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Time evolution of angular momentum coherent state derived by virtue of entangled state representation and a new binomial theorem*

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We study how can an angular momentum coherent state $|\tau\rangle$ keeps its form-invariant during time evolution governed by the Hamiltonian $H = f(t)J_+ + f^*(t)J_- + g(t)J_z$. We discuss this topic in the context of boson realization of $|\tau\rangle$. By employing the entangled state representation $|\zeta\rangle$ and deriving a new binomial theorem involving two-subscript Hermite polynomials, we derive the wave function $\langle\zeta|\tau\rangle$, which turns out to be a single-subscript Hermite polynomial. Based on this result the maintenance of angular momentum coherent state during time evolution is examined, and the value of $\tau(t)$ is totally determined by the parameters involved in the Hamiltonian.

Keywords: invariance of angular momentum coherent state, entangled state representation, new binomial theorem, time evolution

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1. Introduction

Coherent state theory has been widely used in quantum optics and laser physics. The optical coherent state with the amplitude α is defined as[1–4]

$$|\alpha\rangle = \exp(\alpha a^\dagger - \alpha^* a)|0\rangle, \tag{1}$$

where a and $a^\dagger$ are respectively boson annihilation and creation operators, $[a, a^\dagger] = 1$, $a|0\rangle = 0$. It has been common knowledge that the Hamiltonian

$$H_0 = \omega a^\dagger a + f(t)a^\dagger + f^*(t)a \tag{2}$$

can keep form-invariant for coherent state, *i.e.*, if the initial state is a coherent state, it will evolve into still a coherent state. In this work we search for which kind of Hamiltonian can maintain the form of angular momentum coherent state (AMCS), the AMCS is defined as[5–9]

$$|\tau\rangle_j = \exp(\mu J_+ - \mu^* J_-)|j,-j\rangle, \tag{3}$$

where J_+ and J_- are raising and lowering operators of angular momentum, respectively, obeying the following commutative relations

$$[J_+, J_-] = 2J_z, \quad [J_\pm, J_z] = \mp J_\pm, \tag{4}$$

J_z has its eigenstate

$$J_z|j,m\rangle = m|j,m\rangle, \tag{5}$$

and $|j,-j\rangle$ is the lowest weight state annihilated by J_-, *i.e.*, $J_-|j,-j\rangle = 0$. Using the disentangling formula

$$\exp(\mu J_+ - \mu^* J_-) = \mathrm{e}^{\tau J_+}\exp\left[J_z \ln(1+|\tau|^2)\right]\mathrm{e}^{-\tau^* J_-}, \tag{6}$$

where $\mu = (\theta/2)\mathrm{e}^{-\mathrm{i}\varphi}$ and $\tau = \mathrm{e}^{-\mathrm{i}\varphi}\tan(\theta/2)$. Equation (3) becomes

$$|\tau\rangle_j = \frac{1}{(1+|\tau|^2)^j}\mathrm{e}^{\tau J_+}|j,-j\rangle. \tag{7}$$

We shall begin with the boson realization of AMCS to proceed our discussion. We then introduce the entangled state representation and derive a new binomial theorem involving two-subscript Hermite polynomials to go on. It turns out that the wave function of AMCS in the entangled state representation is classified as single-subscript Hermite polynomials H_{2j}. Based on this result and the recursive property of H_{2j} we find that the Hamiltonian $H = f(t)J_+ + f^*(t)J_- + g(t)J_z$ can maintain the form-invariance of AMCS $|\tau(t)\rangle_j$, and the value of $\tau(t)$ is totally determined by $f(t)$, $f^*(t)$, and $g(t)$.

The rest of this paper is arranged as follows. In Section 2 we reform the angular momentum coherent state $|\tau\rangle_j$ in boson realization and show that it is an entangled state. In Section 3 we set up the entangled state representation and introduce the two-subscript Hermite polynomials. In Section 4 we deduce the generalized binomial formula which involves two-subscript Hermite polynomials. In Section 5 we obtain the wave function of $|\tau\rangle_j$ in the entangled state representation.

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In Section 6 we examine the form-invariance of AMCS during time evolution and determine the value of $\tau(t)$.

2. Angular momentum coherent state in boson realization

Using Schwinger boson realization of angular momentum operators

$$J_+ = a^\dagger b,\ \ J_- = ab^\dagger,\ \ \ J_z = \frac{1}{2}\left(a^\dagger a - b^\dagger b\right), \tag{8}$$

the eigenstate of J_z is

$$|j,m\rangle = \frac{a^{\dagger j+m} b^{\dagger j-m}}{\sqrt{(j+m)!\,(j-m)!}}\,|00\rangle, \tag{9}$$

where $|00\rangle$ is two-mode vacuum state in Fock space, $a|00\rangle = 0$, $b|00\rangle = 0$, thus

$$|j,-j\rangle = \frac{b^{\dagger 2j}}{\sqrt{(2j)!}}\,|00\rangle = |0\rangle \otimes |2j\rangle, \tag{10}$$

and we can reform the AMCS in Eq. (7) as

$$\begin{aligned}|\tau\rangle_j &= \frac{1}{(1+|\tau|^2)^j} e^{a^\dagger b\tau} \frac{b^{\dagger 2j}}{\sqrt{(2j)!}}\,|00\rangle \\ &= \frac{1}{\sqrt{(2j)!}\,(1+|\tau|^2)^j}\left(b^\dagger + a^\dagger\tau\right)^{2j}|00\rangle \\ &= \frac{1}{\sqrt{(2j)!}\,(1+|\tau|^2)^j}\sum_{l=0}^{2j}\binom{2j}{l} b^{\dagger l}\left(a^\dagger\tau\right)^{2j-l}|00\rangle \\ &= \frac{1}{(1+|\tau|^2)^j}\sum_{l=0}^{2j}\binom{2j}{l}^{1/2}\tau^{2j-l}\,|2j-l\rangle\otimes|l\rangle, \end{aligned} \tag{11}$$

which exhibits quantum entanglement between the two modes, this enlighten us to introduce the entangled state representation to study $|\tau\rangle_j$.

3. The entangled state representation and the two-subscript Hermite polynomials

In Refs. [10] and [11], the unnormalized continuous-variable entangled state $|\xi\rangle$, which is the common eigenvector of $\left(a+b^\dagger\right)$ and $\left(a^\dagger+b\right)$, noting $\left[\left(a+b^\dagger\right),\left(a^\dagger+b\right)\right]=0$, has been constructed as follows:

$$|\zeta\rangle = \exp\left(a^\dagger\zeta + \zeta^* b^\dagger - a^\dagger b^\dagger\right)|00\rangle,\ \ \zeta = \zeta_1 + \mathrm{i}\zeta_2, \tag{12}$$

which takes the two-subscript Hermite polynomials $\mathrm{H}_{m,n}(\zeta,\zeta^*)$ as its expansion function in the two-mode Fock space

$$|\zeta\rangle = \sum_{m,n=0}^{\infty}\frac{a^{\dagger m}b^{\dagger n}}{m!n!}\mathrm{H}_{m,n}(\zeta,\zeta^*)\,|00\rangle, \tag{13}$$

here $\mathrm{H}_{m,n}(\xi,\xi^*)$ can be either defined through its generating function[12,13]

$$\sum_{m,n=0}^{\infty}\frac{s^m t^n}{m!n!}H_{m,n}(\xi,\xi^*) = \exp\left(-st + s\xi + t\xi^*\right), \tag{14}$$

or by its power-series expansion

$$\begin{aligned}&\mathrm{H}_{m,n}(\xi,\xi^*) \\ &= \frac{\partial^{m+n}}{\partial s^m \partial t^n}\exp\left(-st+s\xi+t\xi^*\right)|_{s=t=0} \\ &= \sum_{l=0}^{\min(m,n)}\frac{m!n!(-1)^l}{l!(m-l)!(n-l)!}\xi^{m-l}\xi^{*n-l}. \end{aligned} \tag{15}$$

Note that $\mathrm{H}_{m,n}$ is not a direct product of two independent single-subscript Hermite polynomials. $\mathrm{H}_{m,n}$ have their own uses in studying quantum optics and light propagation modes in quadratic graded-index media. From Eq. (13) we see

$$\langle\zeta\,|m,n\rangle = \frac{1}{\sqrt{m!n!}}\mathrm{H}_{m,n}(\zeta,\zeta^*). \tag{16}$$

An important insight in $|\zeta\rangle$ is its completeness relation

$$\int\frac{\mathrm{d}^2\zeta}{\pi}\,\mathrm{e}^{-|\zeta|^2}\,|\zeta\rangle\langle\zeta| = 1. \tag{17}$$

4. A generalized binomial formula involved two-subscript Hermite polynomials

From Eqs. (11) and (16) we see the wave function

$$\langle\zeta\,|\tau\rangle_j = \frac{\sqrt{(2j)!}}{(1+|\tau|^2)^j}\sum_{l=0}^{2j}\frac{\tau^{2j-l}}{l!\,(2j-l)!}\mathrm{H}_{2j-l,l}(\zeta,\zeta^*). \tag{18}$$

We are challenged to make the summation in Eq. (18), which involves $\mathrm{H}_{2j-l,l}$. Instead of directly doing this, we try to prove a generalized binomial formula by considering

$$\sum_{l=0}^{2j}\binom{2j}{l}\vdots\,\mathrm{H}_{2j-l,l}(a^\dagger,a)\,\vdots\,\tau^{2j-l}, \tag{19}$$

where $\vdots\,\mathrm{H}_{2j-l,l}(a^\dagger,a)\,\vdots$ is in antinormal ordering form, noting that a and $a^\dagger$ are permuted within the antinormal ordering symbol $\vdots\ \vdots$.[14,15] Using Eq. (14) we have

$$\sum_{m,n=0}^{\infty}\frac{s^m t^n}{m!n!}\vdots\,\mathrm{H}_{m,n}(a^\dagger,a)\,\vdots = \vdots\exp\left(-st+sa^\dagger+ta\right)\vdots = \mathrm{e}^{-st}\mathrm{e}^{ta}\mathrm{e}^{sa^\dagger} = \mathrm{e}^{sa^\dagger}\mathrm{e}^{ta} = \sum_{m,n=0}^{\infty}\frac{s^m t^n}{m!n!}a^{\dagger m}a^n. \tag{20}$$

Comparing $s^m t^n$ on both sides of Eq. (20) we have the operator identity

$$\vdots\,\mathrm{H}_{m,n}(a^\dagger,a)\,\vdots = a^{\dagger m}a^n =: a^{\dagger m}a^n: \ =: a^n a^{\dagger m}:, \tag{21}$$

where : : denotes normal ordering.[16–19] Thus

$$\sum_{l=0}^{2j}\binom{2j}{l}\vdots\,\mathrm{H}_{2j-l,l}(a^\dagger,a)\,\vdots\,\tau^{2j-l} = \sum_{l=0}^{2j}\binom{2j}{l}\tau^{2j-l}a^{\dagger 2j-l}a^l = \sum_{l=0}^{2j}\binom{2j}{l}\tau^{2j-l}:a^{\dagger 2j-l}a^l: \ =: \left(\tau a^\dagger + a\right)^{2j}:. \tag{22}$$

By making up

$$\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}:\left(\tau a^{\dagger}+a\right)^n:=:\exp\left[\lambda\left(\tau a^{\dagger}+a\right)\right]:=\vdots\, \mathrm{e}^{\lambda a}\mathrm{e}^{\lambda\tau a^{\dagger}}\mathrm{e}^{-\lambda^2\tau}\,\vdots=\sum_{n=0}^{\infty}\frac{\left(\sqrt{\tau}\lambda\right)^n}{n!}\vdots\,\mathrm{H}_n\left(\frac{\tau a^{\dagger}+a}{2\sqrt{\tau}}\right)\vdots, \tag{23}$$

noting λ^n on the two sides of Eq. (23) we see

$$:\left(\tau a^{\dagger}+a\right)^n:=\left(\sqrt{\tau}\right)^n\vdots\,\mathrm{H}_n\left(\frac{\tau a^{\dagger}+a}{2\sqrt{\tau}}\right)\vdots, \tag{24}$$

therefore equation (22) becomes

$$\sum_{l=0}^{2j}\binom{2j}{l}\vdots\,\mathrm{H}_{2j-l,l}(a^{\dagger},a)\,\vdots\,\tau^{2j-l}=\tau^j\vdots\,H_{2j}\left(\frac{\tau a^{\dagger}+a}{2\sqrt{\tau}}\right)\vdots. \tag{25}$$

Since the two sides of Eq. (25) are both in antinormal ordering, so we obtain a new generalized binomial formula

$$\sum_{l=0}^{2j}\binom{2j}{l}\mathrm{H}_{2j-l,l}(\zeta,\zeta^*)\tau^{2j-l}=\tau^j\mathrm{H}_{2j}\left(\frac{\tau\zeta+\zeta^*}{2\sqrt{\tau}}\right), \tag{26}$$

its right-hand side appears as a single-subscript Hermite polynomial. Further, let

$$\chi\equiv\frac{\tau\zeta+\zeta^*}{2\sqrt{\tau}}. \tag{27}$$

Equation (18) becomes

$$\begin{aligned}\langle\zeta\,|\tau\rangle_j&=\frac{\sqrt{(2j)!}}{(1+|\tau|^2)^j}\sum_{l=0}^{2j}\frac{\tau^{2j-l}}{l!\,(2j-l)!}\mathrm{H}_{2j-l,l}(\zeta,\zeta^*)\\&=\frac{1}{\sqrt{(2j)!}\,(1+|\tau|^2)^j}\tau^j\mathrm{H}_{2j}(\chi),\end{aligned} \tag{28}$$

here H_{2j} denotes the single-subscript Hermite polynomial of order $2j$. Hence the wave function of angular momentum coherent state in the entangled state representation is classifies as $\mathrm{H}_{2j}(\chi)$. This brings convenience for further studying $|\tau\rangle_j$.

5. Time evolution of AMCS

As an application of the above theory, we consider the system described by the Hamiltonian

$$H=f(t)J_+ + f^*(t)J_- + g(t)J_z, \tag{29}$$

and supposing it initially is in angular momentum coherent state $\langle\tau(0)|$, we wish that it still evolves into an angular momentum coherent state, which means $H|\tau(t)\rangle=E(t)|\tau(t)\rangle$, then the Schrödinger equation is

$$\mathrm{i}\frac{\partial}{\partial t}|\tau(t)\rangle=H|\tau(t)\rangle=E(t)|\tau(t)\rangle. \tag{30}$$

Inserting the completeness relation of entangled states in Eq. (17)

$$\mathrm{i}\frac{\partial}{\partial t}|\tau(t)\rangle=\int\frac{d^2\zeta}{\pi}e^{-|\zeta|^2}|\zeta\rangle\langle\zeta|H|\tau(t)\rangle, \tag{31}$$

and using the following relations

$$\begin{aligned}\langle\zeta|J_+&=\langle\zeta|a^{\dagger}b=\left(\zeta-\frac{\partial}{\partial\zeta^*}\right)\frac{\partial}{\partial\zeta^*}\langle\zeta|,\\\langle\zeta|J_-&=\langle\zeta|b^{\dagger}a=\left(\zeta^*-\frac{\partial}{\partial\zeta}\right)\frac{\partial}{\partial\zeta}\langle\zeta|,\\\langle\zeta|J_z&=\frac{1}{2}\langle\zeta|\left(a^{\dagger}a-b^{\dagger}b\right)=\frac{1}{2}\left(\zeta\frac{\partial}{\partial\zeta}-\zeta^*\frac{\partial}{\partial\zeta^*}\right)\langle\zeta|,\end{aligned} \tag{32}$$

we have

$$\langle\zeta|H|\tau(t)\rangle=\left[f\left(\zeta-\frac{\partial}{\partial\zeta^*}\right)\frac{\partial}{\partial\zeta^*}+f^*\left(\zeta^*-\frac{\partial}{\partial\zeta}\right)\frac{\partial}{\partial\zeta}+\frac{g}{2}\left(\zeta\frac{\partial}{\partial\zeta}-\zeta^*\frac{\partial}{\partial\zeta^*}\right)\right]\langle\zeta\,|\tau(t)\rangle=E\langle\zeta\,|\tau(t)\rangle. \tag{33}$$

Since we assume $|\tau(t)\rangle$ is an angular momentum coherent state, using Eq. (28) the above equation (34) turns to

$$\left[f\left(\zeta-\frac{\partial}{\partial\zeta^*}\right)\frac{\partial}{\partial\zeta^*}+f^*\left(\zeta^*-\frac{\partial}{\partial\zeta}\right)\frac{\partial}{\partial\zeta}+\frac{g}{2}\left(\zeta\frac{\partial}{\partial\zeta}-\zeta^*\frac{\partial}{\partial\zeta^*}\right)\right]\mathrm{H}_{2j}(\chi)=E\,\mathrm{H}_{2j}(\chi). \tag{34}$$

Noting χ in Eq. (27), so

$$\frac{\partial}{\partial\zeta}=\frac{\partial\chi}{\partial\zeta}\frac{\partial}{\partial\chi}=\frac{\sqrt{\tau}}{2}\frac{\partial}{\partial\chi},\qquad\frac{\partial}{\partial\zeta^*}=\frac{1}{2\sqrt{\tau}}\frac{\partial}{\partial\chi}, \tag{35}$$

and using the property of $\mathrm{H}_n(\chi)$ (Refs. [6] and [20])

$$2\chi\mathrm{H}_n'(\chi)-\mathrm{H}_n''(\chi)=2nH_n(\chi), \tag{36}$$

we can respectively express the terms in Eq. (35) as

$$\begin{aligned}I_1&\equiv\left(\zeta-\frac{\partial}{\partial\zeta^*}\right)\frac{\partial}{\partial\zeta^*}\mathrm{H}_{2j}(\chi)\\&=\left(\zeta\frac{1}{2\sqrt{\tau}}\frac{\partial}{\partial\chi}-\frac{1}{4\tau}\frac{\partial^2}{\partial\chi^2}\right)\mathrm{H}_{2j}(\chi)\\&=\zeta\frac{1}{2\sqrt{\tau}}H_{2j}'(\chi)-\frac{1}{4\tau}[2\chi\mathrm{H}_{2j}'(\chi)-4j\mathrm{H}_{2j}(\chi)]\\&=\left(\zeta\frac{1}{2\sqrt{\tau}}-\frac{\chi}{2\tau}\right)\mathrm{H}_{2j}'(\chi)+\frac{j}{\tau}\mathrm{H}_{2j}(\chi),\end{aligned} \tag{37}$$

$$\begin{aligned}I_2&\equiv\left(\zeta^*-\frac{\partial}{\partial\zeta}\right)\frac{\partial}{\partial\zeta}\mathrm{H}_{2j}(\chi)\\&=\left(\zeta^*\frac{\sqrt{\tau}}{2}\frac{\partial}{\partial\chi}-\frac{\tau}{4}\frac{\partial^2}{\partial\chi^2}\right)\mathrm{H}_{2j}(\chi)\\&=\zeta^*\frac{\sqrt{\tau}}{2}\mathrm{H}_{2j}'(\chi)-\frac{\tau}{4}[2\chi\mathrm{H}_{2j}'(\chi)-4j\mathrm{H}_{2j}(\chi)]\\&=\left(\zeta^*\frac{\sqrt{\tau}}{2}-\frac{\tau\chi}{2}\right)\mathrm{H}_{2j}'(\chi)+\tau j\mathrm{H}_{2j}(\chi).\end{aligned} \tag{38}$$

The reverse relation of $\chi=(\zeta^*+\tau\zeta)/2\sqrt{\tau}$ is

$$\zeta=\frac{4|\tau|}{|\tau|^2-1}\left(\frac{\sqrt{\tau^*}}{2}\chi-\frac{\chi^*}{2\sqrt{\tau}}\right). \tag{39}$$

Thus equation (38) becomes

$$I_1 = \left(\zeta\frac{1}{2\sqrt{\tau}} - \frac{\chi}{2\tau}\right)\mathrm{H}'_{2j}(\chi) + \frac{j}{\tau}\mathrm{H}_{2j}(\chi)$$
$$= \frac{\left(|\tau|^2+1\right)\chi - 2\chi^*|\tau|}{2\tau\left(|\tau|^2-1\right)}\mathrm{H}'_{2j}(\chi) + \frac{j}{\tau}\mathrm{H}_{2j}(\chi), \quad (41)$$

while equation (39) takes the form

$$I_2 = \frac{1}{2}\left(\zeta^*\sqrt{\tau} - \tau\chi\right)\mathrm{H}'_{2j}(\chi) + \tau j\mathrm{H}_{2j}(\chi)$$
$$= \frac{\tau\left[2\chi^*|\tau| - \chi\left(|\tau|^2+1\right)\right]}{2\left(|\tau|^2-1\right)}\mathrm{H}'_{2j}(\chi) + j\tau\mathrm{H}_{2j}(\chi), \quad (42)$$

$$I_3 = \frac{1}{2}\left(\zeta\frac{\partial}{\partial\zeta} - \zeta^*\frac{\partial}{\partial\zeta^*}\right)\mathrm{H}_{2j}(\chi)$$
$$= \frac{\left(|\tau|^2+1\right)\chi - 2|\tau|\chi^*}{2\left(|\tau|^2-1\right)}\mathrm{H}'_{2j}(\chi). \quad (43)$$

Then the eigenfunction equation is

$$E\,\mathrm{H}_{2j}(\chi) = fI_1 + f^*I_2 + gI_3$$
$$= \frac{f - \tau^2 f^* + g\tau}{2\left(|\tau|^2-1\right)}\left[\left(|\tau|^2+1\right)\chi - 2\chi^*|\tau|\right]H'_{2j}(\chi) + f\frac{\tau^2+1}{\tau}j\mathrm{H}_{2j}(\chi). \quad (44)$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}\chi}\mathrm{H}_{2j}(\chi) = 4j\mathrm{H}_{2j-1}(\chi),$$

and the Hermite polynomials of different orders are mutual orthogonal, comparing the two sides of Eq. (44) we finally obtain

$$\frac{f - \tau^2 f^* + g\tau}{\left(|\tau|^2-1\right)}\left(\frac{|\tau|^2+1}{2}\chi - \chi^*|\tau|\right) = 0, \quad (45)$$

and

$$E = \frac{f}{\tau}\left(\tau^2+1\right)j. \quad (46)$$

Because the eigenvalue equation cannot depend on the entangled state parameter ζ, then cannot depend on χ either, we have the value of τ, that is

$$f - \tau^2 f^* + g\tau = 0,\ \tau = \frac{g \pm \sqrt{g^2 + 4|f|^2}}{2f^*}, \quad (47)$$

which is totally determined by the parameters in the Hamiltonian (29). Moreover, we know the eigenvalues are

$$E = \left(\frac{f}{\tau} + f^*\tau\right)j = \pm j\sqrt{g^2 + 4|f|^2}. \quad (48)$$

In summary, we have studied how an angular momentum coherent state $|\tau\rangle$ keeps its form-invariant during time evolution governed by the Hamiltonian $H = f(t)J_+ + f^*(t)J_- + g(t)J_z$. By employing the entangled state representation $|\zeta\rangle$ and deriving a new binomial theorem involving two-subscript Hermite polynomials we have derived the wave function $\langle\zeta\,|\tau\rangle$, which is a single-subscript Hermite polynomial. Based on this result the time evolution of angular momentum coherent state is examined, and the value of $\tau(t)$ is determined. It should be noted that our idea can be generalized to many other quantum entangled systems: using recursion relations of new binomial theorem involving special functions to convert differential operation to algebraic operation.

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