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Evolution of quantum states via Weyl expansion in dissipative channel*

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Based on the Weyl expansion representation of Wigner operator and its invariant property under similar transformation, we derived the relationship between input state and output state after a unitary transformation including Wigner function and density operator. It is shown that they can be related by a transformation matrix corresponding to the unitary evolution. In addition, for any density operator going through a dissipative channel, the evolution formula of the Wigner function is also derived. As applications, we considered further the two-mode squeezed vacuum as inputs, and obtained the resulted Wigner function and density operator within normal ordering form. Our method is clear and concise, and can be easily extended to deal with other problems involved in quantum metrology, steering, and quantum information with continuous variable.

Keywords: Weyl expansion, Wigner operator, similar transformation, two-mode squeezed state, integration within ordered product (IWOP) technique

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1. Introduction

In quantum mechanics, the whole quantum system can be totally described when the density operator is known. Considering the abstract of the density operator, some quasi-probability distribution functions, such as Wigner function, Q function as well as P function, are introduced in phase space,[1–3] which is also an appropriate stage connecting classical and quantum mechanics. With the aid of these distribution functions, one can describe the noncalssical properties which are not involved in the classical mechanics. For example, the negative Wigner function is an obvious indicator of nonclassicality.[4] In addition, it is also convenient for expectation calculation of quantum mechanics operators.

Although the operator is abstract, the operator problem is common in quantum physics. For instance, operators' ordering is actually an important topic in quantum optics and quantum field theory, where arranging operators are needed for path integral quantization, and converting quantum master equation (ME) to Fokker–Planck equation in quantum optics, and so on. Furthermore, it will also be beneficial for deeply understanding the essence of the operator from the view point of operator. It is an excellent example that superoperator method is introduced to deal with the dissipative problems.[5] Then these MEs are solved to obtain density operators' infinite operator-sum representation via the thermo-entangled state representation.[6,7] In addition, the operator method has also been applied to the fields of quantum information and preparation of quantum states. For example, an operator describing the quantum-state transfer is derived such that the fidelity of quantum teleportation is thus shown to be a direct consequence of the information obtained in the measurement.[8] Recently, this operator method is used to discuss the preparation of nonclassical states via conditional measurement,[9,10] such as photon-addition/subtraction and the superposition of both, as well as quantum catalysis.

In many cases, density operator is used rather than distribution functions. For instance, quantum entanglement (negativity, entropy) can be calculated by the trace norm of partial transpose of density operator.[11] And the minimum error probability in quantum illumination can be evaluated by two possible output density operators. On one hand, there exist an evolution process for quantum state and an interaction between quantum system and its surroundings. Thus the density operator is usually difficult to obtain. On the other hand, however, it will be relatively easy to describe the resulted state by Wigner function. Thus it is a natural idea how we can obtain the density operator from the Wigner function. In this paper, we shall construct the relation between the Wigner function and density operator by introducing the Wigner operator. Then we pay our attention to the evolution of quantum states through dissipative channel.

The rest of this paper is arranged as follows. In Sec-

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tion 2, we review the Wigner operator and discuss its properties. Based on the property, the evolution of the Wigner function after unitary transformation is considered in Section 3, especially how to obtain density operator from the Wigner function. We use the method above to examine the evolution formula of the Wigner function in dissipative channel in Section 4. Section 5 is devoted to deriving the density operator of the TMSV in different dissipative channels. Our conclusions are drawn in the last section.

2. Wigner operator and its property

2.1. Introduction of Wigner operator

First, we here consider the Wigner function of the density operator. For any single-mode density operator ρ, the Wigner function in coordinate representation $|q\rangle$ is defined as in Refs. [1]–[3]

$$W(q,p)=\int_{-\infty}^{\infty}\frac{\mathrm{d}y}{2\pi}\mathrm{e}^{\mathrm{i}yp}\left\langle q-\frac{y}{2}\middle|\rho\middle|q+\frac{y}{2}\right\rangle. \tag{1}$$

Here, $W(q,p)$ is actually a quasi-probability distribution function satisfying

$$\int_{-\infty}^{\infty}\mathrm{d}q\mathrm{d}pW(q,p)=1, \tag{2}$$

and its marginal distributions, the probability distribution of position and momentum, can be given by performing the integration over the variable p or q, respectively,

$$\int_{-\infty}^{\infty}\mathrm{d}pW(q,p)=\langle q|\rho|q\rangle=W(q), \tag{3}$$

$$\int_{-\infty}^{\infty}\mathrm{d}qW(q,p)=\langle p|\rho|p\rangle=W(p), \tag{4}$$

where $|p\rangle$ is the eigenvalue of the momentum operator.

In order to discuss the quantum system from the view point of operator, it is often convenient to rewrite Eq. (1) as the following form

$$W(q,p)=\mathrm{Tr}\left[\rho\Delta(q,p)\right], \tag{5}$$

by introducing the operator $\Delta(q,p)$, named as Wigner operator,[12] whose form is given by

$$\Delta(q,p)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}y\mathrm{e}^{\mathrm{i}yp}|q+y/2\rangle\langle q-y/2|. \tag{6}$$

From Eq. (6) it is ready to see that the Wigner operator has the following properties: (i) the Wigner operator is Hermite, *i.e.*, $\Delta(q,p)=\Delta^{\dagger}(q,p)$, which protects the real property of the Wigner function; (ii) The trace of the Wigner operator is a constant:

$$\mathrm{Tr}\left[\Delta(q,p)\right]=\frac{1}{2\pi}, \tag{7}$$

and for two same modes, the trace of product is just a Dirac delta function, *i.e.*,

$$\mathrm{Tr}\left[\Delta(q,p)\Delta\left(q',p'\right)\right]=\frac{1}{2\pi}\delta\left(q-q'\right)\delta\left(p-p'\right)$$
$$=\frac{1}{4\pi}\delta\left(\alpha-\alpha'\right)\delta\left(\alpha^{*}-\alpha'^{*}\right), \tag{8}$$

where $\alpha=(q+\mathrm{i}p)/\sqrt{2},\alpha'=(q'+\mathrm{i}p')/\sqrt{2}$ and the overlap relation $\langle q|q'\rangle=\delta(q-q')$ is used.

2.2. Normal ordering form of Wigner operator

On the other hand, one can gain a deep understand if the specific form of the Wigner operator is obtained. For this purpose, we can appeal to the technique of integration within an ordered product (IWOP) of operators,[13–15] and use the coordinate state $|q\rangle$ whose explicit form is given by

$$|q\rangle=\pi^{-1/4}\exp\left\{-\frac{1}{2}q^{2}+\sqrt{2}qa^{\dagger}-\frac{a^{\dagger2}}{2}\right\}|0\rangle, \tag{9}$$

where $Q|q\rangle=q|q\rangle$ and $a^{\dagger}$ and a satisfying $\left[a,a^{\dagger}\right]=1$ are creation and annilihation operators, respectively. With the help of the IWOP technique, it is ready to obtain

$$\Delta(q,p)=\frac{1}{\pi}:\exp\left\{-(q-Q)^{2}-(p-P)^{2}\right\}: \tag{10}$$

or

$$\Delta(q,p)=\frac{1}{\pi}:\exp\left\{-2(a^{\dagger}-\alpha^{*})(a-\alpha)\right\}:$$
$$\equiv\Delta(\alpha,\alpha^{*}), \tag{11}$$

where the symbol : : denotes the normal ordering, and $Q=(a+a^{\dagger})/\sqrt{2}$ and $P=(a-a^{\dagger})/(\sqrt{2}\mathrm{i})$ are the coordinate and momentum operators, satisfying the commutative relation $[Q,P]=\mathrm{i}$. Equation (10) is just the normal ordering form of the Wigner operator, which presents a concise Gaussian type within the normal ordering.

2.3. Weyl ordering and its invariance

Actually, except the normal ordering and anti-normal ordering form of operators, the Weyl ordering form is a special interesting one. Then what is Weyl ordering form of the Wigner operator? Using Eq. (11) and the formula converting the normal ordering form to the Weyl ordering form,[16] *i.e.*,

$$\hat{F}=\vdots\ 2\int_{-\infty}^{\infty}\frac{\mathrm{d}^{2}\beta}{\pi}\langle-\beta|\hat{F}|\beta\rangle\,\mathrm{e}^{2\left(\beta^{*}a-\beta a^{\dagger}+a^{\dagger}a\right)}\ \vdots, \tag{12}$$

where $|\beta\rangle$ is the coherent state, it is ready to see that the Weyl ordering form is just the following delta function, a concise form,

$$\Delta(q,p)=\vdots\ \delta(q-Q)\delta(p-P)\ \vdots$$
$$=\frac{1}{2}\vdots\ \delta(a-\alpha)\delta\left(a^{\dagger}-\alpha^{*}\right)\ \vdots. \tag{13}$$

Here Dirac's δ-operator function within Weyl ordering symbol $\vdots\ \vdots$ can bring much convenience for discussing some problems in quantum optics and quantum statistics.

In addition, there is a very useful property for these operators within Weyl ordering symbol, that the Weyl ordering is invariant under the similar transformations,[17] which means

$$S \vdots\, f\left(a,a^{\dagger}\right) \vdots\, S^{-1} = \vdots\, f\left(SaS^{-1},Sa^{\dagger}S^{-1}\right) \vdots\,. \tag{14}$$

This implies that S operator can run across the symbol $\vdots\ \vdots$. This property can be checked by using the Weyl expression of operator or the definition of Dirac's delta function for Eq. (13). These properties above shall be effective for dealing with the unitary evolution systems and are going to be used in the following calculation.

3. Evolution of Wigner function after unitary transformation

From Eq. (5) one can see that the Wigner function can be formally derived from the Wigner operator. Then how can we obtain the density operator if we know the Wigner function? For any single-mode density operator ρ, combing Eqs. (5) and (8), it is ready to have

$$\rho = 2\pi \int_{-\infty}^{\infty} W(q,p)\,\Delta(q,p)\,\mathrm{d}q\mathrm{d}p, \tag{15}$$

or

$$\rho = 4\pi \int_{-\infty}^{\infty} W(\alpha,\alpha^{*})\,\Delta(\alpha,\alpha^{*})\,\mathrm{d}^{2}\alpha. \tag{16}$$

Here we should notice that $2\mathrm{d}^{2}\alpha = \mathrm{d}q\mathrm{d}p$. Actually, $2\pi W(q,p)$ is just the Weyl correspondence of the density operator ρ.[18] Equation (15) or equation (16) implies that the density operator ρ can be derived from the Wigner function if the Wigner function is easily obtained. Thus equation (15) construct the bridge between the Wigner function in phase space and quantum density operator.

Now, we consider a single mode density operator ρ_{in} going through a unitary evolution characteristics of unitary operator U with the following transformation,

$$U\begin{pmatrix} Q \\ P \end{pmatrix} U^{\dagger} = M\begin{pmatrix} Q \\ P \end{pmatrix}, \tag{17}$$

where $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, $\det M = AD - BC = 1$ required by unitary property, or

$$U\begin{pmatrix} a \\ a^{\dagger} \end{pmatrix} U^{\dagger} = \begin{pmatrix} r & s \\ s^{*} & r^{*} \end{pmatrix}\begin{pmatrix} a \\ a^{\dagger} \end{pmatrix}, \tag{18}$$

where

$$r = (1/2)\left[A + D - \mathrm{i}\,(B - C)\right],$$
$$s = (1/2)\left[A - D + \mathrm{i}\,(B + C)\right],$$

and $|r|^{2} - |s|^{2} = 1$. Actually, for many quantum evolution, the transformation relation of operators are relatively easily obtained. Thus it is convenient to examine the evolved Wigner function from the view point of operator transformation. Under the transformation shown in Eq. (17) and using Eq. (15), then the evolved density operator, denoted as ρ_{out}, can be put into the form

$$\begin{aligned}\rho_{\mathrm{out}} &= U\rho_{\mathrm{in}}U^{\dagger} \\ &= 2\pi \int_{-\infty}^{\infty} W_{\mathrm{in}}(q,p)\left[U\Delta(q,p)U^{\dagger}\right]\mathrm{d}q\mathrm{d}p.\end{aligned} \tag{19}$$

Using Eqs. (13) and (14) it is ready to gain

$$\begin{aligned}U\Delta\left[\begin{pmatrix} q \\ p \end{pmatrix}\right]U^{\dagger} &= U \vdots\, \delta(q-Q)\,\delta(p-P) \vdots\, U^{\dagger} \\ &= \Delta\left[M^{-1}\begin{pmatrix} q \\ p \end{pmatrix}\right],\end{aligned} \tag{20}$$

then substituting Eq. (20) into Eq. (19) yields the output density operator, *i.e.*,

$$\begin{aligned}\rho_{\mathrm{out}} &= 2\pi \int_{-\infty}^{\infty} W_{in}\left[\begin{pmatrix} q \\ p \end{pmatrix}\right]\Delta\left[M^{-1}\begin{pmatrix} q \\ p \end{pmatrix}\right]\mathrm{d}q\mathrm{d}p \\ &= 2\pi \int_{-\infty}^{\infty} W_{\mathrm{in}}\left[M\begin{pmatrix} q \\ p \end{pmatrix}\right]\Delta(q,p)\,\mathrm{d}q\mathrm{d}p.\end{aligned} \tag{21}$$

Comparing Eq. (15) with Eq. (21), it is clear to see that the Wigner function after evolution can be directly obtained by replacing $\left(\begin{smallmatrix} q \\ p \end{smallmatrix}\right)$ with the classical canonical transform $M\left(\begin{smallmatrix} q \\ p \end{smallmatrix}\right)$ in the initial Wigner function in phase space, *i.e.*,

$$W_{\mathrm{in}}\left[M\begin{pmatrix} q \\ p \end{pmatrix}\right] \Rightarrow W_{\mathrm{out}}\left[\begin{pmatrix} q \\ p \end{pmatrix}\right]. \tag{22}$$

Although this conclusion is drawn in the single-mode case, it is also valid for general multi-mode case. For ease of use, for multi-mode case, we show the corresponding formula as follows:

$$W_{\mathrm{in}}[\boldsymbol{M}\boldsymbol{q}] \Rightarrow W_{\mathrm{out}}(\boldsymbol{q}), \tag{23}$$

and

$$\begin{aligned}\rho_{\mathrm{out}} &= (2\pi)^{n} \int_{-\infty}^{\infty} W_{\mathrm{in}}(\boldsymbol{q})\,\Delta\left(\boldsymbol{M}_{n}^{-1}\boldsymbol{q}\right)\mathrm{d}^{n}\boldsymbol{q} \\ &= (2\pi)^{n} \int_{-\infty}^{\infty} W_{\mathrm{in}}(\boldsymbol{M}_{n}\boldsymbol{q})\,\Delta(\boldsymbol{q})\,\mathrm{d}^{n}\boldsymbol{q},\end{aligned} \tag{24}$$

where $\boldsymbol{q} = (q_1, p_1, \ldots q_n, p_n)^{\mathrm{T}}$, and $\Delta(\boldsymbol{q}) = \Delta_1(q_1,p_1) \otimes \Delta_2(q_2,p_2) \cdots \otimes \Delta_n(q_n,p_n)$. These disussions above tell us that when the matrix describing a unitary evolution shown in Eq. (17) is known, then the evolved Wigner function can be directly obtained. According to Eq. (15) or Eq. (21) one can obtain the evolved density operator. That is to say, once the initial Wigner function and the unitary transformation are known, the output state and the Wigner function can be obtained by a replacement and an integration, respectively. From the above discussions, one can clearly see that it is all because of the Weyl ordering form of the Wigner operator and its main invariance property mentioned above in Eq. (14), which is also the reason that we consider the relation between the Wigner function and the density operator. In addition, the Wigner function is a real function and can be measured experimentally.

4. Evolution formula of Wigner function in dissipative channel

Next, we use the method above to examine the evolution formula of the Wigner function in dissipative channel. In most cases, in order to consider the dissipative effect including photon loss (amplitude dissipation), thermal channel and phase sensitive channel, the beam splitter models are widely used, accompanied by two inputs: signal state and accessory state, denoted as $\rho_{\rm in}$ and ρ_m respectively. Thus the output state can be expressed as $\rho^a_{\rm out} = {\rm Tr}_m[\rho_{\rm out}]$, where $\rho_{\rm out} = B\rho_{\rm in}\otimes\rho_m B^\dagger$ and $B = \exp\{\theta(a^\dagger b - ab^\dagger)\}$ which leads to the transformation relation

$$B\begin{pmatrix} a \\ a^\dagger \\ b \\ b^\dagger \end{pmatrix} B^\dagger = S_{ab}\begin{pmatrix} a \\ a^\dagger \\ b \\ b^\dagger \end{pmatrix}, \tag{25}$$

$$S_{ab} = \begin{pmatrix} \sqrt{T} & & -\sqrt{R} & \\ & \sqrt{T} & & -\sqrt{R} \\ \sqrt{R} & & \sqrt{T} & \\ & \sqrt{R} & & \sqrt{T} \end{pmatrix}, \tag{26}$$

where $T = \cos^2\theta$, $R = 1-T$ are the transmissivity and reflectivity, respectively. From Eq. (25), one can see that

$$B\begin{pmatrix} Q_1 \\ P_1 \\ Q_2 \\ P_2 \end{pmatrix} B^\dagger = M_2\begin{pmatrix} Q_1 \\ P_1 \\ Q_2 \\ P_2 \end{pmatrix}, \quad M_2 = S_{ab}, \tag{27}$$

where $Q_1 = (a+a^\dagger)/\sqrt{2}$, $P_1 = (a-a^\dagger)/({\rm i}\sqrt{2})$ and $Q_2 = (b+b^\dagger)/\sqrt{2}$, $P_2 = (b-b^\dagger)/({\rm i}\sqrt{2})$.

In order to realize our purpose by using Eq. (22), one needs to know the initial Wigner function of input states. For any two inputs $\rho_{\rm in}$ and ρ_m which can be chosen to be vacuum, thermal state, *etc*. here, according to Eq. (23), the corresponding Wigner function of $\rho_{\rm out}$ can be given by

$$W_{\rm out}(q_1,p_1,q_2,p_2) = W_{\rm in}(q_1',p_1')\,W_m(q_2',p_2'), \tag{28}$$

where $(q_1',p_1',q_2',p_2')^{\rm T} = M_2(q_1,p_1,q_2,p_2)^{\rm T}$. According to Eqs. (24) and (7), the output $\rho^a_{\rm out}$ is calculated as

$$\begin{aligned}\rho^a_{\rm out} &= (2\pi)^2\int W_{\rm out}(q_1,p_1,q_2,p_2)\,\Delta_{\rm in}(q_1,p_1)\,{\rm Tr}[\Delta_m(q_2,p_2)]\,{\rm d}q_1\,{\rm d}p_1\,{\rm d}q_2\,{\rm d}p_2 \\ &= 2\pi\int W_{\rm in}(q_1',p_1')\,W_m(q_2',p_2')\,\Delta_{\rm in}(q_1,p_1)\,{\rm d}q_1\,{\rm d}p_1\,{\rm d}q_2\,{\rm d}p_2.\end{aligned} \tag{29}$$

Then using Eqs. (5) and (8) one can obtain the Wigner function of $\rho^a_{\rm out}$

$$\begin{aligned}W^a_{\rm out}(q_1'',p_1'') &= {\rm Tr}\left[\rho^a_{\rm out}\Delta_{\rm in}(q_1'',p_1'')\right] = 2\pi\int W_{\rm in}(q_1',p_1')\,W_m(q_2',p_2')\,{\rm Tr}\left[\Delta_{\rm in}(q_1'',p_1'')\,\Delta_{\rm in}(q_1,p_1)\right]{\rm d}q_1\,{\rm d}p_1\,{\rm d}q_2\,{\rm d}p_2 \\ &= \left[\int W_{\rm in}(q_1',p_1')\,W_m(q_2',p_2')\,{\rm d}q_2\,{\rm d}p_2\right]_{q_1\to q_1'',p_1\to p_1''}.\end{aligned} \tag{30}$$

Equations (29) and (30) can be used to derive the density operator and the Wigner function of the output when going through a dissipative channel modeled by beam splitter, respectively. Actually, equations (29) and (30) are general expressions for deriving density operator and Wigner function for any two inputs states when they are combined by a unitary transformation martix.

Now, we use Eq. (30) to derive the evolution of the Wigner function for two kinds of channels: photon loss and thermal channels. For photon loss channel, $\rho_m = |0\rangle\langle 0|$, whose Wigner function is $W_m(q_2,p_2) = (1/\pi)\exp(-q_2^2-p_2^2)$. Substituting it into Eq. (30) one can obtain

$$W^{\rm vac}_{\rm out}(q_1'',p_1'') = \frac{1}{\pi}\int_{-\infty}^{\infty} W_{\rm in}(\sqrt{T}q_1''-\sqrt{R}q_2,\sqrt{T}p_1''-\sqrt{R}p_2)\exp\left[-(\sqrt{R}q_1''+\sqrt{T}q_2)^2-(\sqrt{R}p_1''+\sqrt{T}p_2)^2\right]{\rm d}q_2\,{\rm d}p_2. \tag{31}$$

After making a variable substitution, $\sqrt{T}q_1''-\sqrt{R}q_2\to q$, and $\sqrt{T}p_1''-\sqrt{R}p_2\to p$ which leads to ${\rm d}q_2\,{\rm d}p_2\to(1/R)\,{\rm d}q\,{\rm d}p$, then equation (31) can be reformed as

$$W^{\rm vac}_{\rm out}(q_1'',p_1'') = \int_{-\infty}^{\infty}\frac{{\rm d}q\,{\rm d}p}{\pi R}W_{\rm in}(q,p)\exp\left\{-\frac{1}{R}[(p_1''-\sqrt{T}p)^2+(q_1''-\sqrt{T}q)^2]\right\}. \tag{32}$$

Equation (32) indicates that one can derive the output Wigner function after photon loss only when the initial Wigner function is known. If we take $T = {\rm e}^{-2kt}$, equation (32) has the same form as the result in Ref. [19].

When the channel is a thermal one, *i.e.*,

$$\rho_m = \sum_{m=0}^{\infty}\frac{\bar{n}^m}{(\bar{n}+1)^{m+1}}|m\rangle\langle m|, \tag{33}$$

where $\bar{n}$ is the average photon in the thermal state. The corresponding Wigner function is given by

$$W_{\rm th}(q_2,p_2) = \frac{1}{\pi(2\bar{n}+1)}\exp\left\{-\frac{q_2^2+p_2^2}{2\bar{n}+1}\right\}. \tag{34}$$

In a similar way to derive Eq. (32), by replacing R with $R(2\bar{n}+1)$, it is ready to obtain

$$W_{\rm out}^{\rm th}(q_1'',p_1'')=\frac{1}{(2\bar{n}+1)}\int_{-\infty}^{\infty}\frac{{\rm d}q\,{\rm d}p}{\pi R}W_{\rm in}(q,p)\exp\left\{-\frac{(p_1''-\sqrt{T}p)^2+(q_1''-\sqrt{T}q)^2}{(2\bar{n}+1)R}\right\}, \tag{35}$$

or

$$W_{\rm out}^{\rm th}(\alpha,\alpha^*)=\frac{2}{R}\int\frac{W_{\rm in}(\beta,\beta^*)}{2\bar{n}+1}\exp\left[-\frac{2\left|\alpha-\sqrt{T}\beta\right|^2}{(2\bar{n}+1)R}\right]\frac{{\rm d}^2\beta}{\pi}. \tag{36}$$

Equations (32) and (35) are just the time evolution formula of the Wigner function when going through photon loss and thermal channels, respectively.[19] Here we should mention that there is a Fourier transformation relation between the Wigner function and the characteristic function, thus it will be interesting to bridge the relation between the characteristic function and the transformation, which can also be further extended to multimode case.

5. Density operator of the TMSV in dissipative channel

As an often used entangled resource, the two-mode squeezed vacuum (TMSV) is widely applied in many fields, such as quantum teleportation, quantum key distribution, quantum metrology, and so on.[20–23] Here we pay our attention to the Wigner function and the density operator after the TMSV goes through dissipative channels. It will be useful for further discussion in realistic case.

Theoretically, TMSVs can be shown as $|{\rm TMSV}\rangle=S(r)|00\rangle={\rm sech}\,r\exp\left(a^\dagger b^\dagger\tanh r\right)|00\rangle$, where $S(r)=\exp\left[r\left(a^\dagger b^\dagger-ab\right)\right]$ is the two-mode squeezed operator with r being a squeezing parameter, and $|n,n\rangle$ is two-mode Fock states. Its Wigner function is

$$W_{\rm in}(q_1,p_1,q_2,p_2)=\frac{1}{\pi^2}\exp[2(q_1q_2-p_1p_2)\sinh 2r]\exp[-\left(p_1^2+p_2^2+q_1^2+q_2^2\right)\cosh 2r]. \tag{37}$$

5.1. TMSV through two symmetrical photon-loss channels

Here for simplification, we assume that the two-mode of TMSV go through the same photon loss channel. According to Eq. (32) the Wigner function of dissipative TMSV can be obtained as

$$\begin{aligned}W_{\rm out}(\bar{q}_1,\bar{p}_1,\bar{q}_2,\bar{p}_2)=&\frac{1}{\pi^2R^2}\int_{-\infty}^{\infty}{\rm d}q_1\,{\rm d}p_1\,{\rm d}q_2\,{\rm d}p_2W_{\rm in}(q_1,p_1,q_2,p_2)\exp\left\{-\frac{1}{R}[(\bar{p}_1-\sqrt{T}p_1)^2+(\bar{q}_1-\sqrt{T}q_1)^2]\right\}\\&\times\exp\left\{-\frac{1}{R}[(\bar{p}_2-\sqrt{T}p_2)^2+(\bar{q}_2-\sqrt{T}q_2)^2]\right\}.\end{aligned} \tag{38}$$

Performing directly the integration above, one can obtain

$$W_{\rm out}(\bar{q}_1,\bar{p}_1,\bar{q}_2,\bar{p}_2)=\frac{1}{\pi^2A_1}\exp\left\{2A_3\left(\bar{q}_1\bar{q}_2-\bar{p}_1\bar{p}_2\right)\right\}\exp\left\{-A_2\left(\bar{p}_1^2+\bar{p}_2^2+\bar{q}_1^2+\bar{q}_2^2\right)\right\}, \tag{39}$$

where

$$A_1=1+4RT\sinh^2 r,\quad A_2=\frac{R+T\cosh 2r}{1+4RT\sinh^2 r},$$
$$A_3=\frac{T\sinh 2r}{1+4RT\sinh^2 r}.$$

It is interesting to notice that the Wigner function of output is similar to the initial Wigner function, which indicates that the Gaussianity of TMSV can be kept after photon loss. In other word, the loss channel can be seen as a Gaussian channel, although the output state is no more the pure state. In particular, when $T=1$ corresponding to the case without photon loss, equation (39) just reduces to Eq. (37) as expected. Further using Eq. (15) and the normal ordering form of the Wigner operator shown in Eq. (10) as well as the IWOP technique,[13–15] one can directly obtain the Wigner operator after the dissipation by taking $n=2$ in Eq. (24), *i.e.*,

$$\begin{aligned}\rho_{\rm out}=&\frac{4}{\pi^2A_1}\int_{-\infty}^{\infty}d\bar{q}_1\,{\rm d}\bar{p}_1\,{\rm d}\bar{q}_2\,{\rm d}\bar{p}_2\\&\times\exp\left\{2A_3\left(\bar{q}_1\bar{q}_2-\bar{p}_1\bar{p}_2\right)\right\}\\&\times\exp\left\{-A_2\left(\bar{p}_1^2+\bar{p}_2^2+\bar{q}_1^2+\bar{q}_2^2\right)\right\}\\&\times:\exp\left\{-(\bar{q}_1-Q_1)^2-(\bar{p}_1-P_1)^2\right\}\\&\times\exp\left\{-(\bar{q}_2-Q_2)^2-(\bar{p}_2-P_2)^2\right\}:\\=&B_1:\exp\left\{B_3\left(Q_1Q_2-P_1P_2\right)\right\}\\&\times\exp\left\{-B_2(Q_1^2+Q_2^2+P_1^2+P_2^2)\right\}:,\end{aligned} \tag{40}$$

where

$$B_1=\frac{{\rm sech}^2\,r}{1-R^2\tanh^2 r},\quad B_2=\frac{1-R\tanh^2 r}{2\left(1-R^2\tanh^2 r\right)},$$

$$B_3 = \frac{T\tanh r}{1-R^2\tanh^2 r}.$$

Or rewriting Eq. (40) as

$$\rho_{\rm out} = B_1 :\, {\rm e}^{-2B_2\left(a^\dagger a + b^\dagger b\right) + B_3\left(a^\dagger b^\dagger + ab\right)} :\,, \tag{41}$$

which is a Gaussian form, as expected.[24] Equation (41) is just the normally ordering form of the output state, which is convenient for our further discussions about the normalization, nonclassical properties including the degree of entanglement, and quantum steering.

5.2. TMSV through single-side thermal channel

Next, we further examine the case that one of the TAMV goes through the single-side thermal channel. In fact, this asymmetrical case can be often encountered in quantum information. For instance, quantum illumination can be seen as this model, in which one mode of signal state is going to through a thermal channel. In the process of dealing with the problem, a beam splitter with thermal state as an input is often used. Here we take the TMSV as the inputs and one mode through the thermal channel. Under this case, using Eqs. (35) and (37), the Wigner function of output state can be derived as

$$W_{\rm out}(\bar{q}_1,\bar{p}_1,\bar{q}_2,\bar{p}_2) = \frac{C_1}{\pi^2}\exp\left\{2C_4\left(\bar{q}_1\bar{q}_2 - \bar{p}_1\bar{p}_2\right)\right\}\exp\left\{-C_2\left(\bar{q}_2^2+\bar{p}_2^2\right) - C_3\left(\bar{q}_1^2+\bar{p}_1^2\right)\right\}, \tag{42}$$

where

$$C_1 = \left[T + (2\bar{n}+1)R\cosh 2r\right]^{-1},$$
$$C_2 = \left[(2\bar{n}+1)R + T\cosh 2r\right]C_1,$$
$$C_3 = C_1\cosh 2r, \quad C_4 = \sqrt{T}C_1\sinh 2r.$$

Here we should notice that only one mode goes through the thermal, thus the parameters for other mode in phase space is kept unchanged. Similarly, substituting Eq. (42) into Eq. (24) we finally obtain the density operator after the thermal channel, *i.e.*,

$$\rho_{\rm out} = \frac{\operatorname{sech}^2 r}{1+\bar{n}R} :\, \exp\left\{\frac{\sqrt{T}\tanh r}{1+\bar{n}R}\left(a^\dagger b^\dagger + ab\right)\right\} \times \exp\left\{\frac{-a^\dagger a}{1+\bar{n}R} - \frac{T\cosh^2 r + (1+\bar{n})R}{(1+\bar{n}R)\cosh^2 r}b^\dagger b\right\} :\,. \tag{43}$$

It is obvious that when $T=1$, equations (42) and (43) just reduce to Eq. (37) and the normal ordering form of the TMSV, respectively. In particular, when $\bar{n}=0$ corresponding to the single-side photon loss,[25] the results can be shown in Eqs. (42) and (43). Again, equation (43) is the normally ordering form when one mode of the TMSV through a thermal channel. Actually, for a convenience of our calculation, we often need some different ordering forms such as Weyl ordering form, anti-normal ordering form, and $Q-P$ ordering form, the normally ordering form above can be further converted into other ordering forms by using corresponding formula not shown here.

6. Conclusions

In summary, from the coordinate representation of the Wigner function, we introduced the Wigner operator, which leads to that the Wigner function can be seen as an average value of the Wigner operator within the density operator (the considered quantum state). Then using the explicit form of the coordinate state and the IWOP technique, we derived the normal ordering form of the Wigner operator, and the Weyl ordering form. It is shown that the Wigner operator presents a Gaussian form within normal ordering, while its Weyl ordering form is just a Dirac's delta function. These properties are very useful for discussing the evolution of quantum state from the view point of operator.

Based on these properties above, we further construct the relationship between initial state and output state after a unitary evolution, including Wigner function and density operator, by which it will be convenient and effective to obtain the output density operator. In addition, using this method, we further derived two evolution formula of the Wigner function when any single-mode quantum state goes through photon loss and thermal channels. As applications, we finally considered the two-mode squeezed vacuum going through photon loss and thermal channels, and the output Wigner function and the normal ordering form of output density operator are obtained. These results can provide some convenience for further discussion in other cases, such as quantum steering and quantum metrology as well as quantum cryptography.[26,27]

It should be emphasized that the above method can be extended to multi-mode system although we only considered single- and two-mode cases. Recently, the non-Gaussian operation and conditional measurement are used to improve the quantum correlation, such as the quantum teleportation, the degree of entanglement, quantum steering, and so on.[28] Our method is also valid for non-Gaussian states especially for obtaining the final density operator.[29] In the whole process, the IWOP technique has an important role.

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