The quantum phase transition has also been studied only based on an asymmetrical extended Bose–Hubbard model, which can describe the Josephson oscillation as well as the paramagnetic phase with an energy gap in the spin language qualitatively. In addition, the asymmetry of

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the extended Bose–Hubbard model leads to a number of resonance tunneling processes which is renamed conditional resonance tunneling, and corrects the atom-number parity effect by tuning the bias between double wells.

The rest of this paper is organized as follows. In Section 2 we study the experimental evidence of single-atom tunneling and atom-pair tunneling in an asymmetry double-well trap. In Section 3, we review dynamics and quantum phase transition dependent on atom-pair tunnelling and asymmetry. Finally, in Section 4, we show that quantum effect induced by quantum tunnelling and asymmetry of the extended Bose–Hubbard model, which involves amplitude and energy splitting of quantum tunneling between ground states. In Section 5 we draw some conclusions.

2. Experimental evidence of nonlinear quantum tunneling

Based on the hard-core interaction, the well-known Bose–Hubbard model describes quantum phenomena of ultracold atom-gas clouds in an asymmetrical double-well trap qualitatively. For the many-body case in strong interaction regime, it cannot be ignored that the superexchange interactions between atoms on neighboring lattice sites. Beyond the onsite approximation, the well-known Bose–Hubbard model should be extended. So a peculiar atom-pair hopping term appears naturally in the new Hamiltonian, and leads to a new dynamic process where the strong interaction between atoms can tunnel back and forth as a fragmented pair. Removing a constant-energy term, the extended Bose–Hubbard type of Hamiltonian is reduced to

$$H = \mathrm{H}_0 + H_{T_1} + H_{T_2}, \tag{1}$$

with

$$\begin{cases} H_0 = -\dfrac{\Delta}{2}(n_1 - n_2) + \dfrac{U_0}{2}(n_1^2 + n_2^2 - N) + U_1 n_1 n_2, \\ H_{T_1} = -J(a_1^\dagger a_2 + a_1 a_2^\dagger), \\ H_{T_2} = \dfrac{U_2}{2}(a_1^\dagger a_1^\dagger a_2 a_2 + a_1 a_1 a_2^\dagger a_2^\dagger), \end{cases}$$

where $a_{1,2}^\dagger$ and $a_{1,2}$ are the creative and annihilatie operators, which creates and annihilates an atom in the first well and second well, respectively; $n_{1,2}$ is the count number, which counts the atom number respective well with a total atom number $N = n_1 + n_2$; $\Delta = \varepsilon_1 - \varepsilon_2$ is the bias potential between two wells; U_0 (U_1, U_2) is the onsite (intersite) interaction energy between two atoms; $J = J_0 - (N-1)U_2$ is the tunneling matrix element. The H_{T_1} part of the Hamiltonian Eq. (1) is the single-atom Josephson coupling term, which describes the single-atom tunneling, the H_{T_2} part describes obviously the atom-pair tunneling, which is more important dynamic process in the super-strong interaction regime. These parameters are fitted to $U_1 \approx U_2 \approx 0.02U_0$ in Ref. [13].

2.1. Calculation of single-atom tunneling and atom-pair tunneling

The state of the system can be described in the basis of the Fock states $|n\rangle$,

$$|\Psi\rangle = \sum_{n=0}^{N} C_n |n\rangle .$$

In fact, the amplitude of the atom-number difference is often regarded as the amplitude of quantum tunneling because they are proportional. In physics, the evolution-time of the atom-number difference is entirely due to all quantum tunneling processes. For an N-atom occupation case with an initial state $|N\rangle$, the amplitude of quantum tunneling can be rewritten as

$$A_t = \max[|K_1|^2 + 2|K_2|^2], \tag{2}$$

where

$$\begin{cases} K_1 = \langle N-1|U(t)|N\rangle , \\ K_2 = \langle N-2|U(t)|N\rangle , \end{cases}$$

with $U(t) = \exp(-\mathrm{i}Ht/\hbar)$ being time-evolution operator, K_1 corresponding to the quantum effect of the dominant single-atom tunneling, and K_2 corresponding to quantum effect of the direct atom-pair tunneling. Similarly, the amplitude of the single-atom tunneling and atom-pair tunneling can specifically be defined as

$$\begin{cases} A_1 = \max[|K_1|^2], \\ A_2 = \max[|K_2|^2]. \end{cases} \tag{3}$$

2.2. Experimental evidence of atom-pair tunneling

For single-atom occupation in strong interaction regime, the tunneling matrix element J is reduced to J_0, the onsite (intersite) interaction energy U_0 (U_1, U_2) vanishes, and only single-atom tunneling is left. The energy-gap between the ground state and excited state increases with the bias between double wells increasing, which is marked by the amplitude and frequency of the Rabi oscillation. For investigating the atom-pair tunneling of in an asymmetric double-well trap, we must obtain the bias Δ of the double-well trap and the coefficient J_0 of the hopping term from the single-atom occupation case, which are dependent on the asymmetric double wells. The amplitude of quantum tunneling increases with the bias Δ decreasing, which, importantly, reaches the maximum at $\Delta = 0$ corresponding to resonance tunneling in Fig. 1. The tunneling amplitude in Ref. [1] can be derived from the definition $A_t \equiv 1/2(\max\langle n_1 - n_2\rangle - \max\langle n_1 - n_2\rangle)$ by resolving the

$$\mathrm{i}\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle$$

and expressed as follows:

$$A_{\mathrm{t}} = \frac{4J^2}{\Delta^2 + 4J^2}. \tag{4}$$

From Fig. 1, we can obtain the fit parameters $U_0 \approx 0.70E_{\mathrm{r}}$ and $J_0 \approx 0.14E_{\mathrm{r}}$ (recoil energy).

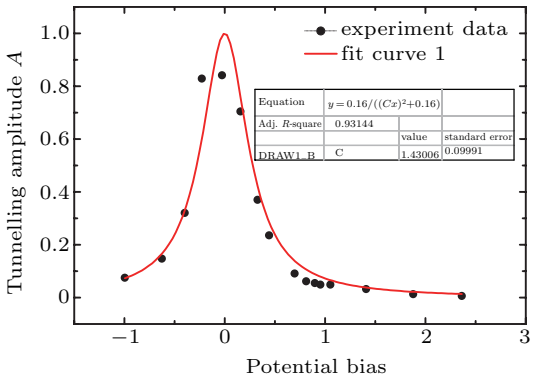


Fig. 1. Tunneling amplitude *versus* potential bias for single atom with initial state $|1\rangle$, where black data points are measured for $V_{\text{short}} = 12E_{\text{r}}$ (recoil energy) with $J/U_0 = 0.2$,[1] and red solid line is fitted from Eq. (4).

For a double-atom case, the quantum tunneling process consists of single-atom tunneling process and atom-pair tunneling process. According to Eq. (3), we plot the amplitude of single-atom tunneling and atom-pair tunneling versus the bias Δ with the fit parameters $U_0 \approx 0.70E_{\text{r}}$ and $J_0 \approx 0.14E_{\text{r}}$ in Fig. 2. Although the intersite interaction energy U_2 is very small, the atom-pair tunneling process corresponding to U_2 leads to the theoretical data better fitting with the experimental data than the case without the atom-pair tunneling process.

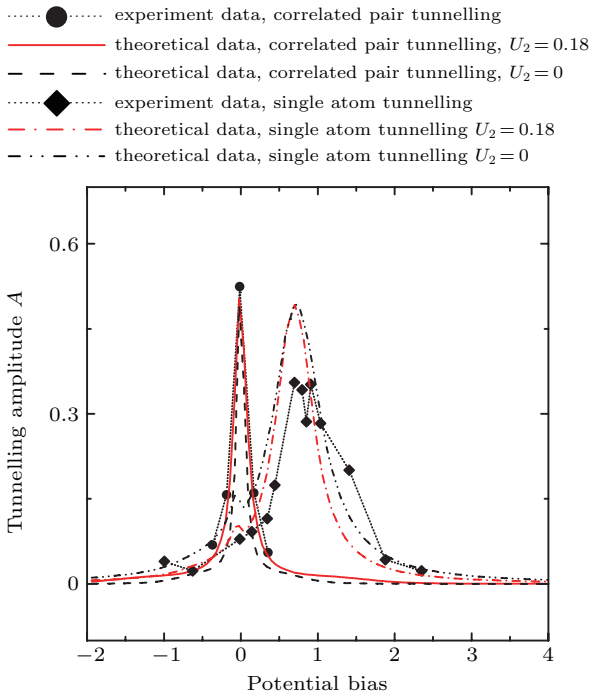


Fig. 2. Plots of tunnelling amplitude *versus* potential bias for doubly occupied case with initial state $|2\rangle$ for different parameters, where data points are measured for $V_{\text{short}} = 12E_{\text{r}}$ (recoil energy) with $J_0/U_0 = 0.2$,[1] and solid line represents theoretical result from Eq. (3).

In the double-atom occupation case, the Fock states $|2\rangle$, $|1\rangle$, and $|0\rangle$ are eigenstates of H_0 part of the Hamiltonian Eq. (1) corresponding to eigenvalues, respectively,

$$\begin{cases} E_2 = -\Delta + U_0, \\ E_1 = U_1, \\ E_2 = \Delta + U_0. \end{cases} \tag{5}$$

The first-order quantum tunneling includes two types of processes, which are single-atom tunneling process from $|2\rangle$ to $|1\rangle$ and atom-pair tunneling from $|2\rangle$ to $|0\rangle$ directly. In the atom-pair tunneling process two atoms can be regarded as a whole cluster in an asymmetric double-well trap, so the amplitude reaches the maximum value at the bias $\Delta = 0$ corresponding to resonance tunneling because $E_2 = E_0$, which is similar to a single-atom occupation case. But the single-atom tunneling process is another story, at the bias $\Delta = 0$ the amplitude of the single-atom tunneling is very small due to the interaction energy between two atoms, which is a type of interaction blockade,[15] at $\Delta = U_0 - U_1$ the amplitude of the single-atom tunneling reaches a maximum value corresponding to the resonance tunneling because $E_2 = E_1$, which is renamed conditional resonance tunneling.

3. Dynamics and quantum phase transition depended on atom-pair tunneling

We, in this section, provide an analytic investigation based on an effective Hamiltonian of single particle with canonical variables: the atom-number difference or population imbalance and phase difference between the two wells.

3.1. Effective Hamiltonian

For the N-atom occupation filling factor N in a double-well trap, we present the pseudoangular momentum operators defined as

$$\begin{cases} S_x = \dfrac{1}{2}(a_1^\dagger a_2 + a_1 a_2^\dagger), \\ S_y = \dfrac{1}{2\mathrm{i}}(a_1^\dagger a_2 - a_1 a_2^\dagger), \\ S_z = \dfrac{1}{2}(n_2 - n_1), \end{cases}$$

with the total angular momentum $s^2 = N/2(N/2+1)$. The Hamiltonian Eq. (1) is rewritten as

$$H = -2JS_x - \Delta S_z + K_1 S_z^2 + K_2 S_x^2, \tag{6}$$

where the parameters are given by $K_1 = U_0 - U_1$ and $K_2 = 2U_2$. In order to describe the physical situation, the spin operator is represented as a classical spin vector,

$$\begin{cases} S_x = s\sin\theta\cos\varphi, \\ S_y = s\sin\theta\sin\varphi, \\ S_z = s\cos\theta. \end{cases}$$

For simplicity, we consider the tunneling near the easy plane, *i.e.*, $\theta_0 = \arccos[\Delta/(2K_1 s)]$. This corresponds to the consideration of the low order quantum tunneling in Section 2. The effective momentum p, one of canonical variables: the atom-number difference and phase difference φ between the two wells, is equal to $s\cos(\theta_0 + \delta)$ with a small quantity δ, and the pseudoangular momentum operators,

$$S_x \approx \frac{s - p\cos\theta_0}{\sin\theta_0}\cos\varphi. \tag{7}$$

Hamiltonian equation (6) is rewritten as

$$H(p,\varphi) \approx \frac{(p-A(\varphi))^2}{2m(\varphi)} + V(\varphi), \tag{8}$$

with a variational mass $m(\varphi)$ and a gauge field $A(\varphi)$:

$$\begin{cases} m(\varphi) = \dfrac{1}{2K_1(1-\lambda_2\cos^2\varphi)}, \\ A(\varphi) = \dfrac{\Delta}{2K_1}\dfrac{1-\lambda_1\cos\varphi}{1-\lambda_2\cos^2\varphi}, \\ V(\varphi) = K_2 s^2 \dfrac{(\cos\varphi-\lambda_1/\lambda_2)^2(\sin^2\theta_0-\lambda_2\cos^2\varphi)}{1-\lambda_2\cos^2\varphi}, \end{cases}$$

where $\lambda_1 = J/(K_1 s\sin\theta_0)$ and $\lambda_1 = K_1/K_2$. The gauge field $A(\varphi)$ has only a quantum effect, *i.e.*, Aharonov–Bohm phase, and can be neglected in classical physics.

In classical physics, by making use of a proper unitary transformation and introducing an incomplete elliptic-integral coordinate such that

$$x = \int \frac{\mathrm{d}\varphi}{\sqrt{1-\lambda_2\cos^2\varphi}} = -F(\pi/2-\varphi,\lambda_2),$$

where $F(\pi/2-\varphi,\lambda_2)$ is the incomplete elliptic integral of the first kind with modulus λ_2. Hamiltonian equation (8) is rewritten as the following expression with constant mass

$$H(p,x) \approx \frac{p^2}{2m} + V(x), \tag{9}$$

where $m = 1/(2K_1)$ is the constant mass, both $V(\varphi)$ and $A(\varphi)$ are rewritten as $V(x)$ and $A(x)$ by $\varphi = \pi/2 + \mathrm{Amplitude}(x,\lambda_2)$ from Eq. (8).

The effective Hamiltonian corresponding to the extended Bose–Hubbard model describes the dynamics of a single particle in a periodic potential with a gauge field. The gauge field A, which is due to the bias potential between two wells, determines the coherence of the quantum tunneling.

3.2. Dynamics and quantum phase transition

The periodic potential in Hamiltonian Eq. (8) possesses two degenerate minima located at $\varphi_0 = \pm\arccos(\lambda_1/\lambda_2)$ due to atom-pair tunneling, which are separated by two typical barriers. The first (second) type of barrier is the centrally located at $\varphi_0 = 0(\pi)$, whose height is given by

$$V_{\max} = K_2 s^2 \frac{(\lambda_1 \pm \lambda_2)^2(\sin^2\theta_0-\lambda_2)}{(1-\lambda_2)\lambda_2}, \tag{10}$$

which corresponds to two types of quantum tunneling processes of ground states and the solutions of two typical instantons in the following sections. Two degenerate minima of the potential correspond to not the ground state $|E_0\rangle$ of Hamiltonian Eq. (1) but a quasi-classical ground state $|\theta_0,\pm\varphi_0\rangle$ of the effective Hamiltonian Eq. (8), which are atomic coherent states.

The barrier height can be controlled by the parameter $\lambda_1 = J/(K_1 s\sin\theta_0)$. Particularly when the Josephson coupling constant J also vanishes, the degenerate minima are located at $\varphi = \pi/2$ and the period of potential $V(\varphi)$ becomes not 2π but π. Obviously, the Josephson coupling breaks the symmetry of the effective periodic potential, while the bias potential between two wells breaks the symmetry of the double wells of trapping-atom. In this way, both of the Josephson coupling and the bias potential are significant for the dynamics and quantum phase transition of the extended Bose–Hubbard model.

The phase-plane portraits corresponding to the classical Hamiltonian of Eq. (8) are plotted in Fig. 3 without the gauge field $A(\varphi)$ because the gauge field $A(\varphi)$ has not classical dynamical effect at all. In Fig. 3, the closed orbits show the oscillations with the fixed phase difference and waved-open lines indicate atom-number self-trapping with the unfixed phase difference. Some of the closed orbits indicate π-phase self-trapping states with the fixed non-zero atom-number difference and phase difference π, which depends on atom-pair tunneling and asymmetry.[16–20]

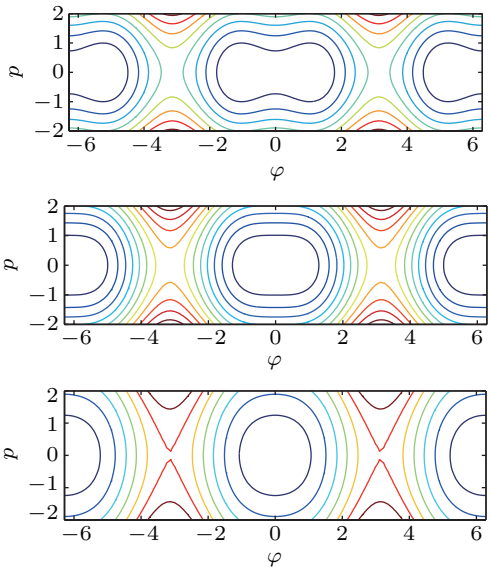


Fig. 3. Phase-space portraits with $s = 10$ and $\Delta = 4$ for different parameters $\lambda_1 = 0.01, \lambda_1 = 0.02$, and $\lambda_1 = 0.03$ (from top to bottom) from Eq. (8).

As the Josephson coupling constant J and the bias Δ of the double-well trap increase, the height of center potential barrier decreases and the twofold degeneracy of the ground state is gradually removed. When the barriers located at a critical point $\lambda_1^c = \lambda_2$ vanish, the disordered phase indicates zero- and π -phase oscillations from Fig. 3. Coinciding with the Landau second-order phase-transition theory, the QPT from the long-range magnetic order to disordered phase is of the second order, where the dimensionless barrier height

$$h = (1-\lambda_1/\lambda_2)^2$$

may be chosen as the order parameter. The highest value of order parameter corresponds to the degenerate $\pi/2$-phase

state. When the order parameter vanishes by properly adjusting the parameters J and Δ, the system approaches to the non-degenerate 0-phase ground state.[13]

4. Amplitude and energy splitting of quantum tunneling between ground states

The instanton method is often used to calculate the amplitude of quantum tunneling and energy splitting of ground state induced by quantum tunneling. From Hamiltonian Eq. (9), the corresponding effective Lagrangian of the extended Bose–Hubbard model is written as

$$L = \frac{1}{2}m\dot{x}^2 - A\dot{x} - V. \tag{11}$$

In the classically forbidden region, namely the potential barriers, the instanton solution in $1+0$ dimensions is a trajectory of the pseudoparticle, and can be obtained from the Euclidean version of the action by Wick rotation $\tau = \mathrm{i}t$, $\beta = \mathrm{i}T$,

$$S_{\mathrm{E}} = \int_{-\beta}^{\beta} (\mathrm{i}s\dot{\varphi} + L^e)\mathrm{d}\tau,$$

with the Euclidean version of effective Lagrangian $L_{\mathrm{E}} = \frac{1}{2}m\dot{x}^2 + A\dot{x} + V$. When $\beta \to \infty$, the Euclidean propagator $K_{\mathrm{E}} = N\int D\varphi\, \mathrm{e}^{-S_{\mathrm{E}}}$ indicates the quantum tunneling between the degenerate ground states, and $|K_{\mathrm{E}}|^2$ can be regarded as the amplitude of the quantum tunneling. The Euclidean version of the action including the Wess–Zuminio term is divided into two parts: real part and imagine part,[21] so

$$S_{\mathrm{E}} = S_{\mathrm{E}}^{\mathrm{R}} + \mathrm{i}S_{\mathrm{E}}^{\mathrm{I}},$$

with

$$\begin{cases} B = S_{\mathrm{E}}^{\mathrm{I}} = \displaystyle\int_{\varphi_i}^{\varphi_f} (s+A)\mathrm{d}\varphi, \\ S_{\mathrm{E}}^{\mathrm{R}} = \displaystyle\int \left(\frac{1}{2}m\dot{x}^2 + V\right)\mathrm{d}x. \end{cases} \tag{12}$$

4.1. Amplitude of quantum tunneling

The Euclidean–Feynman propagator can be evaluated by the stationary-phase perturbation method, in which the zero-order perturbation comes from the action of classical trajectory of pseudoparticles in the barrier region called the instantons. Taking into account the contributions of interference of tunnel paths of two typical instantons with the corresponding two typical boundary conditions Eq. (10), the Feynman propagator is obtained in the one-loop approximation

$$K_{\mathrm{E}} = K_{\mathrm{E}1} + K_{\mathrm{E}2} = NT\exp(-\mathrm{i}B)[G_1 + G_2\exp(-\mathrm{i}\Delta B)],$$

with

$$\Delta B = B_2 - B_1 = -2\pi\left(s + \frac{\Delta}{2K_1\sqrt{1-\lambda_2}}\right), \tag{13}$$

which determines the quantum phase interference of asymmetrical tunneling paths of two types of instantons. The amplitude $|K_{\mathrm{E}}|^2$ of quantum tunneling has an oscillation on the bias Δ of the double-well trap with a period

$$T_\Delta = 2K_1\sqrt{1-\lambda_2} \tag{14}$$

and peaks at $\Delta = 2(l-s)K_1\sqrt{1-\lambda_2}$, $(l = 0, \pm1, \pm2, \ldots)$.

For the N-atom occupation extended Bose–Hubbard model, the Fock states $|N-n\rangle$ ($n = 0, 1, 2, 3, \ldots, N$) are the eigenstates of the H_0 part of the Hamiltonian Eq. (1) corresponding to eigenvalues,

$$E_{N-n} = [n^2 + (\Delta' - N)n](U_0 - U_1 - U_2), \tag{15}$$

where $\Delta' = \Delta/(U_0 - U_1 - U_2)$. In the following analytical investigation, Δ' is always selected as an integer, because amplitude of quantum tunneling reaches extremum. When $(N-\Delta')$ is odd, the eigen-state $|(N-\Delta'+1)/2\rangle$ and $|(N-\Delta'-1)/2\rangle$ (ground states), $|(N-\Delta'+3)/2\rangle$ and $|(N-\Delta'-3)/2\rangle$, …, *etc*., are two-fold degenerate, in which the ground states are coupled each other by only single-atom tunneling directly, and correspond to resonance tunneling because they have the same energy level. But when $(N-\Delta')$ is even, the eigenstates, $|(N-\Delta')/2+1\rangle$ and $|(N-\Delta')/2-1\rangle$ (first excited states), $|(N-\Delta')/2+2\rangle$ and $|(N-\Delta')/2-2\rangle$ (second excited states), …, *etc*., are two-fold degenerate, in which the first excited states are coupled each other directly by only atom-pair tunneling. So the amplitude A_{t} of quantum tunneling between the lower energy eigenstates, regardless of single-atom tunneling and atom-pair tunneling, has an oscillation with a period of

$$T_{\Delta'} = 2(U_0 - U_1 - U_2),$$

which is equivalent to Eq. (14) with a small U_2, because the degenerate ground state in Eqs. (9) and (11) is superposition of these two-fold degenerate Fock states. The atom-number parity effect is also corrected by the asymmetry of the extended Bose–Hubbard model. Here, the amplitude A_t of quantum tunneling is dependent on not only atom-number but also the bias potential between two wells. If Δ' is odd integer, the atom-number parity effect of quantum tunneling has a novel exchange between the odd and even.

4.2. Energy splitting

The Euclidean–Feynman propagator can be evaluated by the stationary-phase perturbation method, in which the zero-order perturbation comes from the action of classical trajectory of pseudoparticles in the barrier region called the instantons. From Eq. (10), we can obtain the exact instanton solutions of two typical instantons

$$
\begin{aligned}
\pm\omega_0\tau_1 &= \operatorname{arctanh}\left(\frac{\sin\beta_1}{\sin\beta_0}\right)+\frac{\lambda_1\sin\beta_0}{\sin^2\theta_0}\left[\frac{\lambda_2(\sin^2\theta_0-\lambda_2)}{\lambda_2^2-\lambda_1^2}\right.\\
&\quad\left.\times\Pi\left(\sin^{-2}\beta_0,\beta_1,\frac{\lambda_2}{\sin^2\theta_0}\right)+F\left(\beta_1,\frac{\lambda_2}{\sin^2\theta_0}\right)\right],\\
\pm\omega_0\tau_2 &= \operatorname{arccoth}\left(\frac{\sin\beta_2}{\sin\beta_0}\right)+\frac{\lambda_2^2-\lambda_1^2}{\lambda_1\lambda_2\sin\beta_0}\\
&\quad\times\Pi\left(\frac{\lambda_2^2}{\lambda_1^2},\frac{\pi}{2}-\varphi_2,\frac{\lambda_2}{\sin^2\theta_0}\right),
\end{aligned}
\tag{16}
$$

where $\Pi(\sin^{-2}\beta_0,\beta_1,((\lambda_2)/(\sin^2\theta_0)))$ is the elliptic integral of the third kind, and

$$
\begin{cases}
\beta_{1,2}=\arcsin\left(\sqrt{\dfrac{\sin^2\theta_0}{\sin^2\theta_0-\lambda_2\cos^2\varphi_{1,2}}}\sin\varphi_{1,2}\right),\\[2ex]
\beta_0=\arcsin\left(\sqrt{\dfrac{\sin^2\theta_0(\lambda_2^2-\lambda_1^2)}{\lambda_2(\lambda_2\sin^2\theta_0-\lambda_1^2)}}\right).
\end{cases}
$$

Taking into account the contributions of both instantons φ_1 and φ_2 with the corresponding boundary conditions Eq. (14) and considering that λ_2 is small, the Feynman propagator is obtained in the one-loop approximation and then the ground-state tunnel splitting is described as

$$
\begin{aligned}
\Delta E &= 2Q\left\{\cosh(F_1+\Delta S-\Delta\pi)\right.\\
&\quad\left.+\cos\left[2\pi\left(\frac{\Delta}{2K_1\sqrt{1-\lambda_2}}+s\right)\right]\right\}^{1/2}\exp(\rho),
\end{aligned}
\tag{17}
$$

with

$$
\begin{aligned}
Q &= \sqrt{\frac{2^4K_1^2s^3(\lambda_2^2-\lambda_1^2)^{5/2}(\lambda_2\sin^2\theta_0-\lambda_1^2)^{5/2}}{\pi\lambda_2^5(\lambda_2-\lambda_1^2)(\sin^2\theta_0-\lambda_2)}},\\
\rho &= \frac{F_1+\pi_1+\pi_2-S_1-S_2}{2},\quad S_2=S_1+\Delta S,\\
\Delta S &= \frac{4\lambda_1 s}{\lambda_2}\sqrt{\frac{\lambda_2}{\sin^2\theta_0-\lambda_2}}\left[K\left(\frac{\lambda_2}{\lambda_2-\sin^2\theta_0}\right)\right.\\
&\quad\left.-\frac{\cos\theta_0}{1-\lambda_2}\Pi\left(-\frac{\lambda_2}{1-\lambda_2},\frac{\lambda_2}{\lambda_2-\sin^2\theta_0}\right)\right],\\
S_1 &= 2\sqrt{\frac{s^2}{1-\lambda_2}}\cos\theta_0\operatorname{arctanh}\left[-\sqrt{\frac{\lambda_2}{1-\lambda_2}}\cot\theta_0\sin\beta_0\right]\\
&\quad+s\ln\left[\frac{\sin\theta_0-\sqrt{\lambda_2}\sin\beta_0}{\sin\theta_0+\sqrt{\lambda_2}\sin\beta_0}\right]\\
&\quad+\frac{4\lambda_1 s}{\sqrt{\lambda_2}\sin2\theta_0}\left[(1-\cos\theta_0)F\left(\beta_0,\frac{\lambda_2}{\sin^2\theta_0}\right)\right.\\
&\quad\left.-\left(1-\frac{\cos^2\theta_0}{1-\lambda_2}\right)\Pi\left(\frac{\lambda_2\cot^2\theta_0}{1-\lambda_2},\beta_0,\frac{\lambda_2}{\sin^2\theta_0}\right)\right],\\
F_1 &= \frac{\lambda_1}{\sin\theta_0}\sqrt{\frac{(\lambda_2^2-\lambda_1^2)}{\lambda_2(\lambda_2\sin^2\theta_0-\lambda_1^2)}}\\
&\quad\times F\left(\beta\left(\arccos\left(\frac{\lambda_1}{\lambda_2}\right)\right),\frac{\lambda_2}{\sin^2\theta_0}\right),
\end{aligned}
$$

$$
\begin{aligned}
\Delta\pi &= \pi_2-\pi_1,\\
\pi_1 &= \frac{\lambda_1\lambda_2^3(\sin^2\theta_0-\lambda_2)\sin^5\beta_0}{16(\lambda_2^2-\lambda_1^2)^2\sin^8\theta_0}\\
&\quad\times\left\{3\lambda_1(\lambda_2^2-\lambda_1^2)\sin\theta_0(\sin^2\theta_0-\lambda_2)^{1/2}\right.\\
&\quad+[3\lambda_1^2\lambda_2+(14\lambda_2^2-9\lambda_1^2)\sin^2\theta_0\\
&\quad\left.-8\lambda_2\sin^4\theta_0]\beta\left(\arccos\left(\frac{\lambda_1}{\lambda_2}\right)\right)\right\},\\
\pi_2 &= \frac{\lambda_1(\lambda_2^2-\lambda_1^2)}{16\lambda_2^3\sin^4\theta_0\sin\beta_0}\left\{3\lambda_1\lambda_2^2\sqrt{\frac{(\sin^2\theta_0-\lambda_2)}{\lambda_2(\lambda_2\sin^2\theta_0-\lambda_1^2)}}\sin\beta_0\right.\\
&\quad\left.-[6\lambda_1^2+\lambda_2(3\lambda_2+8\sin^2\theta_0)]\arcsin\left(\frac{\lambda_1}{\lambda_2}\right)\right\}.
\end{aligned}
$$

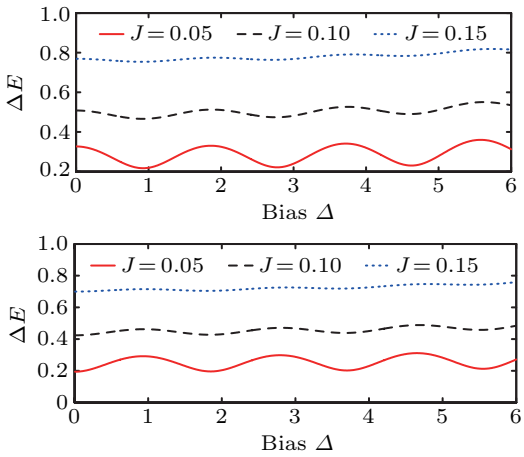


Fig. 4. Plots of numerical result of energy splitting ΔE *versus* potential bias Δ for $U_0=1$, $U_2=0.02$,and different values of $s=10$, 10.5 (from top to bottom) by direct numerical diagonalization of Eq. (1).

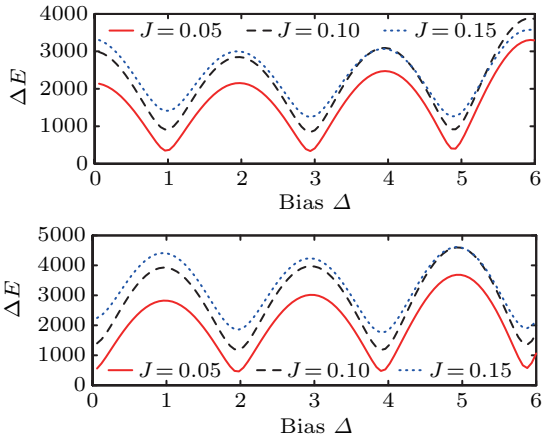


Fig. 5. Plots of energy splitting ΔE *versus* potential bias Δ for parameters $U_0=1$, $U_2=0.02$, and different values of $s=10$, 10.5 (from top to bottom) from Eq. (17).

Figures 4 and 5 show the plots of energy splitting of ground state *versus* bias potential between two wells according to Eq. (1) and direct numerical diagonalization Eq. (17), respectively. Because the degenerate ground states are atomic coherent states

$$
|\theta_0,\varphi_0\rangle=[\cos(\theta_0/2)a_1^\dagger+\mathrm{e}^{\pm\mathrm{i}\varphi_0}\sin(\theta_0/2)a_2^\dagger]^N|vac\rangle,
$$

which are not rigorously the degenerate ground states of H_0 in Eq. (1), they are only consistent qualitatively with the change tendency. So the energy splitting ΔE of the ground state has an oscillation with a period

$$T = 2(U_0 - U_1 - U_2),$$

which is consistent with Eq. (14) in the above subsection.

5. Conclusions

Including the two-body interaction of nearest neighbors, the Bose–Hubbard with atom-pair tunneling is extended in the strong-interaction regime. In this paper, the quantum tunneling effect dependent on asymmetry of an double-well trap is analytically investigated. Due to the existence of atom-pair tunneling, describing quantum phenomena of ultracold atom–gas clouds in an asymmetrical double-well trap, the asymmetrically extended Bose–Hubbard model is better than the previous Bose–Hubbard model model. The asymmetry of the extended Bose–Hubbard model, corresponding to the bias between double wells, leads to a number of resonance tunnelling processes, which is renamed as condition resonance tunnelling. It is analytically investigated that the quantum effect involves the amplitude of quantum tunneling and energy splitting of ground state by nonlinear quantum tunneling.

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