ISSN 1674-1056
http://cpb.iphy.ac.cn
http://iopscience.iop.org/cpb
中国物理 B
Chinese Physics
中国物理
Chinese Physics B

Wigner function for squeezed negative binomial state and evolution of density operator for amplitude decay

Heng-Yun Lv(吕恒云), Ji-Suo Wang(王继锁), Xiao-Yan Zhang(张晓燕), Meng-Yan Wu(吴孟艳), Bao-Long Liang(梁宝龙), Xiang-Guo Meng(孟祥国)

Citation: Chin. Phys. B . 2019, 28(9): 090302 . **doi:** 10.1088/1674-1056/28/9/090302
Journal homepage: http://cpb.iphy.ac.cn; http://iopscience.iop.org/cpb

What follows is a list of articles you may be interested in

Analytical and numerical investigations of displaced thermal state evolutions in a laser process

Chuan-Xun Du(杜传勋), Xiang-Guo Meng(孟祥国), Ran Zhang(张冉), Ji-Suo Wang(王继锁)
Chin. Phys. B . 2017, 26(12): 120301 . **doi:** 10.1088/1674-1056/26/12/120301

Comparison between photon annihilation-then-creation and photon creation-then-annihilation thermal states：Non-classical and non-Gaussian properties

Xu Xue-Xiang, Yuan Hong-Chun, Wang Yan
Chin. Phys. B . 2014, 23(7): 070301 . **doi:** 10.1088/1674-1056/23/7/070301

Mutual transformations between the *P*–*Q*,*Q*–*P*, and generalized Weyl ordering of operators

Xu Xing-Lei, Li Hong-Qi, Fan Hong-Yi
Chin. Phys. B . 2014, 23(3): 030305 . **doi:** 10.1088/1674-1056/23/3/030305

Nonclassicality and decoherence of coherent superposition operation of photon subtraction and photon addition on squeezed state

Xu Li-Juan, Tan Guo-Bin, Ma Shan-Jun, Guo Qin
Chin. Phys. B . 2013, 22(3): 030311 . **doi:** 10.1088/1674-1056/22/3/030311

New approach for deriving the exact time evolution of density operator for diffusive anharmonic oscillator and its Wigner distribution function

Meng Xiang-Guo, Wang Ji-Suo, Liang Bao-Long
Chin. Phys. B . 2013, 22(3): 030307 . **doi:** 10.1088/1674-1056/22/3/030307

Wigner function for squeezed negative binomial state and evolution of density operator for amplitude decay*

Heng-Yun Lv(吕恒云)[1], Ji-Suo Wang(王继锁)[2], Xiao-Yan Zhang(张晓燕)[1], Meng-Yan Wu(吴孟艳)[1], Bao-Long Liang(梁宝龙)[1], and Xiang-Guo Meng(孟祥国)[1,†]

[1] *Shandong Provincial Key Laboratory of Optical Communication Science and Technology, School of Physical Science and Information Engineering, Liaocheng University, Liaocheng 252059, China*

[2] *Shandong Provincial Key Laboratory of Laser Polarization and Information Technology, College of Physics and Engineering, Qufu Normal University, Qufu 273165, China*

(Received 25 June 2019; revised manuscript received 11 July 2019; published online 23 August 2019)

Using the thermal-entangled state representation and the operator-ordering method, we investigate Wigner function (WF) for the squeezed negative binomial state (SNBS) and the analytical evolution law of density operator in the amplitude decay channel. The results show that the analytical WF is related to the square of the module of single-variable Hermite polynomials, which leads to a new two-variable special function and its generating function, and the parameters s and γ play opposite roles in the WF distributions. Besides, after undergoing this channel, the initial pure SNBS evolves into a new mixed state related to two operator Hermite polynomials within normal ordering, and fully loses its nonclassicality and decays to vacuum at long decay time.

Keywords: squeezed negative binomial state, operator ordering, Wigner function, density-operator evolution, amplitude decay

PACS: 03.65.Yz, 42.50.Dv **DOI:** 10.1088/1674-1056/ab38a6

1. Introduction

Squeezed states and their practical applications have been widely explored during the past several decades (see, *e.g.*, the review articles[1,2] and references therein). This is mainly because the noises of squeezed states may be below the vacuum or ground-state noise level in a certain observable, which can be useful for interferometric detection and gravitational waves by means of improving the interferometric and spectroscopic measurements. Especially, two-mode squeezed states served as a new entangled resource is crucial in realizing some vital quantum information processing, such as quantum teleportation,[3,4] quantum computation,[5] and cryptography,[6] and can be readily generated by using experimentally controllable and available sets including phase shifters, beam splitters, and squeezers.[7] For another side, any system in the nature is not absolutely isolated, quantum noise usually occurs and causes the decoherence of system when this system is in the interaction with its surrounding environment. Hence, the environment-induced decoherence aspects in open squeezed optical fields have received great attention from researchers. For instance, the effect of amplitude decay, phase damping, and thermal noise on the decoherence of squeezed states has been analytically investigated in the pioneering works.[8–10]

In probability theory and statistics, the negative binomial distribution reads $\binom{n+s}{n}\gamma^{s+1}(1-\gamma)^n$, where $\gamma\in[0,1]$ and s is a non-negative integer. Corresponding to this definition, there exists a pure state with a negative binomial form of its photon-number probability distribution, which is called the negative binomial state (NBS),[11] that is

$$\rho_0=\sum_{n=0}^{\infty}\binom{n+s}{n}\gamma^{s+1}(1-\gamma)^n|n\rangle\langle n|, \tag{1}$$

where $|n\rangle$ is the Fock state obtained by repeatedly acting the photon-creation operator $a^\dagger$ on the vacuum state $|0\rangle$ in the Fock space. Using the identity $a^s|n\rangle=\sqrt{n!/(n-s)!}\,|n-s\rangle$, we can rewrite ρ_0 as

$$\rho_0=\frac{\gamma^{s+1}}{(1-\gamma)^s s!}a^s\sum_{n=0}^{\infty}(1-\gamma)^n|n\rangle\langle n|a^{\dagger s}. \tag{2}$$

Indeed, the NBS is an intermediate state between a coherent state ($\gamma\to1$ and $s\to\infty$) and a quasi-thermal state ($s=0$). The reason for designating the state ρ_0 for $s=0$ as a quasi-thermal state is that its photon number distribution is the same as that of a mixed thermal state, which is easily proved to be true using the compact form $\rho_0=a^s\rho_{\rm t}a^{\dagger s}/(s!\bar{n}^s)$, regarded as a multi-photon-subtracted thermal state via subtracting s photons from a single-mode thermal field $\rho_{\rm t}=\gamma e^{a^\dagger a\ln(1-\gamma)}$ with the mean number of photons $\bar{n}=(1-\gamma)/\gamma$.[12] This means,

*Project supported by the National Natural Science Foundation of China (Grant No. 11347026) and the Natural Science Foundation of Shandong Province, China (Grant Nos. ZR2016AM03 and ZR2017MA011).

†Corresponding author. E-mail: mengxiangguo1978@sina.com

© 2019 Chinese Physical Society and IOP Publishing Ltd

through subtracting s photons, the photon number of $\rho_{\rm t}$ shows a negative binomial distribution. For the NBS and its derived states, various representations, properties, and quasiprobability distributions have been studied in some detail and their generation schemes in a number of nonlinear processes have been demonstrated.[13–15] Especially lately, the applications of the NBS in quantum optics and its time-evolution in some typical decoherence channels including the amplitude decay channel (ADC),[16] the diffusion process,[17] and the laser channel[18] have been investigated.

As a practical derived state, a squeezed negative binomial state (SNBS) was first introduced by Joshi and Lawande, and its second-order correlation function, Wigner function, Q functions, and photon number distribution have been investigated, and one result that needs to be noted is that the SNBS can be helpful for the transfer of quantum phase-space distributions from squeezed coherent states to squeezed number states.[19] In this paper, we shall introduce a new formula of SNBS, which is defined as an NBS ρ_0 having undergone a two-mode squeezing transform, hence its density operator reads

$$\rho_s = \frac{\gamma^{s+1}}{(1-\gamma)^s s!} S(r)\, a^s \sum_{n=0}^{\infty} (1-\gamma)^n |n\rangle\langle n| a^{\dagger s} S^{-1}(r), \quad (3)$$

where $S(r) = \exp[r(a^{\dagger 2} - a^2)/2]$ is the single-mode squeezing operator with the squeezing r.[20–22] To go a step further, we focus on the theoretical investigation of Wigner function (WF) for SNBS and the evolution behavior of density operator in ADC by fully using the operator-ordering method[23–26] and the thermal entangled state representation (TESR).[27–30] Our motivations for focusing on this topic are explicated as follows. First, the operator-ordering method can help us prove that the WF for SNBS is expressed as the finite-sum representation determined by the non-negative integer s, rather than the infinite-sum form as shown in Ref. [19], which can more accurately describe the nonclassicality of SNBS in phase space (see Section 3 for details). Second, according to our best knowledge, the investigation of the evolution of SNBS in ADC by deriving the analytical results has not been previously discussed because of the complicated calculations resulting from the squeezing operation in SNBS, however TESR can help one to greatly simplify it and derive an accurate and concise result related to two operator Hermite polynomial functions.

2. Operator-ordering products of $\sum (1-\gamma)^n S(r)|n\rangle\langle n| S^{-1}(r)$

To analyze the nonclassicality of SNBS from the aspects of phase space and decoherence evolution, we first use the Bogoliubov transformation caused by the squeezed operator $S(r)$ to change SNBS into

$$\rho_s = \frac{\gamma^{s+1}}{(1-\gamma)^s s!} \left(a\cosh r - a^\dagger \sinh r\right)^s \rho_n \left(a^\dagger \cosh r - a \sinh r\right)^s, \quad (4)$$

where $\rho_n = \sum (1-\gamma)^n S(r)|n\rangle\langle n| S^{-1}(r)$, a weighted quantum superposition of squeezed number states $S(r)|n\rangle$. Next, we derive the normal and antinormal ordering products of ρ_n. For this purpose, using the completeness relation of the coherent state $|\alpha\rangle$ to derive the normal ordering of the operator $\mathrm{e}^{r(a^{\dagger 2}-a^2)/2} a^{\dagger n}$, that is

$$\begin{aligned} \mathrm{e}^{r(a^{\dagger 2}-a^2)/2} a^{\dagger n} &= \mathrm{e}^{r(a^{\dagger 2}-a^2)/2} \int \frac{\mathrm{d}^2\alpha}{\pi} |\alpha\rangle\langle\alpha| a^{\dagger n} \\ &= \operatorname{sech}^{1/2} r \int \frac{\mathrm{d}^2\alpha}{\pi} :\, \exp\left(-|\alpha|^2 + \alpha a^\dagger \operatorname{sech} r + \alpha^* a - \frac{\alpha^2}{2}\tanh r + \frac{a^{\dagger 2}}{2}\tanh r - a^\dagger a\right) \alpha^{*n} :\,, \end{aligned} \quad (5)$$

where the marked symbol $: \; :$ refers to normal ordering.[31] Further, using the integral formula (17) in Ref. [32] to perform the integration (5) over α within normal ordering, and operating this integral result on the vacuum state $|0\rangle$ and using the generating function of n-order Hermite polynomial $\mathrm{H}_n(\cdot)$, we have the compact form of the squeezed number state $S(r)|n\rangle$, that is

$$S(r)|n\rangle = \sqrt{\frac{\operatorname{sech} r \tanh^n r}{2^n n!}}\, \mathrm{H}_n\left(\frac{a^\dagger}{\sqrt{\sinh 2r}}\right) \exp\left(\frac{\tanh r}{2} a^{\dagger 2}\right)|0\rangle, \quad (6)$$

which indicates that a squeezed number state can be viewed as a Hermite polynomial excitation on the squeezed vacuum state. Noting the normal ordering product of the vacuum projector, *i.e.*, $|0\rangle\langle 0| = :\, \mathrm{e}^{-a^\dagger a} :$, we directly obtain the normal ordering of the density operator $S(r)|n\rangle\langle n| S^{-1}(r)$, *i.e.*,

$$S(r)|n\rangle\langle n| S^{-1}(r) = \frac{\operatorname{sech} r \tanh^n r}{2^n n!} :\, \mathrm{H}_n\left(\frac{a^\dagger}{\sqrt{\sinh 2r}}\right) \mathrm{H}_n\left(\frac{a}{\sqrt{\sinh 2r}}\right) \exp\left[\frac{\tanh r}{2}(a^{\dagger 2} + a^2) - a^\dagger a\right] :\,. \quad (7)$$

Using the summation formula concerning two single-variable Hermite polynomials in Eq. (10) in Ref. [33], thus the normal ordering expansion of ρ_n is obtained as the compact exponential form

$$\rho_n = g\colon \exp\left(g_1 a^{\dagger 2} + g_1 a^2 + g_2 a^\dagger a\right)\colon, \tag{8}$$

where we have set $g = [\cosh^2 r - (1-\gamma)^2 \sinh^2 r]^{-1/2}$, $g_1 = \tanh r[1-(1-\gamma)^2 g^2 \tanh r]/2$, and $g_2 = (1-\gamma) g^2 - 1$. Using a method similar to that used to obtain the result (20) in Ref. [33], we can also prove that the normal ordering operator ρ_n is a bivariate distribution operator in phase space. Further, noting the operator identity $\mathrm{e}^{\lambda a^\dagger a} = :\exp[(\mathrm{e}^\lambda - 1)a^\dagger a]:$, equation (8) can be rewritten as

$$\rho_n = g \exp(g_1 a^{\dagger 2}) \exp[a^\dagger a \ln(g_2+1)] \exp(g_1 a^2), \tag{9}$$

which shows that the operator ρ_n is indeed a squeezed thermal state by comparing with the result (11) in Ref. [33].

Further, substituting the normal ordering of ρ_n in Eq. (8) into the antinormal ordering formula[34] and using the mathematical integral formula,[35] we can obtain the antinormal ordering of ρ_n as the following form

$$\rho_n = G \vdots \exp\left[G_1\left(a^2 + a^{\dagger 2}\right) + G_2 a^\dagger a\right] \vdots, \tag{10}$$

where $\vdots\ \vdots$ denotes antinormal ordering[36] and we have set $G = g/[(g_2-1)^2 - 4g_1^2]^{1/2}$, $G_1 = G^2 g_1/g^2$, and $G_2 = 1 - G^2(g_2-1)/g^2$.

3. Wigner function for SNBS

The Wigner distribution function is a useful tool which is used to investigate the nonclassical optical fields from the view point of phase space. In this section, we shall investigate the nonclassicality of the SNBS by analytically and numerically studying its phase-space WF distributions. For the given density operator ρ_s, its WF definition is given by $w(\alpha) = \mathrm{Tr}[\Delta(\alpha)\rho_s]$, where $\Delta(\alpha)$ is the coherent state representation of single-mode Wigner operator, *i.e.*, $\Delta(\alpha) = \pi^{-2}\int \mathrm{d}^2 z\, |\alpha+z\rangle\langle\alpha-z|\, \mathrm{e}^{\alpha z^* - z\alpha^*}$.[37] Considering that $|\alpha \pm z\rangle$ are coherent states,[38,39] hence we first introduce the normal operator identity of the operator $\left(\mu a + v a^\dagger\right)^m$,[10] that is

$$\left(\mu a + v a^\dagger\right)^m = \left(-\mathrm{i}\sqrt{\frac{\mu}{2v}}\right)^m :\mathrm{H}_m\left(\mathrm{i}\frac{\mu a + v a^\dagger}{\sqrt{2\mu v}}\right):, \tag{11}$$

and use the antinormal ordering of ρ_n in Eq. (10), thus the state ρ_s reads

$$\rho_s = \frac{\gamma^{s+1} G}{(1-\gamma)^s s!}\left(-\frac{\sinh 2r}{4}\right)^s :\mathrm{H}_s\left(\varsigma a + \sigma a^\dagger\right): \vdots \exp\left(G_1\left(a^2 + a^{\dagger 2}\right) + G_2 a^\dagger a\right) \vdots :\mathrm{H}_s\left(\varsigma^* a^\dagger + \sigma^* a\right):, \tag{12}$$

where $\varsigma = \mathrm{i}\cosh r/\sqrt{-\sinh 2r}$ and $\sigma = -\mathrm{i}\sinh r/\sqrt{-\sinh 2r}$. Thus, inserting Eq. (12) into the WF definition and using the completeness relation of the coherent state $|\beta\rangle$ and the generating function of $\mathrm{H}_s(\cdot)$, *i.e.*,

$$\mathrm{H}_s(x) = \frac{\partial^s}{\partial t^s}\exp\left(-t^2 + 2tx\right)\big|_{t=0}, \tag{13}$$

we therefore have

$$\begin{aligned} w(\alpha) = {} & \frac{\gamma^{s+1} G}{(1-\gamma)^s s!}\left(\frac{\sinh 2r}{4}\right)^s \mathrm{e}^{-|\alpha|^2}\frac{\partial^{2s}}{\partial t^s \partial \tau^s}\mathrm{e}^{-t^2-\tau^2}\int\frac{\mathrm{d}^2 z}{\pi^2}\mathrm{e}^{-|z|^2+\alpha z^* - z\alpha^*}\int\frac{\mathrm{d}^2\beta}{\pi}\exp\left[G_1\left(\beta^2+\beta^{*2}\right) + (G_2-1)|\beta|^2\right. \\ & \left. + (\alpha^* - z^*)\beta + (\alpha+z)\beta^*\right]\mathrm{H}_s\left[2\varsigma\beta t + 2\sigma(\alpha^* - z^*)t\right]\mathrm{H}_s\left[2\varsigma^*\beta^*\tau + 2\sigma^*(\alpha+z)\tau\right]\big|_{t=\tau=0}. \end{aligned} \tag{14}$$

Further, using the integral formula[40,41] to complete the two integrations over β and z in the whole phase space, the WF $w(\alpha)$ can be calculated as

$$w(\alpha) = \mathrm{e}^{\Xi}\frac{\partial^{2s}}{\partial t^s \partial \tau^s}\mathrm{e}^{-f_1 t^2 - f_1^* \tau^2 + f_2 t\tau + f_3 t + f_3^* \tau}\big|_{t=\tau=0}, \tag{15}$$

where we have set

$$\mathrm{e}^{\Xi} = \frac{\gamma^{s+1} G}{\pi s!\sqrt{KL}}\left(-\frac{\sinh 2r}{4(1-\gamma)}\right)^s \mathrm{e}^{f_4|\alpha|^2 + f_5(\alpha^2 + \alpha^{*2})}, \tag{16}$$

with

$$K = (G_2-1)^2 - 4G_1^2, \quad K_1 = 1 + \frac{1-G_2}{K},$$

$$\begin{aligned} K_2 &= 1 - \frac{1-G_2}{K}, \qquad K_3 = \frac{4G_1\varsigma}{K} + 2\sigma, \\ K_4 &= \frac{2(1-G_2)\varsigma}{K}, \qquad K_5 = \frac{G_1}{K}, \quad L = K_1^2 - 4K_5^2, \\ f_1 &= \frac{K_1 K_3 K_4}{L} - \frac{K_3^2 K_5}{L} - \frac{K_4^2 K_5}{L} - \frac{4G_1\varsigma^2}{K} + 1, \\ f_2 &= \frac{2K_5}{L}\left(K_3^* K_4 + K_3 K_4^*\right) - \frac{K_1|K_3|^2}{L} \\ &\quad - \frac{K_1|K_4|^2}{L} + \frac{4(1-G_2)|\varsigma|^2}{K}, \\ f_3 &= \left[\frac{K_1}{L}\left(K_2 K_4 - 2K_3 K_5\right) - \frac{2K_5}{L}\left(K_2 K_3 - 2K_4 K_5\right)\right. \end{aligned}$$

$$+\frac{2(1-G_2)\varsigma}{K}\Bigg]\alpha+\Bigg[\frac{K_1}{L}(K_2K_3-2K_4K_5)$$
$$-\frac{2K_5}{L}(K_2K_4-2K_3K_5)+2\left(\sigma+\frac{2G_1\varsigma}{K}\right)\Bigg]\alpha^*,$$
$$f_4=\frac{1-G_2}{K}-\frac{K_1K_2^2}{L}-\frac{4K_1K_5^2}{L}-\frac{8K_2K_5^2}{L}-1,$$
$$f_5=\frac{K_5}{L}\left(2K_1K_2+4K_5^2+K_2^2\right)+\frac{G_1}{K}.$$

Further, using the exponential expansion of cross product term $f_2t\tau$, it then follows that

$$w(\alpha)=\mathrm{e}^{\Xi}\sum_{l=0}^{\infty}\frac{f_2^l}{l!}\frac{\partial^{2l}}{\partial f_3^l\partial f_3^{*l}}\frac{\partial^{2s}}{\partial t^s\partial\tau^s}\,\mathrm{e}^{-f_1^*t^2-f_1\tau^2+f_3t+f_3^*\tau}\Big|_{t=\tau=0}. \tag{17}$$

Noting the generation function of $\mathrm{H}_n(\cdot)$ in Eq. (13) and its l-order differential relation of $\mathrm{H}_s(\cdot)$, we finally obtain

$$w(\alpha)=\mathrm{e}^{\Xi}\sum_{l=0}^{s}\frac{|f_1|^{s-l}f_2^l(s!)^2}{l!\left[(s-l)!\right]^2}\left|H_{s-l}\left(\frac{f_3}{2\sqrt{f_1}}\right)\right|^2, \tag{18}$$

which is the analytical WF of the SNBS related to the square of the module of $(s-l)$-th single-variable Hermite polynomial. Obviously, when $s=0$, $w(\alpha)$ becomes the WF for squeezed quasi-thermal field, *i.e.*,

$$w_{s=0}(\alpha)=\frac{\gamma G}{\pi\sqrt{KL}}\exp\left[f_4|\alpha|^2+f_5\left(\alpha^2+\alpha^{*2}\right)\right]. \tag{19}$$

Mathematically, it is well known that two-variable Hermite polynomials $\mathrm{H}_{m,n}(x,y)$ can be introduced either by its generating function

$$\sum_{m,n=0}^{\infty}\frac{t^mt'^n}{m!n!}\mathrm{H}_{m,n}(x,y)=\mathrm{e}^{-tt'+tx+t'y}, \tag{20}$$

or by its power-series expansion

$$\mathrm{H}_{m,n}(x,y)=\frac{\partial^{m+n}}{\partial t^m\partial t'^n}\mathrm{e}^{-tt'+tx+t'y}|_{t=t'=0}$$
$$=\sum_{l=0}^{\min(m,n)}\frac{m!n!(-1)^l}{l!(m-l)!(n-l)!}x^{m-l}y^{n-l}, \tag{21}$$

where $\mathrm{H}_{m,n}(x,y)$ represents the transfer of the amplitude of number state in the time evolution process of a forced Harmonic oscillator,[42] and can be useful for studying light propagation modes in quadratic graded-index media.[43] Hence, a comparison of the right sides of Eqs. (15) and (18) directly leads to the result that

$$\frac{\partial^{2s}}{\partial t^s\partial\tau^s}\mathrm{e}^{-t^2-\tau^2+f_2t\tau/f_1+f_3t/\sqrt{f_1}+f_3^*\tau/\sqrt{f_1^*}}\Big|_{t=\tau=0}$$
$$=\sum_{l=0}^{s}\frac{(f_2/|f_1|)^l(s!)^2}{l!\left[(s-l)!\right]^2}\left|\mathrm{H}_{s-l}\left(\frac{f_3}{2\sqrt{f_1}}\right)\right|^2, \tag{22}$$

If we replace $f_3/\sqrt{f_1}\to 2x$, $f_3^*/\sqrt{f_1^*}\to 2y$, and $f_2/|f_1|\to\omega$ in Eq. (22), we therefore have

$$\mathrm{e}^{-t^2-\tau^2+\omega t\tau+2xt+2y\tau}=\sum_{s=0}^{\infty}\frac{t^s\tau^s}{s!s!}\mathfrak{F}_{s,s}(x,y;\omega), \tag{23}$$

where

$$\mathfrak{F}_{s,s}(x,y;\omega)=\sum_{l=0}^{s}\frac{\omega^l(s!)^2}{l!\left[(s-l)!\right]^2}\mathrm{H}_{s-l}(x)\,\mathrm{H}_{s-l}(y) \tag{24}$$

is a new two-variable special function whose generating function is $\exp(-t^2-\tau^2+\omega t\tau+2xt+2y\tau)$, a new and interesting result. Thus, the differential relation of the special function $\mathfrak{F}_{s,s}(x,y;\omega)$ can be obtained as

$$\frac{\partial^{m+n}}{\partial x^m\partial y^n}\mathfrak{F}_{s,s}(x,y;\omega)=\frac{(s!)^2}{(s-m)!(s-n)!}\mathfrak{F}_{s-m,s-n}(x,y;\omega), \tag{25}$$

which is the same form as the well-known differential relation of $\mathrm{H}_{m,n}(x,y)$.[29] Further, using the special function $\mathfrak{F}_{s,s}(x,y;\omega)$, we find that the right side of Eq. (22) can be represented as $\mathfrak{F}_{s,s}\left(f_3/(2\sqrt{f_1}),f_3^*/(2\sqrt{f_1^*});f_2/|f_1|\right)$, thus the analytical WF for SNBS can be simplified as

$$w(\alpha)=\mathrm{e}^{\Xi}\mathfrak{F}_{s,s}\left(f_3/(2\sqrt{f_1}),f_3^*/(2\sqrt{f_1^*});f_2/|f_1|\right). \tag{26}$$

To clearly observe the nonclassicality of SNBS, we plot its WF in $(\mathrm{Re}\,\alpha,\mathrm{Im}\,\alpha)$ phase space for a fixed squeezing r and different values of the parameters s and γ as shown in Fig. 1. Obviously, the SNBS always shows squeezing effect in one of the quadratures for $r\neq 0$ as a feature of nonclassicality of this state. For $s\to\infty$ (*e.g.*, $s=100$) and $\gamma\to 1$ (or $s=0$), the WF becomes the standard Gaussian distribution of squeezed coherent state (or squeezed quasi-thermal state) [see Figs. 1(a) and 1(d)], which is completely accordant with the analytical result outlined in Ref. [19]. Comparing Figs. 1(a), 1(b), and 1(c), we find that, with the increase of s, the WF gradually shows an inverted peak in the center of phase space, however it slowly loses the inverted peak as γ increases and finally becomes Gaussian when γ tends to one. Moreover, the larger value of s can lead to the smaller peak value of WF, but the variation of the peak value with γ is just a fully contrary trend. This result can easily be understood via analyzing the result in Eq. (2) since more and smaller number states are annihilated and the probabilities of finding number states in this field becomes small with increasing s (especially at center part in the phase space), however the larger γ causes the higher weights of the existing number states. Besides, for the limited case ($\gamma\to 1$), the WF distribution seems basically unchanged with the increase of s, which is because the value of parameter s has a negligible effect on the weights of the existing number states.

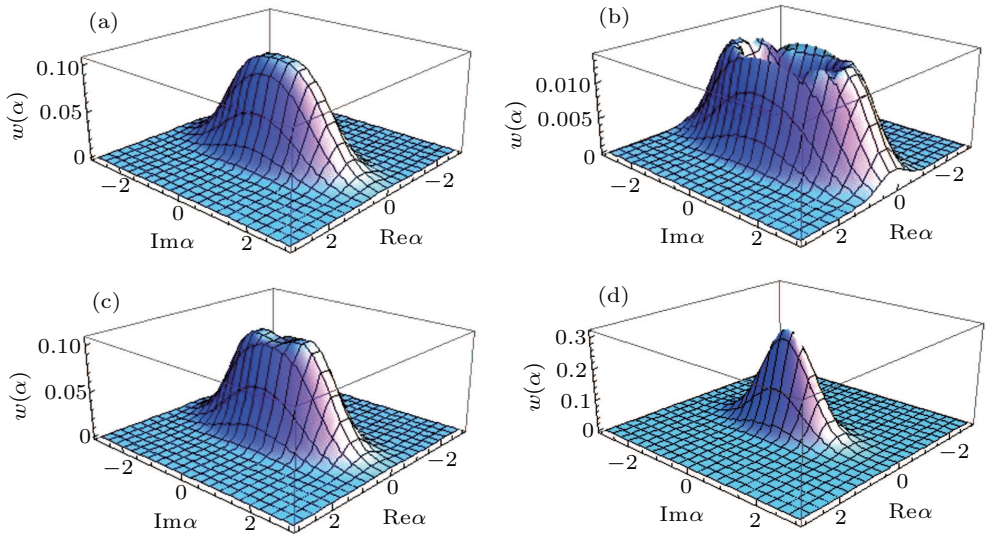


Fig. 1. The WF distributions for SNBS with a fixed squeezing $r = 0.4$ and the different values of the parameters (s, γ): (a) $(0, 0.5)$, (b) $(5, 0.5)$, (c) $(5, 0.9)$ and, (d) $(100, 1)$.

4. Evolution of SNBS in ADC

For any open system, noise always arises in fundamental dynamics processes. For instance, the purely dissipative noise in ADC leads to the nonclassicality deterioration of the system. In this section, we shall analytically obtain the evolution formula of SNBS in ADC and investigate how does the amplitude dissipation affect the nonclassicality of SNBS.

4.1. Kraus operator solutions to master equation describing ADC

In the interaction picture, the density-operator evolution of the system in ADC can be characterized by[44–47]

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = \kappa(2a\rho(t)a^{\dagger} - a^{\dagger}a\rho(t) - \rho(t)a^{\dagger}a), \tag{27}$$

where κ is the decay rate of the system. Indeed, this process completely differs from the phase-decay process with a purely dephasing noise. To obtain the solutions of the above operator equation, we review the two-mode TESR describing quantum entanglement between system and heat reservoir, which is defined in enlarged Fock space as

$$\begin{aligned}|\eta\rangle &= \exp\left(-\frac{1}{2}|\eta|^2 + \eta a^{\dagger} - \eta^{*}\tilde{a}^{\dagger} + a^{\dagger}\tilde{a}^{\dagger}\right)|0\tilde{0}\rangle \\ &= D(\eta)|\eta = 0\rangle,\end{aligned} \tag{28}$$

where $D(\eta)$ is a displacement operator with the complex variable η, the operators $a^{\dagger}$ and $\tilde{a}^{\dagger}$ are respectively the real and fictitious creation operators, which yield the commutative relations, *i.e.*, $[\tilde{a}, \tilde{a}^{\dagger}] = 1$ and $[\tilde{a}, a^{\dagger}] = 0$. Under the basis of the entangled state $|\eta = 0\rangle$, there exist the following extremely useful interchange relations between the real and fictitious modes, *i.e.*, $a \Leftrightarrow \tilde{a}^{\dagger}$, $a^{\dagger} \Leftrightarrow \tilde{a}$, and $a^{\dagger}a \Leftrightarrow \tilde{a}^{\dagger}\tilde{a}$, which play a vital role of solving the master equation (27) since it can help convert the equation for the density operator $\rho(t)$ in Eq. (27) into the evolution equation for the state vector $|\rho(t)\rangle = \rho(t)|\eta = 0\rangle$, that is

$$\frac{\mathrm{d}}{\mathrm{d}t}|\rho(t)\rangle = \kappa(2a\tilde{a} - a^{\dagger}a - \tilde{a}^{\dagger}\tilde{a})|\rho(t)\rangle, \tag{29}$$

whose formal solution is

$$|\rho(t)\rangle = \exp\left[\kappa t(2a\tilde{a} - a^{\dagger}a - \tilde{a}^{\dagger}\tilde{a})\right]|\rho(0)\rangle. \tag{30}$$

Fruther, through calculating the scalar product $\langle\eta\,|\rho(t)\rangle$, we find that it is proportional to the inner product between an exponentially decaying entangled state $\langle\eta\,\mathrm{e}^{-\kappa t}|$ and the initial state vector $|\rho(0)\rangle$, *i.e.*, $\langle\eta\,\mathrm{e}^{-\kappa t}\,|\rho(0)\rangle$, and its proportionality coefficient is just a Gaussian decaying factor $\mathrm{e}^{-T|\eta|^2/2}$, $T = 1 - \mathrm{e}^{-2\kappa t}$. This point clearly reveals the decoherence behavior of the channel, namely, the amplitude decay. To take off the density operator $\rho(t)$ from $\langle\eta\,|\rho(t)\rangle$, we need to use the completeness relation $\pi^{-1}\int \mathrm{d}^2\eta\,|\eta\rangle\langle\eta| = 1$, thus after strict derivations (for the detailed formula derivations see Ref. [27]), its infinitive Kraus operator-sum representation of $\rho(t)$ as the solutions of the master equation (27) can be calculated as

$$\rho(t) = \sum_{m=0}^{\infty} M_m \rho_0 M_m^{\dagger}, \tag{31}$$

where $M_m \equiv (T^m/m!)^{1/2}\,\mathrm{e}^{-\kappa t a^{\dagger}a}a^m$ is the Kraus operator of $\rho(t)$. Hence, for a known initial state ρ_0, the evolved density operator $\rho\,(t)$ in ADC can be freely obtained from Eq. (31). Also, using the operator identity $\mathrm{e}^{\lambda a^{\dagger}a}a\,\mathrm{e}^{-\lambda a^{\dagger}a} = \mathrm{e}^{-\lambda}a$, we can prove the normalization condition $\sum_m M_m^{\dagger}M_m = 1$.

4.2. Density-operator evolution

Noting the marshalling sequence of the operators a^m, ρ_0, and $a^{\dagger m}$ in Eq. (31), we need to introduce the anti-normal operator identity of the operator $\left(\mu a + \nu a^{\dagger}\right)^m$,[32] that is

$$\left(\mu a + \nu a^{\dagger}\right)^m = \left(\sqrt{\frac{\mu\nu}{2}}\right)^m : \mathrm{H}_m\left(\frac{\mu a + \nu a^{\dagger}}{\sqrt{2\mu\nu}}\right) :, \tag{32}$$

and use the normal ordering of ρ_n in Eq. (8). Hence, the state ρ_s reads

$$\rho_s = \frac{\gamma^{s+1}}{(1-\gamma)^s s!}\left(-\frac{\sinh 2r}{4}\right)^s : \mathrm{H}_s\left(\frac{a\cosh r - a^\dagger \sinh r}{\sqrt{-\sinh 2r}}\right) : \rho_n : \mathrm{H}_s\left(\frac{a^\dagger \cosh r - a\sinh r}{\sqrt{-\sinh 2r}}\right) : . \tag{33}$$

Substituting Eqs. (8) and (33) into Eq. (31) leads to the form

$$\rho(t) = \frac{\gamma^{s+1} g}{(1-\gamma)^s s!}\left(-\frac{\sinh 2r}{4}\right)^s \sum_{m=0}^{s} \frac{T^m}{m!}\, \mathrm{e}^{-\kappa t a^\dagger a} : a^m \mathrm{H}_s\left(\rho a + \upsilon a^\dagger\right) : : \exp\left(g_1 a^{\dagger 2} + g_1 a^2 + g_2 a^\dagger a\right) : : \mathrm{H}_s\left(\rho a^\dagger + \upsilon a\right) a^{\dagger m} : \mathrm{e}^{-\kappa t a^\dagger a}, \tag{34}$$

where we have set $\rho = \mathrm{i}\,\varsigma$ and $\upsilon = \mathrm{i}\,\sigma$. Inserting the completeness relations of coherent states $|\alpha\rangle$ and $|\beta\rangle$ into Eq. (34),[38,39] and using the operator identity $\mathrm{e}^{-\kappa t a^\dagger a}|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2}\mathrm{e}^{\alpha e^{-\kappa t} a^\dagger}|0\rangle$ and the generating function of $\mathrm{H}_s(\cdot)$ in Eq. (13),[48] we obtain

$$\begin{aligned}\rho(t) = {} & \frac{\gamma^{s+1} g}{(1-\gamma)^s s!}\left(-\frac{\sinh 2r}{4}\right)^s \frac{\partial^{2s}}{\partial \tau^s \partial \tau'^s}\int \frac{\mathrm{d}^2\alpha \mathrm{d}^2\beta}{\pi^2} : \exp\left(\alpha \mathrm{e}^{-\kappa t} a^\dagger + \beta^* \mathrm{e}^{-\kappa t} a - a^\dagger a\right) : \\ & \times \exp\left[-|\alpha|^2 - |\beta|^2 + g_1\alpha^{*2} + g_1\beta^2 + T\alpha\beta^* + (g_2+1)\alpha^*\beta\right] \\ & \times \exp\left[-\tau^2 + 2(\rho\alpha + \upsilon\alpha^*)\tau - \tau'^2 + 2(\rho\beta^* + \upsilon\beta)\tau'\right]\big|_{\tau=\tau'=0}.\end{aligned} \tag{35}$$

Using the integration technique within an operator-ordering product[49–52] to perform the two integrals over α and β, then $\rho(t)$ becomes

$$\rho(t) =: \mathrm{e}^{\Omega}\frac{\partial^{2s}}{\partial \tau^s \partial \tau'^s}\exp\left(-h_1\tau^2 - h_1\tau'^2 + h_2\tau\tau' + h_3\tau + h_3^\dagger\tau'\right)\Big|_{\tau=\tau'=0} : , \tag{36}$$

where e^{Ω} can be represented as

$$\mathrm{e}^{\Omega} = \frac{\gamma^{s+1} g h^{1/2}}{(1-\gamma)^s s!}\left(-\frac{\sinh 2r}{4}\right)^s : \exp\left\{h g_1 \mathrm{e}^{-2\kappa t}\left(a^{\dagger 2} + a^2\right) + \left[4hg_1^2 T\mathrm{e}^{-2\kappa t} + h(g_2+1)(1-(g_2+1)T)\mathrm{e}^{-2\kappa t} - 1\right]a^\dagger a\right\} : , \tag{37}$$

since the normal ordering symbol within another normal ordering symbol can be canceled, *i.e.*, $: W : U : : = : WU :$, and the parameters h, h_1, h_2, and h_3 respectively are

$$\begin{aligned} h &= \left\{[1-(g_2+1)T]^2 - 4g_1^2T^2\right\}^{-1}, \\ h_1 &= 1 - 4h[1-(g_2+1)T]\rho\upsilon + 4hg_1\left(\rho^2 + T^2\upsilon^2\right), \\ h_2 &= 4h[1-(g_2+1)T]\left(T\upsilon^2 + 2g_1T\rho\upsilon + (g_2+1)\rho^2\right) \\ &\quad + 8hg_1T\rho\left(\upsilon + 2g_1\rho + T(g_2+1)\upsilon\right), \\ h_3 &= 4hg_1T\left(T(g_2+1)\upsilon + 2g_1\rho\right)\mathrm{e}^{-\kappa t}a \\ &\quad + 2\left((1-(g_2+1)T)\upsilon + 2hg_1\rho\right)\mathrm{e}^{-\kappa t}a^\dagger. \end{aligned}$$

Comparing the right sides of Eqs. (15) and (36), and using the new special function in Eq. (24), we immediately obtain the analytical evolution of SNBS in ADC, that is

$$\rho(t) =: \mathrm{e}^{\Omega}\mathfrak{F}_{s,s}\left(h_3/(2\sqrt{h_1}), h_3^\dagger/(2\sqrt{h_1}); h_2/h_1\right) : , \tag{38}$$

which means that the pure SNBS evolves into a new mixed state in ADC, related to two operator Hermite polynomial functions whose variables are the linear combinations of creation and annihilation operators within normal ordering. Especially, when $r = 0$, thus $g = 1$, $g_1 = 0$, and $g_2 = -\gamma$, equation (38) becomes the evolution of NBS in ADC, *i.e.*,

$$\rho(t) = \sum_{n=0}^{\infty}\binom{n+s}{n}\gamma'^{s+1}(1-\gamma')^n|n\rangle\langle n| \tag{39}$$

with $\gamma' = \gamma\mathrm{e}^{2\kappa t}/\left[1-\gamma\left(1-\mathrm{e}^{2\kappa t}\right)\right]$, which shows that, after undergoing ADC, the resultant state is still an NBS, but the parameter γ decays to γ' and $\gamma' < \gamma$. It is in agreement with the result in Ref. [16]. For the case of $s = 0$, $\rho(t)$ becomes

$$\rho(t) = \gamma g h^{1/2} : \exp\left\{h g_1 \mathrm{e}^{-2\kappa t}\left(a^{\dagger 2} + a^2\right) + \left[4hg_1^2T\mathrm{e}^{-2\kappa t} + h(g_2+1)(1-(g_2+1)T)\mathrm{e}^{-2\kappa t} - 1\right]a^\dagger a\right\} : , \tag{40}$$

which is the evolution of squeezed quasi-thermal field in ADC. In addition, when the decay time $\kappa t = 0$, thus $T = 0$, $h = 1$, $h_1 = 1 - 4h\rho\upsilon + 4hg_1\rho^2$, $h_2 = 4h(g_2+1)\rho^2$, and $h_3 = 2(\upsilon + 2hg_1\rho)a^\dagger$, so equation (38) returns to SNBS in Eq. (3). For the limited decay time, *i.e.*, $\kappa t \to \infty$, $T \to 1$, thus SNBS finally decays to vacuum.

5. Conclusions

In summary, we have used TESR and the operator-ordering method to derive the analytical evolution law of SNBS in ADC. It is found that the analytical WF is the finite-dimensional sum over the square of the module of single-variable Hermite polynomials. Numerically, the WF for SNBS can exhibit a similar Gaussian distribution in both coherent and quasi-thermal limited cases, and the larger s can result in an inverted peak with the smaller peak value in the WF phase space but the larger γ plays an opposite role. For another case, after passing through ADC, the pure SNBS evolves into

a new mixed state, whose normal ordering product of the density operator is related to two operator Hermite polynomials of the linear combinations of creation and annihilation operators, however the SNBS finally decays to vacuum as a result of amplitude decay when the decay time κt is long enough. From the above results, we have clearly found that, in the study of the environment-induced decoherence aspects, the TESR and the operator-ordering theory provide a privileged tool to investigate the analytical evolution of density operator and characterize destruction level of nonclassicality. Besides, through deriving the WF for SNBS, we have found a new two-variable special function $\mathfrak{F}_{s,s}(x,y;\omega)$ and its generating function, with which the concise formulas of WF and density-operator evolution can be obtained.

References

[1] Kimble H J 1992 *Phys. Rep.* **219** 227
[2] Dodonov V V 2002 *J. Opt. B: Quantum Semiclass. Opt.* **4** R1
[3] Wang S, Hou L L, Chen X F and Xu X F 2015 *Phys. Rev. A* **91** 063832
[4] Sohma M and Hirota O 2003 *Phys. Rev. A* **68** 022303
[5] Barenco A, Dutch D, Ekert A and Jozsa R 1995 *Phys. Rev. Lett.* **74** 4083
[6] Ekert A 1991 *Phys. Rev. Lett.* **67** 661
[7] Mandel L and Wolf E 1995 *Optical Coherence and Quantum Optics* (Cambridge: Cambridge University Press) p. 649
[8] Biswas A and Agarwal G S 2007 *Phys. Rev. A* **75** 032104
[9] Hu L Y and Fan H Y 2008 *J. Opt. Soc. Am. B* **25** 1955
[10] Meng X G, Wang Z, Fan H Y and Wang J S 2012 *J. Opt. Soc. Am. B* **29** 3141
[11] Berry M V 1984 *Proc. R. Soc. A* **392** 45
[12] Fan H Y, Zaidi H R and Klauder J R 1987 *Phys. Rev. D* **35** 1831
[13] Agarwal G S 1992 *Phys. Rev. A* **45** 1787
[14] Matsuo K 1990 *Phys. Rev. A* **41** 519
[15] Wang X G and Fu H C 2000 *Int. J. Theor. Phys.* **39** 1437
[16] Chen F and Fan H Y 2014 *Chin. Phys. B* **23** 030304
[17] Liu T K, Wu P P, Shan C J, Liu J B and Fan H Y 2015 *Chin. Phys. B* **24** 090302
[18] Da C, Chen Q F and Fan H Y 2014 *Int. J. Theor. Phys.* **53** 4372
[19] Joshi A and Lawande S V 1991 *J. Mod. Opt.* **38** 2009
[20] Meng X G, Wang Z, Fan H Y and Wang J S 2012 *J. Opt. Soc. Am. B* **29** 1835
[21] Xu Y J and Meng X G 2013 *Int. J. Theor. Phys.* **52** 4155
[22] Wang Z, Meng X G and Fan H Y 2012 *J. Opt. Soc. Am. B* **29** 397
[23] Song J, He R, Yuan H, Zhou J and Fan H Y 2016 *Chin. Phys. Lett.* **33** 110302
[24] Fan H Y 2003 *J. Opt. B: Quantum Semiclass. Opt.* **5** R147
[25] Meng X G, Fan H Y and Wang J S 2018 *Int. J. Theor. Phys.* **57** 1202
[26] Zhou J, Fan H Y and Song J 2017 *Chin. Phys. B* **26** 070301
[27] Fan H Y and Hu L Y 2008 *Mod. Phys. Lett. B* **22** 2435
[28] Fan H Y 2008 *Opt. Commun.* **281** 2023
[29] Meng X G, Wang Z, Wang J S and Fan H Y 2013 *J. Opt. Soc. Am. B* **30** 1614
[30] Meng X G, Wang J S and Gao H C 2016 *Int. J. Theor. Phys.* **55** 3630
[31] Du C X, Meng X G, Zhang R and Wang J S 2017 *Chin. Phys. B* **26** 120301
[32] Meng X G, Wang J S, Liang B L and Du C X 2018 *J. Exp. Theor. Phys.* **127** 383
[33] Xu X L, Li H Q and Fan H Y 2015 *Chin. Phys. B* **24** 070306
[34] Methta C L 1967 *Phys. Rev. Lett.* **18** 752
[35] Puri R R 2001 *Mathematical Methods of Quantum Optics* (Berlin: Springer-Verlag) p. 21
[36] Zhang B L, Meng X G and Wang J S 2012 *Chin. Phys. B* **21** 030304
[37] Meng X G, Wang J S and Liang B L 2013 *Chin. Phys. B* **22** 030307
[38] Glauber R J 1963 *Phys. Rev.* **130** 2529
[39] Klauder J R 1963 *J. Math. Phys.* **4** 1005
[40] Wang Z, Meng X G and Fan H Y 2013 *J. Phys. A: Math. Theor.* **46** 135305
[41] Meng X G and Xu X J 2014 *Int. J. Theor. Phys.* **53** 1239
[42] Fan H Y and Jiang T F 2007 *Mod. Phys. Lett. B* **21** 475
[43] Fan H Y and Xu X F 2004 *Opt. Lett.* **29** 1048
[44] Meng X G, Goan H S, Wang J S and Zhang R 2018 *Opt. Commun.* **411** 15
[45] Gardiner C W and Zoller P 2000 *Quantum Noise*, 2nd edn. (Berlin: Springer-Verlag) p. 147
[46] Meng X G, Wang J S, Liang B L and Han C X 2018 *Front. Phys.* **13** 130322
[47] Xiang S H, Zhu X X and Song K H 2018 *Chin. Phys. B* **27** 100305
[48] Wang Z, Meng X G, Li H M and Yuan H C 2014 *Int. J. Mod. Phys. B* **28** 1450115
[49] Wang J S, Meng X G and Fan H Y 2015 *Chin. Phys. B* **24** 014203
[50] Wang J S, Meng X G and Fan H Y 2011 *Chin. Phys. Lett.* **28** 104209
[51] Meng Y, Wang J S and Meng X G 2011 *Int. J. Theor. Phys.* **50** 3348
[52] Meng X G, Liu J M, Wang J S and Fan H Y 2019 *Eur. Phys. J. D* **73** 32