ISSN 1674-1056
http://cpb.iphy.ac.cn
http://iopscience.iop.org/cpb
中国物理 B
Chinese Physics
中国物理
Chinese Physics B

Painlevé integrability of the supersymmetric Ito equation

Feng-Jie Cen(岑锋杰), Yan-Dan Zhao(赵燕丹), Shuang-Yun Fang(房霜韵), Huan Meng(孟欢), Jun Yu(俞军)
Citation: Chin. Phys. B . 2019, 28(9): 090201 . **doi:** 10.1088/1674-1056/28/090201
Journal homepage: http://cpb.iphy.ac.cn; http://iopscience.iop.org/cpb

What follows is a list of articles you may be interested in

A nonlocal Burgers equation in atmospheric dynamical system and its exact solutions

Xi-Zhong Liu(刘希忠), Jun Yu(俞军), Zhi-Mei Lou(楼智美), Xian-Min Qian(钱贤民)
Chin. Phys. B . 2019, 28(1): 010201 . **doi:** 10.1088/1674-1056/28/1/010201

A macroscopic traffic model based on weather conditions

Zawar H. Khan, Syed Abid Ali Shah, T. Aaron Gulliver
Chin. Phys. B . 2018, 27(7): 070202 . **doi:** 10.1088/1674-1056/27/7/070202

Multi-soliton solutions for the coupled modified nonlinear Schrödinger equations via Riemann-Hilbert approach

Zhou-Zheng Kang(康周正), Tie-Cheng Xia(夏铁成), Xi Ma(马茜)
Chin. Phys. B . 2018, 27(7): 070201 . **doi:** 10.1088/1674-1056/27/7/070201

Integrability classification and exact solutions to generalized variable-coefficient nonlinear evolution equation

Han-Ze Liu(刘汉泽), Li-Xiang Zhang(张丽香)
Chin. Phys. B . 2018, 27(4): 040202 . **doi:** 10.1088/1674-1056/27/4/040202

Painlevé integrability of generalized fifth-order KdV equation with variable coefficients: Exact solutions and their interactions

Xu Gui-Qiong
Chin. Phys. B . 2013, 22(5): 050203 . **doi:** 10.1088/1674-1056/22/5/050203

Painlevé integrability of the supersymmetric Ito equation*

Feng-Jie Cen(岑锋杰), Yan-Dan Zhao(赵燕丹), Shuang-Yun Fang(房霜韵), Huan Meng(孟欢), and Jun Yu(俞军)†

Department of Physics, Shaoxing University, Shaoxing 312000, *China*

(Received 20 May 2019; revised manuscript received 23 June 2019; published online 30 August 2019)

A supersymmetric version of the Ito equation is proposed by extending the independent and dependent variables for the classic Ito equation. To investigate the integrability of the $N = 1$ supersymmetric Ito (sIto) equation, a singularity structure analysis for this system is carried out. Through a detailed analysis in two cases by using Kruskal's simplified method, the sIto system is found to pass the Painlevé test, and thus is Painlevé integrable.

Keywords: supersymmetric Ito equation, Painlevé analysis, integrability

PACS: 02.30.Jr, 02.30.Ik, 05.45.Yv, 47.35.Fg **DOI:** 10.1088/1674-1056/ab38a7

1. Introduction

The conception of supersymmetry was first proposed by physicists aiming to unify all field theories into one,[1–3] inspired by which mathematicians developed many efficient methods to construct supersymmetric systems, often with good integrability. For instance, the supersymmetric versions of the Sine–Gordon equation, the Kadomtsev–Petviashvili (KP) hierarchy, the Korteweg–de Vries (KdV) hierarchy, the Boussinesq equation, and a number of other systems were established and their soliton solutions were obtained by different methods.[4–10]

It is well known that for bosonic systems, through Painlevé analysis not only integrable properties could be investigated, but also Bäcklund transformations, Hirota's bilinear form, and even Lax representations of many nonlinear systems could be obtained.[11,12] The Painlevé integrability demands that there exist Laurent series on open sets of the complex time variable solutions and these solutions should be consistent, or match on the overlapping pieces of the sets on which they are defined. In other words, this notion of integrability is the existence of a meromorphic solution. Painlevé analysis has also been extended to super nonlinear systems to study the super KdV equation[4] and many other important nonlinear systems.[8,13–18]

It is well known that the Ito equation

$$u_{tt} + 6(u_x u_t)_x + u_{xxxt} = 0 \tag{1}$$

can be transformed into the bilinear form

$$D_t(D_t + D_x^3) f \cdot f = 0, \tag{2}$$

by the transformation

$$u = (\ln f)_x. \tag{3}$$

In this paper, we concentrate on the Ito equation (1) to find its supersymmetric form and take the standard Painlevé test by using Kruskal's simplification. Different properties of the Ito equation (1) or its different form of variation have been discussed in many literatures. Ito has obtained the first explicitly expressed N-soliton solutions of Eq. (2). For the Ito integro-differential equation, Ebadi *et al.*[19] obtained its N-soliton solutions and conserved quantities. The integrable properties of this equation such as the nonlinear superposition formula, Kac–Moody algebra, bi-Hamiltonian structure have also been found in Refs. [20]–[22].

This paper will be devoted to the integrability of the supersymmetric Ito (sIto) equation. In the next section, we supersymmetrize the Ito equation by simply introducing a new Grassmann variable and a superderivative. In Section 3, a detailed Painlevé test is carried on the supersymmetric Ito equation, which is categorized into to two cases. The last section is devoted to a short summary and discussion.

2. Supersymmetrization of the Ito equation

In order to supersymmetrize the Ito equation (1), in the first step, we have to introduce some new independent variables by extending the variable t to a doublet (t,θ) with θ being an Grassmann odd variable. So the original independent variables (x,t) are generalized to super case (x,t,θ) and the associated superderivative is

$$\mathscr{D}_t = \theta\partial_t + \partial_\theta. \tag{4}$$

In the next step, we have to extend the dependent variable $u(x,t)$. There exist two ways to do so, one of which is to introduce a bosonic superfield

$$\Psi(x,t,\theta) = u(x,t) + \theta\eta(x,t), \tag{5}$$

*Project supported by the National Natural Science Foundation of China (Grant Nos. 11975156 and 11775146) and the Natural Science Foundation of Zhejiang Province, China (Grant No. LY18A050001).

†Corresponding author. E-mail: junyu@usx.edu.cn

© 2019 Chinese Physical Society and IOP Publishing Ltd

and the other one is to introduce a fermionic superfield

$$\Phi(x,t,\theta)=\zeta(x,t)+\theta v(x,t), \tag{6}$$

where $u(x,t),v(x,t)$ and $\eta(x,t),\zeta(x,t)$ are Grassmann even and odd variables, respectively.

In this paper, we concentrate on the first extension way. To accomplish the symmetrization procedure, we first multiply each term of the Ito equation (1) by the Grassmann odd variable θ and then rewrite them in terms of the corresponding superfields, i.e.,

$$u_{tt}\to\mathscr{D}_t\Psi_t,\quad u_{xxxt}\to\mathscr{D}_t\Psi_{xxx},\quad 6(u_xu_t)_x\to 6(\Psi_x(\mathscr{D}_t\Psi))_x. \tag{7}$$

By substituting, we finally obtain the supersymmetric Ito equation in the form

$$\mathscr{D}_t\Psi_t+6(\Psi_x(\mathscr{D}_t\Psi))_x+\mathscr{D}_t\Psi_{xxx}=0, \tag{8}$$

which can be written in component form by using Eq. (5) as

$$u_{tt}+6(u_xu_t+\eta_x\eta)_x+u_{xxxt}=0, \tag{9a}$$

$$\eta_t+6(u_x\eta)_x+\eta_{xxx}=0. \tag{9b}$$

3. Painlevé integrablity of the supersymmetric Ito equation

To explore the Painlevé integrability property of the Ito Eq. (1), we start with the following series expansions for the field variables:

$$u=\varphi^{\alpha}\sum_{j=0}^{\infty}u_j\varphi^j, \tag{10}$$

$$\eta=\varphi^{\beta}\sum_{j=0}^{\infty}\eta_j\varphi^j, \tag{11}$$

where $\phi\equiv\phi(x,t)$ is a bosonic singular manifold, and $u_j\equiv u_j(x,t)$ and $\eta_j\equiv\eta_j(x,t)$ are bosonic and fermionic functions, respectively. There exist various concrete methods to carry out the Painlevé test, such as the Weiss–Tabor–Carnevale approach,[12] the Kruskal's simplification, the Conte's invariant method,[23] and the Lou's extended approach.[24] Here we use Kruskal's simplification method, which means we take the following simplifications:

$$u_j(x,t)=u_j(t), \tag{12}$$

$$\eta_j(x,t)=\eta_j(t), \tag{13}$$

$$\varphi(x,t)=x+g(t), \tag{14}$$

with the additional condition $u_0\neq 0$ and $\eta_0\neq 0$.

To determine the constants α and β, we use the leading order analysis, i.e., substituting

$$u=u_0\varphi^{\alpha},\quad \eta=\eta_0\varphi^{\beta} \tag{15}$$

into Eqs. (9a) and (9b) and balancing the dispersion term and nonlinear term

$$6(u_xu_t)_x\sim u_{xxxt},\quad 6(u_x\eta)_x\sim\eta_{xxx}, \tag{16}$$

two solutions are obtained,

$$\textbf{case 1:}\qquad \alpha=-1,\quad \beta=2; \tag{17}$$

$$\textbf{case 2:}\qquad \alpha=-1,\beta=3, \tag{18}$$

where $u_0=1$ and η_0 is an arbitrary function in both cases.

While for another case of balancing nonlinear terms and dispersions terms

$$6(\eta_x\eta)_x\sim u_{xxxt},\quad 6(u_x\eta)_x\sim\eta_{xxx}, \tag{19}$$

there is no solution and should be abandoned.

The next step is to determine the resonant points, which should be discussed for the two cases, respectively. For case 1, we substitute Eqs. (10) and (11) with Eqs. (12), (13, (17), i.e.,

$$u=\sum_{j=0}^{\infty}u_j(t)\varphi^{j-1},\quad \eta=\sum_{j=0}^{\infty}\eta_j(t)\varphi^{j+2} \tag{20}$$

into the supersymmetric Ito equation (9) and obtain

$$\begin{aligned}&\sum_{j=0}^{\infty}\{u_{j,tt}\varphi^{j-1}+2(j-1)u_{j,t}\varphi^{j+2}g'+(j-1)(j-2)u_j\varphi^{j-3}(g'^2+(j-1)u_j\varphi^{j-2}g'')+(j-1)(j-2)(j-3)u_{j,t}\varphi^{j-4}\\&+(j-1)(j-2)(j-3)(j-4)u_j\varphi^{j-5}g'\}+6\sum_{j=0}^{\infty}[(j-1)(j-2)u_j\varphi^{j-3}]\sum_{i=0}^{\infty}\left[u_{i,t}\varphi^{i-1}+(i-1)u_i\varphi^{i-2}g'\right]\\&+6\sum_{j=0}^{\infty}[(j-1)u_j\varphi^{j-2}]\sum_{i=0}^{\infty}[(i-1)u_{i,t}\varphi^{i-2}+(i-1)(i-2)u_i\varphi^{i-3}g']+6\sum_{j=0}^{\infty}[(j+2)(j+1)\eta_j\varphi^j]\sum_{i=0}^{\infty}[\eta_i\varphi^{i+2}]=0,\end{aligned} \tag{21}$$

$$\begin{aligned}&\sum_{j=0}^{\infty}[\eta_{j,t}\varphi^{j+2}+(j+2)\eta_j\varphi^{j+1}g'+(j+1)(j+2)j\eta_j\varphi^{j-1}]+6\sum_{j=0}^{\infty}[(j-1)(j-2)u_j\varphi^{j-3}]\sum_{i=0}^{\infty}\eta_i\varphi^{i+2}\\&+6\sum_{j=0}^{\infty}[(j-1)u_j\varphi^{j-2}]\sum_{i=0}^{\infty}[(i+2)\eta_i\varphi^{i+1}]=0.\end{aligned} \tag{22}$$

To simplify the above equations, we use the following identities:

$$\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}u_j v_i=\sum_{j=0}^{\infty}\sum_{i=0}^{j}v_i u_{j-i}, \tag{23}$$

and extract the common summation $\sum_{j=0}^{\infty}$ to obtain

$$\begin{aligned}&u_{j,tt}\varphi^{j-1}+2(j-1)u_{j,t}\varphi^{j-2}g'+(j-1)(j-2)u_j\varphi^{j-3}g'^2+(j-1)u_j\varphi^{j-2}g''+(j-1)(j-2)(j-3)u_{j,t}\varphi^{j-4}\\&+(j-1)(j-2)(j-3)(j-4)u_j\varphi^{j-5}g'+6\sum_{i=0}^{j}[(j-i-1)(j-i-2)u_{j-i}u_{i,t}\varphi^{j-4}+(j-i-1)(j-i-2)(i-1)u_{j-i}u_i\varphi^{j-5}g'\\&+(j-i-1)(i-1)u_{j-i}u_{i,t}\varphi^{j-4}+(j-i-1)(i-1)(i-2)u_{j-i}u_i\varphi^{j-5}g'+(j-i+2)(j-i+1)\eta_{j-i}\eta_i\varphi^{j+2}]=0,\end{aligned} \tag{24}$$

$$\begin{aligned}&\eta_{j,t}\varphi^{j+2}+(j+2)\eta_j\varphi^{j+1}g'+(j+1)(j+2)j\eta_j\varphi^{j-1}\\&+6\sum_{i=0}^{j}[(j-i-1)(j-i-2)u_{j-i}\eta_i\varphi^{j-1}+(j-i-1)(i+2)u_{j-i}\eta_i\varphi^{j-1}]=0.\end{aligned} \tag{25}$$

Now by collecting the coefficients of φ^{j-5} and φ^{j-1} and making them to be zero, we respectively obtain

$$\begin{aligned}&u_{j-4,tt}\varphi^{j-5}+2(j-4)u_{j-3,t}\varphi^{j-5}g'+(j-3)(j-4)u_{j-2}\varphi^{j-5}g'^2+(j-2)(j-3)(j-4)u_{j-1,t}\varphi^{j-5}\\&+(j-1)(j-2)(j-3)(j-4)u_j\varphi^{j-5}g'+6\sum_{i=0}^{j}[(j-i-3)(j-i-2)u_{j-i-1}u_{i,t}\varphi^{j-5}\\&+(j-i-1)(j-i-2)(i-1)u_{j-i}u_i\varphi^{j-5}g'+(j-i-2)(i-1)u_{j-i-1}u_{i,t}\varphi^{j-5}\\&+(j-i-1)(i-1)(i-2)u_{j-i}u_i\varphi^{j-5}g'+(j-i-5)(j-i-6)\eta_{j-i-7}\eta_i\varphi^{j-5}]=0,\end{aligned} \tag{26}$$

$$\begin{aligned}&\eta_{j-3,t}\varphi^{j-1}+j\eta_{j-2}\varphi^{j-1}g'+(j+1)(j+2)j\eta_j\varphi^{j-1}\\&+6\sum_{i=0}^{j}[(j-i-1)(j-i-2)u_{j-i}\eta_i\varphi^{j-1}+(j-i-1)(i+2)u_{j-i}\eta_i\varphi^{j-1}]=0.\end{aligned} \tag{27}$$

For Eqs. (26) and (27), selecting the terms containing u_j and η_j and leaving other terms be considered as one term, we obtain

$$\begin{aligned}&(j-1)(j-2)(j-3)(j-4)u_jg'+6[(j-1)(j-2)(-1)u_ju_0g'+(-1)(-2)(j-1)u_0u_jg'\\&+(j-1)(-1)(-2)u_ju_0g'+(-1)(j-1)(j-2)u_ju_0g']=F_1,\end{aligned} \tag{28}$$

$$(j+1)(j+2)j\eta_j+6[(j-1)(j-2)u_j\eta_0+(-1)(-2)u_0\eta_j+(j-1)(2)u_j\eta_0+(-1)(j+2)u_0\eta_j]=F_2, \tag{29}$$

where $F_i\equiv F_i(u_1,u_2,\ldots,u_{j-1},\eta_1,\eta_2,\ldots,\eta_{j-1},x,t,\theta)$, $(i=1,2)$.

By substituting $u_0=1$ into Eqs. (28) and (29), they transform into

$$(j+1)(j-1)(j-4)(j-6)g'u_j=F_1, \tag{30}$$

$$(j-1)(j+4)j\eta_j+6j(j-1)\eta_0u_j=F_2, \tag{31}$$

which can be written in matrix form

$$\begin{pmatrix}(j+1)(j-1)(j-4)(j-6)g' & 0\\ 6j(j-1)\eta_0 & j(j-1)(j+4)\end{pmatrix}\begin{pmatrix}u_j\\ \eta_j\end{pmatrix}=\begin{pmatrix}F_1\\ F_2\end{pmatrix}. \tag{32}$$

The resonant points are determined by making the determinant of the coefficient matrix in Eq. (32) to be zero, i.e.,

$$\begin{vmatrix}(j+1)(j-1)(j-4)(j-6)g' & 0\\ 6j(j-1)\eta_0 & j(j-1)(j+4)\end{vmatrix}=0, \tag{33}$$

which leads to

$$(j+1)(j-1)(j-4)(j-6)j(j-1)(j+4)g'=0. \tag{34}$$

For case 2, we substitute Eqs. (10) and (11) with Eqs. (12), (13), (18), i.e.,

$$u=\sum_{j=0}^{\infty}u_j(t)\varphi^{j-1},\quad \eta=\sum_{j=0}^{\infty}\eta_j(t)\varphi^{j+3} \tag{35}$$

into the supersymmetric Ito equation (9) and respectively obtain

$$\sum_{j=0}^{\infty}[u_{j,tt}\varphi^{j-1}+2(j-1)u_{j,t}\varphi^{j-2}g'+(j-1)(j-2)u_j\varphi^{j-3}g'^2+(j-4)u_{j-3}\varphi^{j-5}g''+(j-2)(j-3)(j-4)u_{j-1,t}\varphi^{j-5}$$
$$+(j-1)(j-2)(j-3)(j-4)u_j\varphi^{j-5}g']+6\sum_{j=0}^{\infty}[(j-i-2)(j-i-3)u_{j-i-1}u_{i,t}\varphi^{j-5}+(j-i-1)(j-i-2)(i-1)u_{j-i}u_i\varphi^{j-5}g'$$
$$+(j-i-2)(i-1)u_{j-i-1}u_{i,t}\varphi^{j-5}+(j-i-1)(i-1)(i-2)u_{j-i}u_i\varphi^{j-5}g'+(j-i-6)(j-i-7)\eta_{j-i-9}\eta_i\varphi^{j-5}]=0, \quad (36)$$
$$\sum_{j=0}^{\infty}[\eta_{j,t}\varphi^{j+3}+(j+3)\eta_j\varphi^{j+2}g'+(j+3)(j+2)(j+1)\eta_j\varphi^j]$$
$$+6\sum_{j=0}^{\infty}[(j-1)(j-2)u_j\varphi^{j-3}]\sum_{i=0}^{\infty}\eta_i\varphi^{i+3}+6\sum_{j=0}^{\infty}[(j-1)u_j\varphi^{j-2}]\sum_{i=0}^{\infty}[(i+3)\eta_i\varphi^{i+2}]=0. \quad (37)$$

Using the same identities as in Eq. (23) and performing similar procedures as in case 1, we obtain

$$\begin{pmatrix}(j+1)(j-1)(j-4)(j-6)g' & 0\\ 6(j+1)(j-1)\eta_0 & j(j+1)(j+5)\end{pmatrix}\begin{pmatrix}u_j\\ \eta_j\end{pmatrix}=\begin{pmatrix}F_1\\ F_2\end{pmatrix}. \quad (38)$$

We have the determinant equation

$$\begin{vmatrix}(j+1)(j-1)(j-4)(j-6)g' & 0\\ 6(j+1)(j-1)\eta_0 & j(j+1)(j+5)\end{vmatrix}=0, \quad (39)$$

which leads to

$$(j+1)(j-1)(j-4)(j-6)j(j-1)(j+4)g'=0. \quad (40)$$

For case 1, the resonant point arises at $j=-1$, 1, 4, 6, 0, -4. It is known that $j=-1$ corresponds to the arbitrariness in the location of the singularity surface. The other four non-negative resonance values correspond exactly to four arbitrary functions

$$\eta_0,\ \eta_1,\ u_4,\ u_6, \quad (41)$$

and the other lower order coefficients satisfy

$$u_2 = -\frac{g'^2+6u_{1,t}}{12g'}, \quad (42)$$
$$u_3 = \frac{-6u_{2,t}+g''}{24g'}, \quad (43)$$
$$u_5 = (u_{1,tt}+2u_{2,t}g'+u_2g''+2u_3g''+6u_{4,t}+12u_3u_{1,t}$$
$$-6u_{4,t}+24u_2u_3g'+6u_2u_{2,t})/24g', \quad (44)$$
$$u_{2,tt}+4u_{3,t}g'+2u_3g''+6u_4g'^2+24u_{5,t}$$
$$+36u_4u_{1,t}+24u_3u_{2,t}-12u_0u_{5,t}+12u_2u_4g'$$
$$+48u_3^2g'+12u_2u_{3,t}=0. \quad (45)$$

For case 2, a similar conclusion holds, which is omitted here.

From the above result, we know that equation (9) passes the Painlevé test. Therefore, the sIto system is a candidate for the integrable supersymmetric equation.

4. Discussion and conclusion

In summary, the supersymmetric Ito equation is constructed by introducing new Grassmann odd independent variables and a new bosonic super field. A detailed Painlevé test using Kruskal's simplification method is performed on the sIto equation, it demonstrates that this equation could pass the Painlevé test and we call it possess Painlevé integrability. The methods to supersymmetrize classical equations and Painlevé test procedure are applicative to abundant other integrable systems, meanwhile through some Painlevé analyses we can obtain Lax pair and Bäcklund transformations, which deserve to be studied in the future.

Acknowledgment

The authors would like to thank Dr. Xi-zhong Liu and Bo Ren for their helpful discussions.

References

[1] Ramond P 1971 *Phys. Rev. D* **3** 2415
[2] Wess J and Zumino B 1974 *Nucl. Phys. B* **70** 39
[3] Neveu A and Schwarz J H 1971 *Nucl. Phys. B* **31** 86
[4] Mathieu P 1988 *Phys. Lett. A* **128** 169
[5] Kupershmidt B A 1986 *Mech. Res. Commun.* **13** 47
[6] Martin Y I and Radul A O 1985 *Commun. Math. Phys.* **98** 65
[7] Liu Q P 2010 *J. Math. Phys.* **51** 93511
[8] Zhang M X, Liu Q P, Wang J and Wu K 2008 *Chin. Phys. B* **17** 10
[9] Ren B, Lin J and Yu J 2013 *AIP Advances* **3** 042129
[10] Liu X Z and Yu J 2013 *Z. Naturforsch. A* **68** 539
[11] Ablowitz M, Ramani A and Segur H 1980 *J. Math. Phys.* **21** 715
[12] Weiss J, Tabor M and Carnevale G 1983 *J. Math. Phys.* **24** 522
[13] Hlavat'y L 1989 *Phys. Lett. A* **137** 173
[14] Brunelli J C and Das A 1995 *J. Math. Phys.* **36** 268
[15] Bourque S and Mathieu P 2001 *J. Math. Phys.* **42** 3517
[16] Zhang H P, Li B and Chen Y 2010 *Appl. Math. Comput.* **217** 1555
[17] Lou S Y, Chen C L and Tang X Y 2002 *J. Math. Phys.* **43** 4078
[18] Tang X Y and Hu H C 2002 *Chin. Phys. Lett.* **19** 1225
[19] Ebadi G, Kara A H, Petkovic M D, Yildirim A and Biswas A 2012 *P. Romanian Acad. A* **13** 215
[20] Hu X B and Li Y 1991 *J. Phys. A* **24** 1979
[21] Drinfeld V G and Sokolov V 1985 *J. Sov. Math.* **30** 1975
[22] Liu Q P 2000 *Phys. Lett. A* **277** 31
[23] Conte R 1989 *Phys. Lett. A* **140** 383
[24] Lou S Y 1998 *Z. Naturforsch.* **53a** 251