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Exact solutions of a (2+1)-dimensional extended shallow water wave equation*

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We give the bilinear form and n-soliton solutions of a (2+1)-dimensional [(2+1)-D] extended shallow water wave (eSWW) equation associated with two functions v and r by using Hirota bilinear method. We provide solitons, breathers, and hybrid solutions of them. Four cases of a crucial $\phi(y)$, which is an arbitrary real continuous function appeared in f of bilinear form, are selected by using Jacobi elliptic functions, which yield a periodic solution and three kinds of doubly localized dormion-type solution. The first order Jacobi-type solution travels parallelly along the x axis with the velocity $(3k_1^2+\alpha,0)$ on (x,y)-plane. If $\phi(y)=\mathrm{sn}(y,3/10)$, it is a periodic solution. If $\phi(y)=\mathrm{cn}(y,1)$, it is a dormion-type-I solutions which has a maximum $(3/4)k_1p_1$ and a minimum $-(3/4)k_1p_1$. The width of the contour line is $\ln[(2+\sqrt{6}+\sqrt{2}+\sqrt{3})/(2+\sqrt{6}-\sqrt{2}-\sqrt{3})]$. If $\phi(y)=\mathrm{sn}(y,1)$, we get a dormion-type-II solution (26) which has only one extreme value $-(3/2)k_1p_1$. The width of the contour line is $\ln[(\sqrt{2}+1)/(\sqrt{2}-1)]$. If $\phi(y)=\mathrm{sn}(y,1/2)/(1+y^2)$, we get a dormion-type-III solution (21) which shows very strong doubly localized feature on (x,y) plane. Moreover, several interesting patterns of the mixture of periodic and localized solutions are also given in graphic way.

Keywords: (2+1)-dimensional extended shallow water wave equation, Hirota bilinear method, dormion-type solution

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1. Introduction

Nonlinear phenomena are ubiquitous in fields of engineering, physics, and even in social sciences. A wide variety of processes in physics can be described by nonlinear partial differential equations (PDEs). In recent decades, nonlinear science has been highly developed and applied in many areas. Integrable nonlinear systems have been interested in many mathematicians and physicists. One important task is to look for exact wave solutions of nonlinear evolution equations. These exact solutions are conducive for us to understand the physical mechanism of nature, such as solitons propagating with finite speed. Thus, in recent years, various approaches have been established to construct the exact solutions in closed form of the nonlinear PDEs, including Lie group method,[1,2] inverse scattering method,[3] Hirota bilinear method,[4–7] the tanh-function method,[8] Darboux transformation,[9–12] the Jacobi elliptic function expansion method,[13] extended Jacobi elliptic function expansion method,[14–16] and so on. Among these famous methods, the Hirota bilinear method is a direct approach to solve nonlinear PDEs. Its advantage is that if we obtain the corresponding bilinear form of the equation, the multi-soliton solutions can be constructed in a simple and algebraic way.

The shallow water wave (SWW) equation is applied to study the surface wave in shallow water. The SWW equations are well known as a flow of shallow water at the free surface under gravity, or below the surface of horizontal pressure in a fluid.[17–19] Stokes, a pioneers of hydrodynamics, derived the equation of motion of an incompressible inviscid fluid under constant vertical gravity.[19] From these basic equations, various shallow water wave models can be obtained by further simplifying assumptions. These shallow water models are widely applied in oceanography and atmospheric science.

The (1+1)-dimensional SWW equation[20,21] arises from the Boussinesq approximation is in the form

$$u_{xxt}+\alpha uu_t+\beta u_x\partial_x^{-1}u_t-u_t-u_x=0, \tag{1}$$

where α and β are arbitrary nonzero constants. By taking $u_x=v$, equation (1) can be written as

$$v_{xxxt}+\alpha v_xv_{xt}+\beta v_{xx}v_t-v_{xt}-v_{xx}=0. \tag{2}$$

In the case $\alpha=2\beta$, equation (2) becomes SWW–Ablowitz–Kaup–Newell–Segur (SWW–AKNS) equation,[21] while in the case $\alpha=\beta$ it becomes SWW–Hirota–Satsuma (SWW–HS) equation.[20] Both these two equations are completely integrable and exist Lax pair.[21,22] The (2+1)-D SWW equation which is a (2+l)-D generalization of the shallow water wave equation[23] has the form

$$u_{yt}+u_{xxxy}-3u_{xx}u_y-3u_xu_{xy}=0. \tag{3}$$

This is formulated as a nonlocal Riemann–Hilbert problem. A set of studies about Eq. (3) has been done in Refs. [24]–[26]. In addition, several generalized equations about the (2+1)-D SWW equation have been studied, such as the generalized (2+1)-D SWW equation.[27]

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In this paper, inspired the above results of shallow water wave equations, we study a new integrable nonlinear equation, namely a (2+1)-D extended shallow water wave (eSWW) equation,[28] and further discover new patterns of nonlinear waves due to the appearance of an arbitrary function. This newly introduced eSWW equation[28] is given by a form as

$$u_{yt}+3u_{xxxy}-3u_{xx}u_y-3u_xu_{xy}+\alpha u_{xy}=0. \tag{4}$$

If setting $x=y$, $v=r$, and $\alpha=0$, equation (4) can be reduced to KdV equation.[29] In Ref. [28], exact periodic wave solution of eSWW equation was constructed by using the generalized $D_{\bar{p}}$ -operator[30] and Riemann theta function[31] in terms of the Hirota bilinear method. In Ref. [32] Wronskian, Pfaffian, and periodic wave solutions of Eq. (4) has been given. Letting

$$v=u_y,\quad r=u_x, \tag{5}$$

then the eSWW equation is written as

$$v_t+v_{xxx}-3(rv)_x+\alpha v_x=0,\quad r_y=v_x, \tag{6}$$

which will be studied in the following context to get four kinds of solutions including soliton, breather, hybrid, and Jacobi-type solutions.

This paper is organized as follows: In Section 2, we give the bilinear form and n-soliton of eSWW equation, *i.e.*, Eq. (6), and analyze the amplitudes and extreme values of bright soliton and dark soliton. In addition, we obtain the breathers and hybrid solutions by the complexification method.[33–36] In Section 3, we get new periodic solution and three kinds of dormion-type solution by setting an arbitrary function $\phi(y)$ as a Jacobi elliptic function. Note that $\phi(y)$ is appeared in the f for the bilinear form of the eSWW equation. Finally, we conclude this paper in Section 4.

2. Bilinear form and n-soliton solution

As we know, the Hirota bilinear D-operator plays a very important role in Hirota bilinear method which is defined as[4]

$$\begin{aligned}&D_t^lD_x^mD_y^nf(x,y,t)\cdot g(x,y,t)\\ =\;&(\partial_t-\partial_{t'})^l(\partial_x-\partial_{x'})^m(\partial_y-\partial_{y'})^n\\ &\times f(x,y,t)g(x,y,t)|_{t'=t,x'=x,y'=y}.\end{aligned}$$

Through the dependent variable transformation

$$u=-6(\ln f)_x, \tag{7}$$

equation (4) is transformed into a bilinear form

$$D_y(D_t+\alpha D_x+3D_x^3)f\cdot f=0. \tag{8}$$

Here f is a real function of x, y, and t. The N-soliton solution of Eq. (8) is expressed as

$$f^{[N]}=\sum_{\mu=0,1}\exp\left(\sum_{i<j}^{N}\mu_i\mu_jA_{ij}+\sum_{i=1}^{N}\mu_i\eta_i\right), \tag{9}$$

where

$$\begin{aligned}&\alpha_{ij}\triangleq\exp(A_{ij})=\frac{(k_i-k_j)(p_i-p_j)}{(k_i+k_j)(p_i+p_j)},\\ &\eta_i=k_ix+p_iy-\omega_it+\eta_i^0,\\ &\omega_i=3k_i^3+\alpha k_i,\end{aligned} \tag{10}$$

and k_i, p_i, η_i^0 are constant. And then we obtain the N-soliton solution of Eq. (6)

$$\begin{aligned}v^{[N]}&=-6(\ln(f^{[N]}))_{xy},\\ r^{[N]}&=-6(\ln(f^{[N]}))_{xx}.\end{aligned} \tag{11}$$

2.1. Soliton solutions

In order to obtain one-soliton solutions, we gain $f^{[1]}=1+\mathrm{e}^{k_1x+p_1y-(3k_1^3+\alpha k_1)t+\eta_1^0}$ by substituting $N=1$ into Eq. (9). Then we gain the solution from Eq. (11) as follows:

$$\begin{aligned}v^{[1]}&=-\frac{6k_1p_1\,\mathrm{e}^{k_1x+p_1y-(3k_1^3+\alpha k_1)t+\eta_1^0}}{(1+\mathrm{e}^{k_1x+p_1y-(3k_1^3+\alpha k_1)t+\eta_1^0})^2},\\ r^{[1]}&=-\frac{6k_1^2\,\mathrm{e}^{k_1x+p_1y-(3k_1^3+\alpha k_1)t+\eta_1^0}}{(1+\mathrm{e}^{k_1x+p_1y-(3k_1^3+\alpha k_1)t+\eta_1^0})^2}.\end{aligned} \tag{12}$$

It can be seen from this formula that $v^{[1]}$ and $r^{[1]}$ have the same extreme line $k_1x+p_1y-(3k_1^3+\alpha k_1)t+\eta_1^0=0$, but different amplitudes. The amplitude of $v^{[1]}$ is $(3/2)k_1p_1$, but the amplitude of $r^{[1]}$ is $-(3/2)k_1^2$. Thus $v^{[1]}$ is a dark soliton when $k_1p_1>0$ (Fig. 1(a)) and a bright soliton when $k_1p_1<0$ (Fig. 1(b)). However, $r^{[1]}$ generates only a dark soliton as shown in Fig. 1(c). In addition, $v^{[1]}$ and $r^{[1]}$ have the same velocity on (x,y)-plane, which is given by $((3k_1^3+\alpha k_1)k_1/(k_1^2+p_1^2),(3k_1^3+\alpha k_1)p_1/(k_1^2+p_1^2))$.

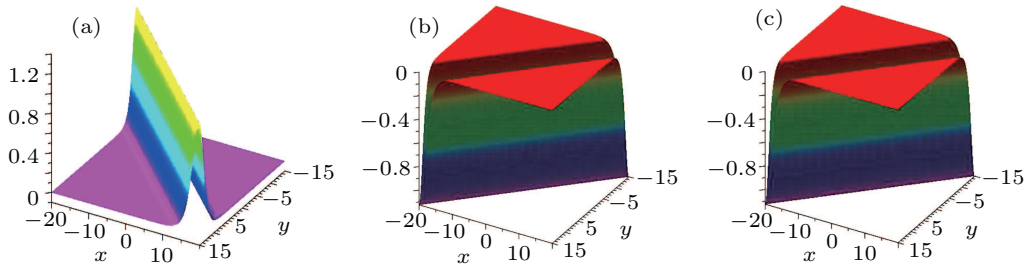


Fig. 1. One-soliton equation (12) with $\alpha=1$, $\eta_1^0=0$, $t=0$. (a) Bright soliton $v^{[1]}$: $k_1=-3/4$, $p_1=1$; (b) Dark soliton $v^{[1]}$: $k_1=3/4$, $p_1=1$; (c) Dark soliton $r^{[1]}$: $k_1=3/4$, $p_1=1$.

In the same way, we obtain two-soliton solution by setting $N=2$ in Eqs. (9) and (11). Two formulas of the solutions are given by

$$v^{[2]} = -6\frac{k_1p_1\mathrm{e}^{\eta_1}+k_2p_2\mathrm{e}^{\eta_2}+2(k_1-k_2)(p_1-p_2)\mathrm{e}^{\eta_1+\eta_2}+\alpha_{12}(k_1p_1\mathrm{e}^{\eta_2}+k_2p_2\mathrm{e}^{\eta_1})\mathrm{e}^{\eta_1+\eta_2}}{(1+\mathrm{e}^{\eta_1}+\mathrm{e}^{\eta_2}+\alpha_{12}\mathrm{e}^{\eta_1+\eta_2})^2},$$
$$r^{[2]} = -6\frac{k_1^2\mathrm{e}^{\eta_1}+k_2^2\mathrm{e}^{\eta_2}+((k_1-k_2)^2+\alpha_{12}(k_1+k_2)^2)\mathrm{e}^{\eta_1+\eta_2}+\alpha_{12}(k_1^2\mathrm{e}^{\eta_2}+k_2^2\mathrm{e}^{\eta_1})\mathrm{e}^{\eta_1+\eta_2}}{(1+\mathrm{e}^{\eta_1}+\mathrm{e}^{\eta_2}+\alpha_{12}\mathrm{e}^{\eta_1+\eta_2})^2}, \tag{13}$$

where η_1, η_2, and α_{12} are given in Eq. (10).

The profiles of two-soliton given by Eq. (13) are shown in Fig. 2. In this case, $r^{[2]}$ is always a dark soliton (Fig. 2(d)), but $v^{[2]}$ is not. If $k_1p_1>0$ and $k_2p_2>0$, $v^{[2]}$ is a dark soliton (see Fig. 2(a)); If $k_1p_1/k_2p_2<0$, $v^{[2]}$ becomes a mixed-soliton (dark–bright form) (see Fig. 2(b)); and if $k_1p_1<0$ and $k_2p_2<0$, $v^{[2]}$ turns into a bright soliton as shown in Fig. 2(c).

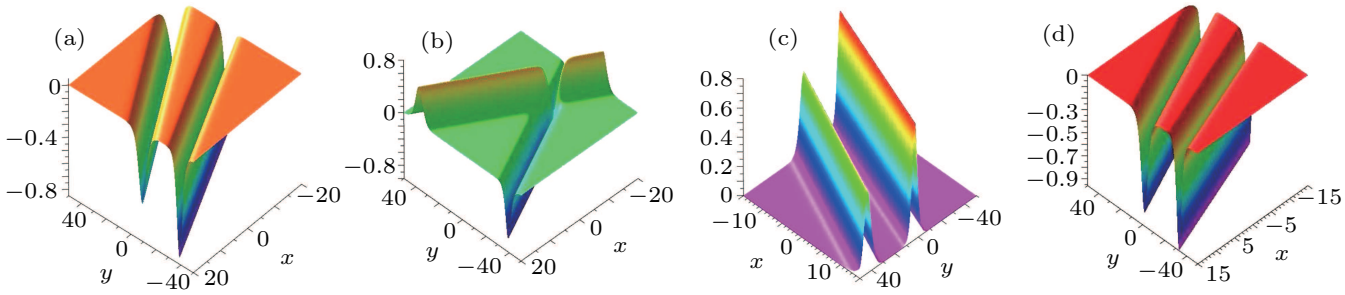


Fig. 2. Two-soliton with $\alpha=-1/2$, $t=0$, $p_1=1/2$, $\eta_1^0=-5$, $p_2=2/3$, $\eta_2^0=10$. (a) Dark soliton: $v^{[2]}$ with $k_1=3/4$, $k_2=4/5$; (b) Mixed soliton: $v^{[2]}$ with $k_1=-3/4$, $k_2=4/5$; (c) Bright soliton: $v^{[2]}$ with $k_1=-3/4$, $k_2=-4/5$; (d) Dark soliton: $r^{[2]}$ with $k_1=3/4$, $k_2=4/5$.

Similarly, we can obtain the N-dark-soliton $r^{[N]}$ or N-bright-soliton $r^{[N]}$ or N-mixed-soliton which consists of dark and bright solitons. Two examples of three-soliton are shown in Fig. 3.

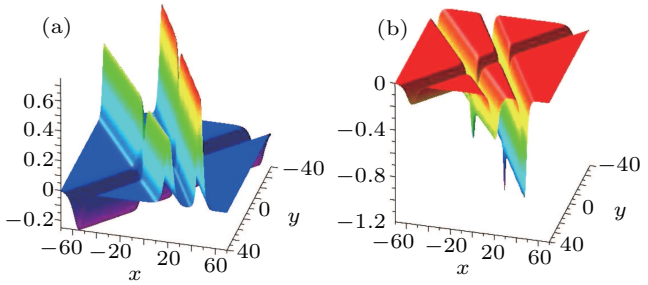


Fig. 3. Three-soliton with $\alpha=1$, $t=0$, $k_1=-3/4$, $k_2=-1/2$, $k_3=1/3$, $p_1=2/3$, $p_2=3/5$, $p_3=1/2$, $\eta_1^0=10$, $\eta_2^0=-8$, $\eta_3^0=0$. (a) Mixed soliton $v^{[3]}$; (b) Dark soliton $r^{[3]}$.

2.2. Breather solutions

If we let $N=2m$ $(m\in \boldsymbol{N}_+)$, and $\eta_{2n}=\eta_{2n-1}^*$ $(n=1,\ldots,m)$ in Eq. (9), the $2m$-soliton solutions will become m-breather solutions. Hereinafter we denote the breather as $v_b^{[m]}$ and $r_b^{[m]}$. First, we set

$$\begin{aligned} &k_{2n-1}=a_n+\mathrm{i}b_n, \quad k_{2n}=a_n-\mathrm{i}b_n,\\ &p_{2n-1}=c_n+\mathrm{i}d_n, \quad p_{2n}=c_n-\mathrm{i}d_n,\\ &\eta_{2n-1}^0=\eta_{2n}^0=0. \end{aligned} \tag{14}$$

The real and imaginary parts of η_i given in Eq. (10) are

$$\begin{aligned} R_n&=\mathrm{Re}(\eta_{2n-1})=a_nx+c_ny-(3(a_n^2-3b_n^2)+\alpha)a_nt,\\ I_n&=\mathrm{Im}(\eta_{2n-1})=b_nx+d_ny-(3(3a_n^2-b_n^2)+\alpha)b_nt. \end{aligned} \tag{15}$$

In the case $N=2$, the one-breather of the eSWW equation can be generated with the function f as follows:

$$\begin{aligned} f_b^{[1]}&=1+2\mathrm{e}^{R_1}\cos I_1+\alpha_{12}\mathrm{e}^{2R_1},\\ \alpha_{12}&=-\frac{b_1d_1}{a_1c_1}. \end{aligned} \tag{16}$$

Substituting Eq. (16) into Eq. (11), we have the following expressions:

$$\begin{aligned} v_b^{[1]}&=\frac{-12\mathrm{e}^{R_1}((1+\alpha_{12}\mathrm{e}^{2R_1})(a_1c_1-b_1d_1)\cos I_1-(1-\alpha_{12}\mathrm{e}^{2R_1})(a_1d_1+b_1c_1)\sin I_1-2\mathrm{e}^{R_1}(a_1d_1+b_1c_1))}{4\mathrm{e}^{R_1}\cos I_1(1+\mathrm{e}^{R_1}\cos I_1+\alpha_{12}\mathrm{e}^{2R_1})+(1+\alpha_{12}\mathrm{e}^{2R_1})^2},\\ r_b^{[1]}&=\frac{-12\mathrm{e}^{R_1}((1+\alpha_{12}\mathrm{e}^{2R_1})(a_1^2-b_1^2)\cos I_1-2a_1b_1\sin I_1(1-\alpha_{12}\mathrm{e}^{2R_1})-2\mathrm{e}^{R_1}(b_1^2+a_1^2\alpha_{12}))}{4\mathrm{e}^{R_1}\cos I_1(1+\mathrm{e}^{R_1}\cos I_1+\alpha_{12}\mathrm{e}^{2R_1})+(1+\alpha_{12}\mathrm{e}^{2R_1})^2}. \end{aligned} \tag{17}$$

Their profiles are plotted in Fig. 4. The trajectory of $v_b^{[1]}$ and $r_b^{[1]}$ is $R_1=0$. And through computing, we get that their period is $T_{[x]}=|2c_1\pi/a_1d_1-b_1c_1|$, $T_{[y]}=|2a_1\pi/a_1d_1-b_1c_1|$, and then the distance between two adjacent peaks is

$$T=\sqrt{T_{[x]}^2+T_{[y]}^2}=\left|\frac{2\pi\sqrt{a_1^2+c_1^2}}{a_1d_1-b_1c_1}\right|. \tag{18}$$

It implies that the breather moves parallelly on the (x,y)-plane as t changing, while its shape keeps unchanged. $v_b^{[1]}$ and $r_b^{[1]}$ have the same velocity $((3(a_1^2-3b_1^2)+\alpha)a_1^2/(a_1^2+c_1^2), (3(a_1^2-3b_1^2)+\alpha)a_1c_1/(a_1^2+c_1^2))$ on (x,y)-plane.

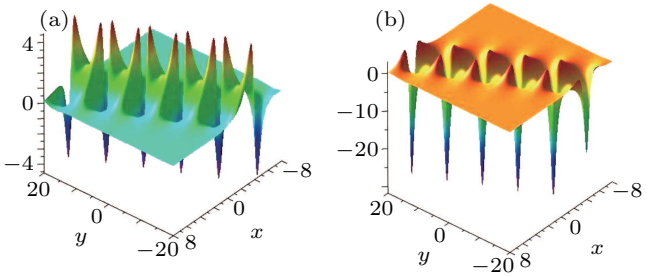


Fig. 4. One-breather Eq. (12) with $t=0$, $\alpha=-1/2$, $a_1=-1$, $b_1=1/2$, $c_1=1/5$, $d_1=2/3$, $\eta_1^0=\eta_2^0=0$. (a) $v_b^{[1]}$, (b) $r_b^{[1]}$.

In the case $N=4$, two-breathers $v_b^{[2]}$ and $r_b^{[2]}$ are generated by the following function through Eq. (11), namely,

$$\begin{aligned}f_b^{[2]} =\ & 1+2\mathrm{e}^{R_1}\cos I_1+2\mathrm{e}^{R_2}\cos I_2+\alpha_{12}\mathrm{e}^{2R_1}+\alpha_{34}\mathrm{e}^{2R_2}\\ &+\alpha_{12}\alpha_{34}(R_{\alpha13}^2+I_{\alpha13}^2)(R_{\alpha14}^2+I_{\alpha14}^2)\mathrm{e}^{2R_1+2R_2}\\ &+2\mathrm{e}^{R_1+R_2}(R_{\alpha13}\cos(I_1+I_2)-I_{\alpha13}\sin(I_1+I_2)\\ &+R_{\alpha14}\cos(I_1-I_2)-I_{\alpha14}\sin(I_1-I_2))\\ &+2\alpha_{12}\mathrm{e}^{2R_1+R_2}((R_{\alpha13}R_{\alpha14}+I_{\alpha13}I_{\alpha14})\cos I_2\\ &+(R_{\alpha13}I_{\alpha14}-R_{\alpha14}I_{\alpha13})\sin I_2)\\ &+2\alpha_{34}\mathrm{e}^{R_1+2R_2}((R_{\alpha13}R_{\alpha14}-I_{\alpha13}I_{\alpha14})\cos I_1\\ &-(R_{\alpha13}I_{\alpha14}+R_{\alpha14}I_{\alpha13})\sin I_1). \end{aligned}\tag{19}$$

Here,

$$\begin{aligned}&R_{\alpha_{13}}=\frac{\Theta_1+\Theta_2+\Theta_3}{\Theta_8},\quad R_{\alpha14}=\frac{\Theta_1+\Theta_4+\Theta_5}{\Theta_9},\\ &I_{\alpha_{13}}=\frac{\Theta_6}{\Theta_8},\quad I_{\alpha14}=\frac{\Theta_7}{\Theta_9},\\ &\alpha_{12}=-\frac{b_1d_1}{a_1c_1},\quad \alpha_{34}=-\frac{b_2d_2}{a_2c_2},\\ &\Theta_1=(a_1^2-a_2^2+b_1^2-b_2^2)(c_1^2-c_2^2+d_1^2-d_2^2),\\ &\Theta_2=2(a_1-a_2)(b_1+b_2)(c_2d_1-c_1d_2),\\ &\Theta_3=-2(c_1+c_2)(d_1-d_2)(a_2b_1-a_1b_2),\\ &\Theta_4=2(a_1-a_2)(b_1-b_2)(c_2d_1+c_1d_2),\\ &\Theta_5=-2(c_1+c_2)(d_1+d_2)(a_2b_1+a_1b_2),\\ &\Theta_6=-2(a_1d_2+a_2d_1+b_1c_2+b_2c_1),\\ &\Theta_7=-2(a_2d_1-a_1d_2+b_1c_2-b_2c_1),\\ &\Theta_8=((a_1+a_2)^2+(b_1+b_2)^2)((c_1+c_2)^2+(d_1+d_2)^2),\\ &\Theta_9=((a_1+a_2)^2+(b_1-b_2)^2)((c_1+c_2)^2+(d_1-d_2)^2). \end{aligned}\tag{20}$$

The profiles of the above two-breather solutions are plotted in Fig. 5.

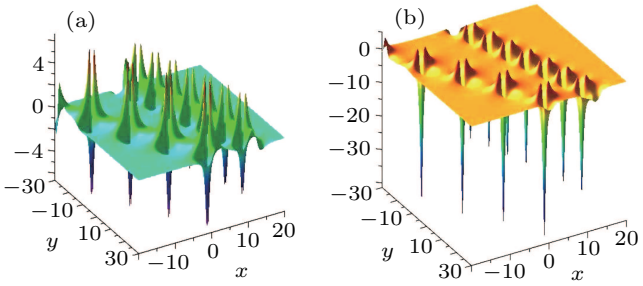


Fig. 5. Two-breather solutions with $t=0$, $\alpha=-1/2$, $a_1=-1/2$, $b_1=1/3$, $c_1=1/5$, $d_1=1/3$, $a_2=-1$, $b_2=1/2$, $c_2=1/5$, $d_2=2/3$, $\eta_1^0=\eta_2^0=-3$, $\eta_3^0=\eta_4^0=5$. (a) $v_b^{[2]}$, (b) $r_b^{[2]}$.

2.3. Hybrid solutions

If $N\geqslant 3$, through partial complexification we can obtain hybrid solutions, containing breathers and solitons. For example, in the case $N=3$, we can obtain a solution containing a breather and a soliton if setting $\eta_1=\eta_2^*$ and η_3 as a real function; and in the case $N=4$, we can obtain a solution containing a breather and two solitons if setting $\eta_1=\eta_2^*$ and η_3, η_4 as real functions. Two examples of the three-hybrid solution are depicted in Fig. 6.

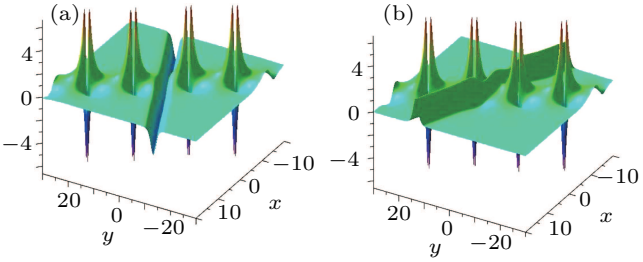


Fig. 6. Hybrid solution $v^{[3]}$ with $\alpha=-1/2$, $t=0$, $k_1=-1/2+\mathrm{i}/3$, $p_1=1/5+\mathrm{i}/3$, $\eta_1^0=\mathrm{i}/4$, $k_2=-1/2-\mathrm{i}/3$, $p_2=1/5-\mathrm{i}/3$, $\eta_2^0=-\mathrm{i}/4$. (a) One breather and one dark soliton with parameters $k_3=-1$, $p_3=3/2$, $\eta_3^0=0$; (b) One breather and one bright soliton with parameters $k_3=1$, $p_3=3/2$, $\eta_3^0=0$.

3. Jacobi-type solution

In this section, we shall give another kind of new solution. One crucial observation is that $v^{[N]}$ and $r^{[N]}$ are also solutions of the eSWW equation if we set $\eta_i=k_ix+p_i\phi-\omega_it+\eta_i^0$ in Eq. (10) by inserting a continuous arbitrary real function of y, *i.e.*, $\phi=\phi(y)$. This fact is also mentioned in Ref. [37]. According to this fact and setting ϕ be a Jacobi elliptic function, then equation (11) yields $v_J^{[N]}$ and $r_J^{[N]}$ which are hereinafter called Jacobi-type solution. We mainly discuss $v_J^{[N]}$ in this section, which will provide a periodic solution and three kinds of dormion-type solutions.

3.1. Case 1: $N=1$

Using above modified η_1 with ϕ, the first order Jacobi-type solution of Eq. (6) is in the form of

$$v_J^{[1]}=-\frac{6k_1p_1\phi'(y)\mathrm{e}^{k_1x+p_1\phi-(3k_1^3+\alpha k_1)t+\eta_1^0}}{(1+\mathrm{e}^{k_1x+p_1\phi(y)-(3k_1^3+\alpha k_1)t+\eta_1^0})^2}. \tag{21}$$

The character of the solution depends on the specific choice of ϕ. The velocity of $v_J^{[1]}$ on (x,y)-plane is $(3k_1^2+\alpha,0)$ because of the term $x-(3k_1^2+\alpha)t$ and the periodicity of ϕ with respect to y. That means the first order Jacobi-type solution parallelly travels along the x axis.

If we choose $\phi(y)=\mathrm{sn}(y,3/10)$, the corresponding solution is given by

$$v_{Jp}^{[1]}=-\frac{6k_1p_1\mathrm{cn}(y,3/10)\mathrm{dn}(y,3/10)\exp(k_1x+p_1\mathrm{sn}(y,3/10)-(3k_1^3+\alpha k_1)t+\eta_1^0)}{(1+\exp(k_1x+p_1\mathrm{sn}(y,3/10)-(3k_1^3+\alpha k_1)t+\eta_1^0))^2}. \tag{22}$$

This is a periodic solution because of the appearance of elliptic functions, and its extreme value is $\pm(3/2)k_1p_1$, which is confirmed by Fig. 7(a). The contour lines on different hights are plotted in Fig. 7(a), where h is the hight and $-|(3/2)k_1p_1|<h<|(3/2)k_1p_1|$.

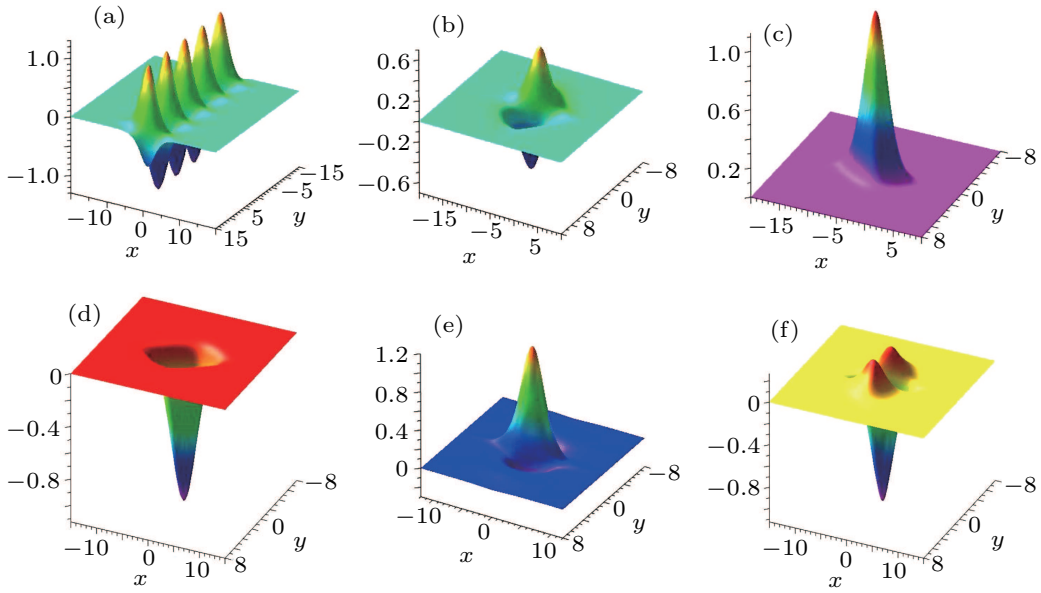


Fig. 7. $v_J^{[1]}$ with $\alpha=1$, $\eta_1^0=0$, $t=0$. (a) Periodic solution (22): $\phi(y)=\mathrm{sn}(y,3/10)$, $k_1=-3/4$, $p_1=1$; (b) Dormion-type-I soliton (23): $\phi(y)=\mathrm{cn}(y,1)$, $k_1=-3/4$, $p_1=1$; (c) Bright dormion-type-II soliton (26): $\phi(y)=\mathrm{sn}(y,1)$, $k_1=-3/4$, $p_1=1$; (d) Dark dormion-type-II soliton (26): $\phi(y)=\mathrm{sn}(y,1)$, $k_1=3/4$, $p_1=1$; (e) Bright dormion-type-III soliton (21): $\phi(y)=\mathrm{sn}(y,1/2)/(1+y^2)$, $k_1=-3/4$, $p_1=1$; (f) Dark dormion-type-III soliton (21): $\phi(y)=\mathrm{sn}(y,1/2)/(1+y^2)$, $k_1=3/4$, $p_1=1$.

If we choose $\phi(y)=\mathrm{cn}(y,1)$, and then we obtain a doubly localized solution on (x,y) plane which is called a dormion-type-I solution and is shown in Fig. 7(b). The expression of this solution is given by

$$v_{J\mathrm{I}}^{[1]}=\frac{6k_1p_1\,\mathrm{sech}\,y\tanh y\exp(k_1x+p_1\,\mathrm{sech}\,y-(3k_1^3+\alpha k_1)t+\eta_1^0)}{(1+\exp(k_1x+p_1\,\mathrm{sech}\,y-(3k_1^3+\alpha k_1)t+\eta_1^0))^2}, \tag{23}$$

which is traveling along $y=0$ with a velocity $(3k_1^2+\alpha)$ on (x,y) plane. It is interesting to note that $v_{J\mathrm{I}}^{[1]}$ has one maximum $(3/4)k_1p_1$ located at $((3k_1^2+\alpha)t-(\sqrt{2}p_1+2\eta_1^0)/2k_1,\ \mathrm{arccosh}\sqrt{2})$ and minimum $-(3/4)k_1p_1$ located at $((3k_1^2+\alpha)t-(\sqrt{2}p_1+2\eta_1^0)/2k_1,\ -\mathrm{arccosh}\sqrt{2})$. The contour lines on different hights are plotted in Fig. 8(b), where $-|(3/4)k_1p_1|<h<|(3/4)k_1p_1|$. The contour line of Eq. (23) on the hight $h=(3/8)k_1p_1$ (the half amplitude) is

$$\frac{2\,\mathrm{sech}\,y\tanh y\exp(k_1x+p_1\,\mathrm{sech}\,y-(3k_1^3+\alpha k_1)t+\eta_1^0)}{(1+\exp(k_1x+p_1\,\mathrm{sech}\,y-(3k_1^3+\alpha k_1)t+\eta_1^0))^2}-\frac{1}{8}=0. \tag{24}$$

Taking the derivative of x in Eq. (24), and making $\mathrm{d}y/\mathrm{d}x=0$, we can get two endpoints

$$\left(-\frac{\eta_1^0}{k_1}-\frac{(2+\sqrt{6}-\sqrt{2}-\sqrt{3})p_1}{(8+3\sqrt{6}-5\sqrt{2}-4\sqrt{3})k_1},\ \ln(2+\sqrt{6}-\sqrt{2}-\sqrt{3})\right),$$
$$\left(-\frac{\eta_1^0}{k_1}-\frac{(2+\sqrt{6}+\sqrt{2}+\sqrt{3})p_1}{(8+3\sqrt{6}+5\sqrt{2}+4\sqrt{3})k_1},\ \ln(2+\sqrt{6}+\sqrt{2}+\sqrt{3})\right)$$

on (x,y)-plane. In addition, we obtain two tangent lines perpendicular to the y axis,

$$\begin{aligned} l_{11}:&\quad y=\ln(2+\sqrt{6}-\sqrt{2}-\sqrt{3}),\\ l_{12}:&\quad y=\ln(2+\sqrt{6}+\sqrt{2}+\sqrt{3})). \end{aligned} \tag{25}$$

If we define the distance of l_{11} and l_{12} as width, we obtain: the width is $\ln[(2+\sqrt{6}+\sqrt{2}+\sqrt{3})/(2+\sqrt{6}-\sqrt{2}-\sqrt{3})]$. Namely, the width has nothing to do with any parameters. The end points and tangent lines are plotted in Fig. 9(a).

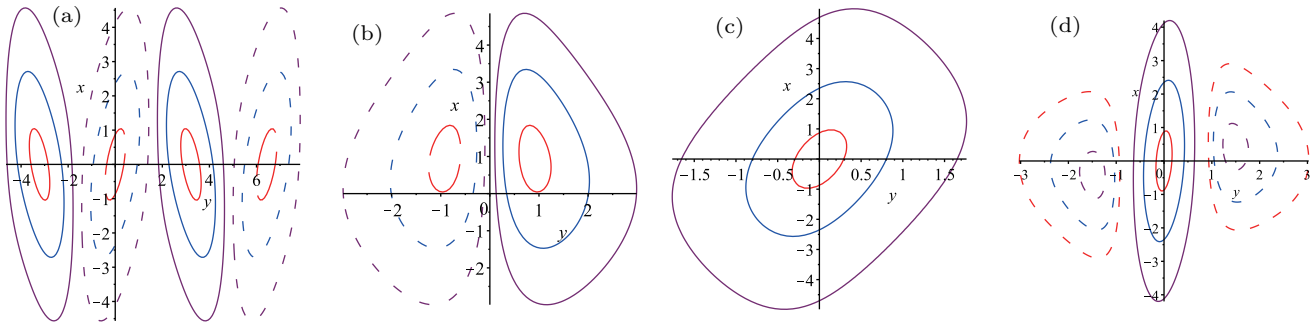


Fig. 8. Parameters: $\alpha=1$, $\eta_1^0=0$, $k_1=-3/4$, $p_1=1$, $t=0$. (a) The contour line $v_{Jp}^{[1]}-h$ with $h=(27/20)k_1p_1$ (solid, red), $h=(3/4)k_1p_1$ (solid, blue), $h=(1/4)k_1p_1$ (solid, purple), $h=-(27/20)k_1p_1$ (dash, red), $h=-(3/4)k_1p_1$ (dash, blue), $h=-(1/4)k_1p_1$ (dash, purple); (b) The contour line $v_{JI}^{[1]}-h$ with $h=(27/40)k_1p_1$ (solid, red), $h=(3/8)k_1p_1$ (solid, blue), $h=(3/20)k_1p_1$ (solid, purple), $h=-(27/40)k_1p_1$ (dash, red), $h=-(3/8)k_1p_1$ (dash, blue), $h=-(3/20)k_1p_1$ (dash, purple); (c) The contour line $v_{JII}^{[1]}-h$ with $h=-(27/20)k_1p_1$ (red), $h=-(3/4)k_1p_1$ (blue), $h=-(1/6)k_1p_1$ (purple); (d) The contour line $v_{JIII}^{[1]}-h$ with $h=-(27/20)k_1p_1$ (solid, red), $h=-(3/4)k_1p_1$ (solid, blue), $h=-(1/4)k_1p_1$ (solid, purple), $h=(1/6)k_1p_1$ (dash, red), $h=(1/4)k_1p_1$ (dash, blue), $h=(1/3)k_1p_1$ (dash, purple).

If we choose $\phi(y)=\mathrm{sn}(y,1)$, and then we obtain a doubly localized solution on (x,y) plane which is called a dormion-type-II solution and is shown in Fig. 7(c). The formula of this solution is given by

$$v_{JII}^{[1]}=-\frac{6k_1p_1\,\mathrm{sech}^2\,y\exp(k_1x+p_1\tanh y-(3k_1^3+\alpha k_1)t+\eta_1^0)}{(1+\exp(k_1x+p_1\tanh y-(3k_1^3+\alpha k_1)t+\eta_1^0))^2}. \tag{26}$$

This solution is different from dormion-type-I because $v_{JII}^{[1]}$ just has one extreme value $-(3/2)k_1p_1$ located at $((3k_1^2+\alpha)t-\eta_1^0/k_1,\,0)$ of (x,y) plane. We can see from formula of $v_{JII}^{[1]}$ that it is also a traveling wave along $y=0$ with the same velocity as $v_{JI}^{[1]}$. In addition, $v_{JII}^{[1]}$ generates a bright dromion when $k_1p_1<0$ (Fig. 7(c)), and a dark dromion when $k_1p_1>0$ (Fig. 7(d)). The contour lines on different hights are plotted in Fig. 8(c), where h is between 0 and $-(3/2)k_1p_1$. The contour line of Eq. (26) on the hight $h=-(3/4)k_1p_1$ (the half amplitude) is

$$\frac{2\,\mathrm{sech}^2\,y\exp(k_1x+p_1\tanh y-(3k_1^3+\alpha k_1)t+\eta_1^0)}{(1+\exp(k_1x+p_1\tanh y-(3k_1^3+\alpha k_1)t+\eta_1^0))^2}-\frac{1}{4}=0. \tag{27}$$

Using the same method as Eqs. (24) and (25), we can obtain two endpoints $((\sqrt{2}\eta_1^0-p_1)/\sqrt{2}k_1,\ \ln(\sqrt{2}-1))$ and $((\sqrt{2}\eta_1^0-p_1)/\sqrt{2}k_1,\ \ln(\sqrt{2}+1))$ on $(x,\,y)$-plane. The tangent lines perpendicular to the y axis are

$$l_{21}:\quad y=\ln(\sqrt{2}-1),\qquad l_{22}:\quad y=\ln(\sqrt{2}+1). \tag{28}$$

The width is $\ln(\sqrt{2}+1)/(\sqrt{2}-1)$ and also has nothing to do with any parameters. The end points and tangent lines are plotted in Fig. 9(b).

If we choose $\phi(y)=\mathrm{sn}(y,1/2)/(1+y^2)$, equation (21) yields a dormion-type-III solution $v_{JIII}^{[1]}$ which shows very strong doubly localized feature (Figs. 7(e) and 7(f)) on (x,y) plane, and the profile of this solution is invariant during the propagation along $y=0$, although there exist Jacobi elliptic functions in the representation of solution. In other words, the periodicity of $v_{JIII}^{[1]}$ is disappeared remarkably such that it shows behavior as a dormion, because the $(1+y^2)$ in denominator of $\phi(y)$ completely depresses the amplitude when $y>10$.

A simple calculation shows that $v_{JIII}^{[1]}$ has a significant extreme value $-(3/2)k_1p_1$ at centeral point $((3k_1^2+\alpha)t-\eta_1^0/k_1,\,0)$ of $(x,\,y)$ plane and other two small extreme values located in two sides. It is too long to write out the formulas of the above two small extreme values, but it can give approximately as $0.3568320478k_1p_1$. Furthermore, the centeral point is a maximum if $k_1p_1<0$ which implies that this solution is a bright dromion (Fig. 7(e)), however it is a minimum if $k_1p_1>0$ so that this solution is a dark dormion (Fig. 7(f)). The contour line $v_{JIII}^{[1]}-h=0$ is plotted in Fig. 8(d), where h is between the peak value and valley value. Because it is too complicated to write, we do not give the figure and expressions of the end points, the tangent lines perpendicular to the y axis, and the width of the contour lines.

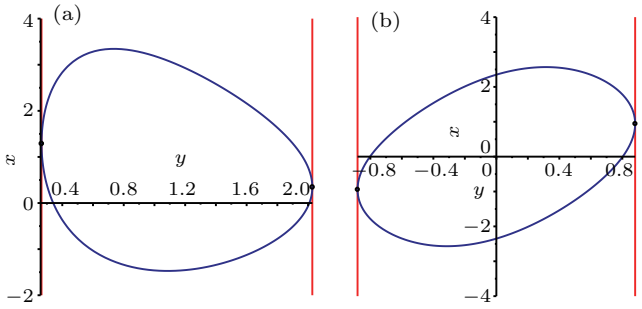


Fig. 9. The curve line is the contour line on the height of half amplitude. The two black points are the end points, and the two red lines are the tangent lines perpendicular to the y axis. Parameters: $\alpha=1$, $\eta_1^0=0$, $k_1=-3/4$, $p_1=1$, $t=0$. (a) The contour line is Eq. (24), and the tangent lines are Eq. (25); (b) The contour line is Eq. (27), and the tangent lines are Eq. (28).

3.2. Case 2: $N=2$

We can obtain the second order Jacobi-type solution as follows:

$$v_J^{[2]} = -6\phi'(y)\frac{k_1p_1\,\mathrm{e}^{\eta_1} + k_2p_2\,\mathrm{e}^{\eta_2} + (2(k_1-k_2)(p_1-p_2) + \alpha_{12}(k_1p_1\,\mathrm{e}^{\eta_2} + k_2p_2\,\mathrm{e}^{\eta_1}))\,\mathrm{e}^{\eta_1+\eta_2}}{(1+\mathrm{e}^{\eta_1}+\mathrm{e}^{\eta_2}+\alpha_{12}\,\mathrm{e}^{\eta_1+\eta_2})^2}. \tag{29}$$

If we choose $\phi(y) = \mathrm{sn}(y, 3/10)$, it shows obviously a periodic structure in Fig. 10(a). If we choose $\phi(y) = \mathrm{cn}(y, 1)$, it is dormion-type-I solution as shown in Fig. 10(b). It has two maximum values and two minimum values on (x, y) plane. If we choose $\phi(y) = \mathrm{sn}(y, 1)$, it is a dormion-type-II solution. Its extreme values have three scenarios: letting $k_1p_1 > 0$ and $k_2p_2 > 0$ it is a dark form as shown in Fig. 10(c), which has only two minimum values; letting $k_1p_1/k_2p_2 < 0$ it is a mixed form as shown in Fig. 10(d), which has a minimum value and a maximum value; letting $k_1p_1 < 0$ and $k_2p_2 < 0$ we can get a bright solution as shown in Fig. 10(e), which has only two maximum values.

If we choose $\phi(y) = \mathrm{sn}(y, 1/2)/(1+y^2)$, $v_J^{[2]}$ becomes dormion-type-III solution. Its extreme values also have three scenarios: letting $k_1p_1 > 0$ and $k_2p_2 > 0$ it is a dark form as shown in Fig. 10(f) which has four maximum values and two minimum values; letting $k_1p_1/k_2p_2 < 0$, it is a mixed form as shown in Fig. 10(g) which has three minimum values and three maximum values; letting $k_1p_1 < 0$ and $k_2p_2 < 0$, it is a bright form as shown in Fig. 10(h) which has four minimum values and two maximum values.

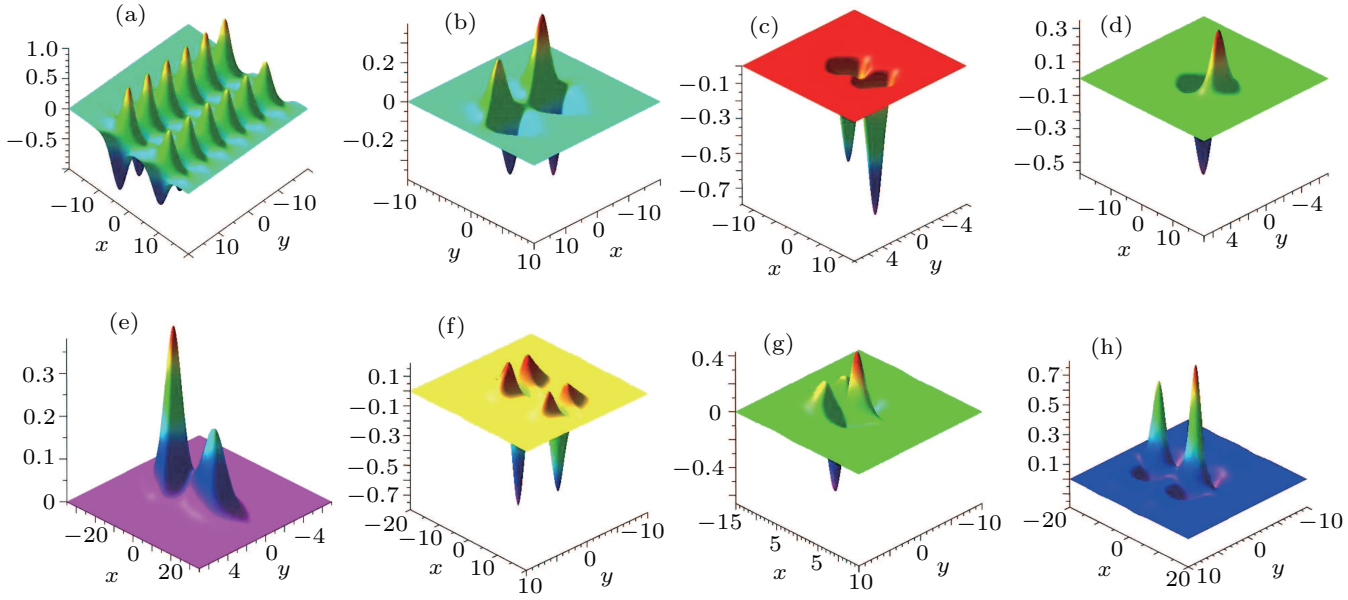


Fig. 10. $v_J^{[2]}$ with $t = 0$, $\alpha = -1/2$. (a) Periodic solution: $\phi(y) = \mathrm{sn}(y, 3/10)$, $k_1 = 3/4$, $k_2 = 4/5$, $p_1 = 1/2$, $p_2 = 2/3$, $\eta_1^0 = -5$, $\eta_2^0 = 10$; (b) Dormion-type-I soliton: $\phi(y) = \mathrm{cn}(y, 1)$, $k_1 = 3/4$, $k_2 = 4/5$, $p_1 = 1/2$, $p_2 = 2/3$, $\eta_1^0 = -2$, $\eta_2^0 = 3$; (c) Dark dormion-type-II soliton: $\phi(y) = \mathrm{sn}(y, 1)$, $k_1 = 3/4$, $k_2 = 4/5$, $p_1 = 1/2$, $p_2 = 2/3$, $\eta_1^0 = 2$, $\eta_2^0 = -2$; (d) Mixed dormion-type-II soliton: $\phi(y) = \mathrm{sn}(y, 1)$, $k_1 = -3/4$, $k_2 = 4/5$, $p_1 = 1/2$, $p_2 = 2/3$, $\eta_1^0 = 2$, $\eta_2^0 = -2$; (e) Bright dormion-type-II soliton: $\phi(y) = \mathrm{sn}(y, 1)$, $k_1 = -1/4$, $k_2 = -2/5$, $p_1 = 1/2$, $p_2 = 2/3$, $\eta_1^0 = 2$, $\eta_2^0 = -2$; (f) Dark dormion-type-III soliton: $\phi(y) = \mathrm{sn}(y, 1/2)/(y^2+1)$, $k_1 = 3/4$, $k_2 = 4/5$, $p_1 = 1/2$, $p_2 = 2/3$, $\eta_1^0 = -2$, $\eta_2^0 = 3$; (g) Mixed dormion-type-III soliton: $\phi(y) = \mathrm{sn}(y, 1/2)/(y^2+1)$, $k_1 = -3/4$, $k_2 = 4/5$, $p_1 = 1/2$, $p_2 = 2/3$, $\eta_1^0 = -2$, $\eta_2^0 = 3$; (h) Bright dormion-type-III soliton: $\phi(y) = \mathrm{sn}(y, 1/2)/(y^2+1)$, $k_1 = -1/4$, $k_2 = -2/5$, $p_1 = 1/2$, $p_2 = 2/3$, $\eta_1^0 = -2$, $\eta_2^0 = 3$.

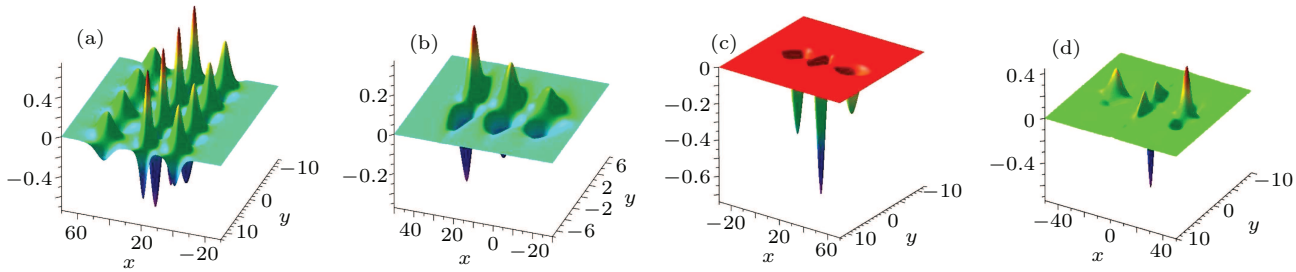


Fig. 11. $v_J^{[3]}$ with $\alpha = 1$, $t = 0$, $p_1 = 2/3$, $p_2 = 3/5$, $p_3 = 1/2$. (a) Periodic solution: $\phi(y) = \mathrm{sn}(y, 3/10)$, $k_1 = 3/4$, $k_2 = 1/2$, $k_3 = 1/3$, $\eta_1^0 = -10$, $\eta_2^0 = 5$, $\eta_3^0 = -8$; (b) Dormion-type-I soliton: $\phi(y) = \mathrm{cn}(y, 1)$, $k_1 = 3/4$, $k_2 = 1/2$, $k_3 = 1/3$, $\eta_1^0 = -5$, $\eta_2^0 = 5$, $\eta_3^0 = 7$; (c) Dark dormion-type-II soliton: $\phi(y) = \mathrm{sn}(y, 1)$, $k_1 = 3/4$, $k_2 = 1/2$, $k_3 = 1/3$, $\eta_1^0 = -10$, $\eta_2^0 = 0$, $\eta_3^0 = -8$; (d) Mixed dormion-type-III soliton: $\phi(y) = \mathrm{sn}(y, 1/2)/(y^2+1)$, $k_1 = 3/4$, $k_2 = -1/2$, $k_3 = -1/3$, $\eta_1^0 = 0$, $\eta_2^0 = 16$, $\eta_3^0 = -4$.

3.3. Case 3: $N=3$

By setting

$$f = 1+\mathrm{e}^{\eta_1}+\mathrm{e}^{\eta_2}+\mathrm{e}^{\eta_3}+\alpha_{12}\mathrm{e}^{\eta_1+\eta_2}+\alpha_{13}\mathrm{e}^{\eta_1+\eta_3} + \alpha_{23}\mathrm{e}^{\eta_2+\eta_3}+\alpha_{123}\mathrm{e}^{\eta_1+\eta_2+\eta_3}, \tag{30}$$

in Eqs. (9), (10), and (11), we can obtain the third-order Jacobi-type solution $v_J^{[3]}$ and $r_J^{[3]}$, where

$$\eta_i = k_i x + p_i\phi(y) - \omega_i t + \eta_i^0,\ \ \omega_i = 3k_i^3 + \alpha k_i,$$
$$\alpha_{ij} = (k_i-k_j)(p_i-p_j)/(k_i+k_j)(p_i+p_j),$$
$$\alpha_{123} = \alpha_{12}\alpha_{13}\alpha_{23}.$$

By the same way used above, we obtain more kinds of solutions through choosing different function $\phi(y)$ and η_i. Part of the situation is shown in Fig. 11.

4. Summary

In this paper, we obtained the bilinear form and the n-soliton solution of an eSWW equation by using Hirota method. The solution $v^{[1]}$ is dark when $k_1p_1>0$ while it is bright when $k_1p_1<0$. The solution $r^{[1]}$ is always a dark soliton. By using complexification method, breathers and hybrid solutions are constructed which are all travelling waves.

More importantly, we obtained Jacobi-type solution associated with a certain given $\phi(y)$ which is an arbitrary real continuous function appeared in f of bilinear form. It is selected by using Jacobi elliptic functions, and the character of the solutions depends on its specific choice. We stress on the case $N=1$. The first-order Jacobi-type solution $v_J^{[1]}$ parallelly travels along the x axis with the velocity $(3k_1^2+\alpha,\ 0)$ on $(x,\ y)$-plane.

(i) When $\phi(y)=\mathrm{sn}(y,3/10)$, equation (22) is a periodic solution and the period depends on $\phi(y)$.

(ii) When $\phi(y)=\mathrm{cn}(y,1)$, we obtained a dormion-type-I solutions. Equation (23) has one maximum $(3/4)k_1p_1$ located at $((3k_1^2+\alpha)t-(\sqrt{2}p_1+2\eta_1^0)/2k_1,\ \mathrm{arccosh}\sqrt{2})$ and one minimum $-(3/4)k_1p_1$ located at $((3k_1^2+\alpha)t-\sqrt{2}p_1+2\eta_1^0)/2k_1,\ -\mathrm{arccosh}\sqrt{2})$. The width of the contour line on the hight $h=(3/8)k_1p_1$ (the half amplitude) is $\ln(2+\sqrt{6}+\sqrt{2}+\sqrt{3})/(2+\sqrt{6}-\sqrt{2}-\sqrt{3})$.

(iii) When $\phi(y)=\mathrm{sn}(y,1)$, we obtained a dormion-type-II solution equation (26) which has only one extreme value $-(3/2)k_1p_1$ located at $((3k_1^2+\alpha)t-\eta_1^0/k_1,0)$ of (x,y) plane. The width of the contour line on the hight $h=-(3/4)k_1p_1$ (the half amplitude) is $\ln(\sqrt{2}+1)/(\sqrt{2}-1)$.

(iv) When $\phi(y)=\mathrm{sn}(y,1/2)/(1+y^2)$, we obtained a dormion-type-III solution (21) which shows very strong doubly localized feature on (x,y) plane, and the profile of this solution is invariant during the propagation along $y=0$.

Moreover, we gave several figures of the mixture of periodic and localized solutions.

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