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On a biseparability criterion of bipartite qudit state

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In this paper, we propose a new approach to tackle the separability problem for bipartite qudit mixed-states. This is based on the Majorana representation which allows to represent a N-spinors (qudit) as a symmetric state of N spin-1/2. We also discuss how we can exploit such representation and the notion of the biseparability of multipartite qubit states in the sense to establish new criteria of the separability problem based on the PPT and concurrence.

Keywords: separability, bipartite qubit state, partial separability, Majorana representation

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1. Introduction

The entanglement properties constitute a useful resource for several quantum information processing. Indeed, this counter-intuitive concept has been used to implement various quantum protocols such as quantum key distribution,[1–3] quantum teleportation,[4,5] and quantum communication.[6] Most quantification attempts have concentrated on bipartite qubit states, which is well understood, but matters get more complicated in mixed and multipartite cases and many problems regarding the quantification entanglement in multipartite systems are still not well understood. In this paper, we will focus on the entanglement in bipartite qudit mixed-states. We especially consider the criterion of separability which is one of the principal problems in quantum entanglement. In the content of multipartite entanglement, the Peres and Horodecki separability criteria for bipartite states[7,8] is necessary and sufficient in the (2×2)- and (2×3)-dimensional cases and only necessary for higher dimensional systems. This paper will take a new approach to the problem of separability using Majorana representation[9] which has recently attracted a great deal of attention in connection with multipartite entanglement. We will first introduce the Majorana representation and use it to decompose a qudit state into a multipartite qubit state which will be the subject of Section 2 using the connection between N-spinors (qudit) and symmetric state of N spin-1/2.[10] In Section 3, we will define the notion of partial separability in multipartite qubit systems, especially the biseparability.[11–13] We will also discuss how we can exploit it in order to have new criteria of separability in bipartite qudit states. As we will see in Section 4, using the positive partial Franspose criterion[7,8] (PPT) on all subsystems of the form 2×2 will reveal a necessary criterion of the biseparability of the multipartite qubit state and consequently the separability of the bipartite qudit state. Also in Section 5, we will do the same but with the concurrence measure instead of the PPT which is an easier calculation. Concluding remarks close this paper.

2. The relation between N-spinors and symmetric pure state representation

Majorana has suggested a very nice connection between N-spinors and a pure symmetric state.[9] In this representation a symmetric state can be seen geometrically as a set of points in the Bloch sphere.[9] Any symmetric state can be written as

$$|\Psi_{\rm sym}\rangle = M\sum_{p}\hat{P}\{|\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_N\rangle\}, \tag{1}$$

where

$$|\varepsilon_l\rangle = \cos(\beta_l/2)\,{\rm e}^{-{\rm i}\alpha_l/2}|0\rangle + \sin(\beta_l/2)\,{\rm e}^{{\rm i}\alpha_l/2}|1\rangle, \quad (l=0,1,2,\ldots,N), \tag{2}$$

$\hat{P}$ corresponds to the set of all $N!$ permutations of qubits $\{|\varepsilon_l\rangle\}$ and M is the normalization factor. Every symmetric state of N qubits can be expressed in a unique way over the Dicke basis formed by the $N+1$ joined eigenstates $\{|N/2,l-N/2\rangle$, with $(l=0,1,2,\ldots,N)\}$ of the collective operators $\hat{S}_z=\sum_{i=1}^{N}\hat{\sigma}_i^z$ and $\hat{S}^2$, where $\hat{S}=\sum_{i=1}^{N}\hat{\sigma}_i$ with

$$\left|\frac{N}{2},l-\frac{N}{2}\right\rangle = \sqrt{\frac{l!(N-l)!}{N!}}\left(|\underbrace{0,0\ldots,0}_{l},\underbrace{1,1\ldots,1}_{N-l}\rangle + \text{permutations}\right). \tag{3}$$

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For $N=2$, equation (1) reduces to

$$|\Psi_{\rm sym}\rangle = M\left(|\varepsilon_1,\varepsilon_2\rangle + |\varepsilon_2,\varepsilon_1\rangle\right),$$

the pure symmetric state describes a spin-1 particle. In the Dicke basis $\{|1,l-1\rangle,\ l=0,1,2\}$ for each value of l corresponds to a basis state $|j,m\rangle$ of spin-1 particle

$$\left\{|j=1,m=-1\rangle = |0,0\rangle, |j=1,m=0\rangle = (1/\sqrt{2})(|1,0\rangle + |0,1\rangle), |j=1,m=1\rangle = |1,1\rangle\right\}.$$

An arbitrary pure state of a spin-j can be seen as a pure symmetric state that can be expressed in the Dicke basis as follows:

$$\sum_{m=-j}^{j} c_l\,|j,m\rangle = M\sum_p \hat{P}\{|\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_N\rangle\} = \sum_{l=0}^{N} c_l \left|\ \frac{N}{2}=j, l-\frac{N}{2}\right\rangle. \tag{4}$$

In this method, we can decompose state of a spin-j into $2j$ spin-1/2 in the Dicke basis formed by $\{|N/2,l-N/2\rangle\}$ that can be seen as symmetric state of $N=2j$ spin-1/2. This can be geometrically represented by $2j$ points in the Bloch sphere which offers a suitable geometric picture of the system in terms of points on the unit sphere S^2. Note that each point on the Majorana sphere are characterized with the pair of angles (α_l,β_l) determining the orientation of each point on the sphere.

There is a simpler way to express the coefficients c_l in terms of the Majorana spinors orientations (α_l,β_l).[10] We first notice that a rotation $R\otimes R\cdots\otimes R$ on the symmetric state transforms it into another symmetric state. Choosing $R^{-1}(\alpha_s,\beta_s)\otimes R^{-1}(\alpha_s,\beta_s)\otimes\cdots\otimes R^{-1}(\alpha_s,\beta_s)$, takes one of the spinors $|\varepsilon_s\rangle$ with the orientation angles (α_s,β_s) to $|0\rangle$

$$\langle 1,1,\ldots,1|R^{-1}(\alpha_s,\beta_s)\otimes R^{-1}(\alpha_s,\beta_s)\otimes\ldots\otimes R^{-1}(\alpha_s,\beta_s)|\Psi_{\rm sym}\rangle = 0. \tag{5}$$

There exist N rotations $R_s^{-1}, s=1,2,\ldots,N$ which lead to the same result

$$\langle 1,1,\ldots,1|R_s^{-1}\otimes R_s^{-1}\otimes\ldots\otimes R_s^{-1}\left\{\sum_{l=0}^{N} c_l\left|\ \frac{N}{2},l-\frac{N}{2}\right\rangle\right\} = 0. \tag{6}$$

or

$$\sum_{l=0}^{N} c_l\left\langle \frac{N}{2},-\frac{N}{2}\ \middle|\ \boldsymbol{R}_s^{-1}\ \middle|\ \frac{N}{2},l-\frac{N}{2}\right\rangle = 0 \tag{7}$$

with $\boldsymbol{R}_s^{-1}\equiv R_s^{-1}\otimes R_s^{-1}\otimes\ldots\otimes R_s^{-1}$. This condition can be expressed as follows:

$$\sum_{l=0}^{N} c_l\langle\frac{N}{2},-\frac{N}{2}|\boldsymbol{R}_s^{-1}|\frac{N}{2},l-\frac{N}{2}\rangle = \sum_{l=0}^{N} c_l(-1)^l\sqrt{C_l^N}\left(\cos\left(\frac{\beta_s}{2}\right)\right)^{N-l}\left(\sin\left(\frac{\beta_s}{2}\right)\right)^{l}\exp\left[\mathrm{i}\left(l-\frac{N}{2}\right)\alpha_s\right] = 0. \tag{8}$$

This result can be re-equated as

$$P(\varepsilon) = \sum_{l=0}^{N} c_l(-1)^l\sqrt{C_l^N}\varepsilon^l = 0, \tag{9}$$

where

$$\varepsilon_s = \tan(\beta_s/2)\,\mathrm{e}^{(\mathrm{i}\alpha_s)}.$$

Given the parameters c_l, the N-roots ε_s of this polynomial determine the orientations (α_s,β_s) of the spinors constituting the N-qubit symmetric state. In other words, using an inverse stereographic projection, we can find the orientation of the spin on the Bloch sphere.

$$|\varepsilon_s\rangle = \cos(\beta_s/2)\,\mathrm{e}^{-\mathrm{i}\alpha_s/2}|0\rangle + \sin(\beta_s/2)\,\mathrm{e}^{\mathrm{i}\alpha_s/2}|1\rangle. \tag{10}$$

Our interest in this representation is due to the fact that we can decompose a qudit into a multipartite qubit symmetric state as in Eq. (4) and Fig. 1.

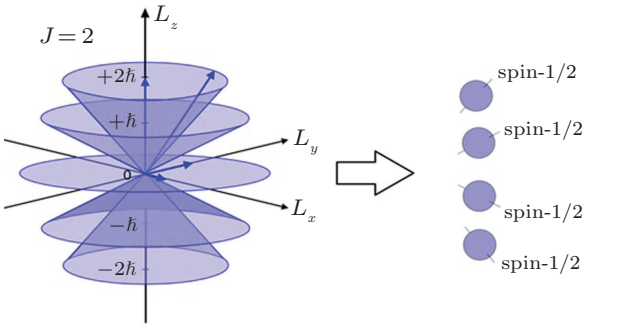


Fig. 1. The representation of a spin-j as a multipartite pure symmetric state.

In what follows we employ this decomposition for bipartite qudit state. For a pure state, assume that $|\psi\rangle$ is a bipartite qudit state in the Hilbert space $H=H_1\otimes H_2$, $\{|e_i^A\rangle\}$ and $\{|e_j^B\rangle\}$ are respectively a set of basis state in the Hilbert space H_1 and H_2 of dimension N and N'

$$|\psi\rangle = \sum_{i,j} c_{i,j}\,|e_i^A\rangle\,|e_j^B\rangle. \tag{11}$$

The basis states can be decomposed into a multipartite qubit symmetric states known as Dicke states introduced before in Eq. (3), $|e_i^A\rangle = |N/2, i-N/2\rangle$ such decomposition allows us to rewrite the state $|\psi\rangle$ in the following form

$$|\psi\rangle = \sum_{i,j} c_{i,j} \frac{1}{\sqrt{C_i^N C_j^{N'}}} \left(|\underbrace{0,0\ldots,0}_{i},\underbrace{1,1\ldots,1}_{N-i}\rangle + \text{permutations} \right) \otimes \left(|\underbrace{0,0\ldots,0}_{j},\underbrace{1,1\ldots,1}_{N'-j}\rangle + \text{permutations} \right). \tag{12}$$

For a mixed state $\rho = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|$, with $\sum p_\alpha = 1$, the density matrix can be written as follows:

$$\rho = \sum_\alpha p_\alpha \sum_{i,j,k,l} c_{i,j}^{(\alpha)} \bar{c}_{k,l}^{(\alpha)} \left| \frac{N}{2}, i-\frac{N}{2} \right\rangle \left\langle \frac{N}{2}, k-\frac{N}{2} \right| \otimes \left| \frac{N'}{2}, j-\frac{N'}{2} \right\rangle \left\langle \frac{N'}{2}, l-\frac{N'}{2} \right|. \tag{13}$$

Such representation can preserve naturally the separability. We assume that we have a mixed bipartite qudit state $\rho = \sum_i^N p_i |\psi_i\rangle\langle\psi_i|$ composed by M pure separable state $|\psi_i\rangle$ with different probabilities p_i, $\sum_{i=1}^N p_i = 1$. The separable mixed state ρ preserve the separability after its decomposition into a multipartite qubit state. The separability is only between the two parts associated with each qudit in the state ρ as follows:

$$\begin{aligned}
&|\psi_1\rangle = |\varphi_1\rangle \otimes |\xi_1\rangle \longrightarrow |\psi_1\rangle = \sum_{l_1=0}^{N} c_{l_1} \left| \frac{N}{2}, l_1 - \frac{N}{2} \right\rangle \otimes \sum_{k_1=0}^{N'} c_{k_1} \left| \frac{N'}{2}, k_1 - \frac{N'}{2} \right\rangle, \\
&|\psi_2\rangle = |\varphi_2\rangle \otimes |\xi_2\rangle \longrightarrow |\psi_2\rangle = \sum_{l_2=0}^{N} c_{l_2} \left| \frac{N}{2}, l_2 - \frac{N}{2} \right\rangle \otimes \sum_{k_2=0}^{N'} c_{k_2} \left| \frac{N'}{2}, k_2 - \frac{N'}{2} \right\rangle, \\
&\vdots \\
&|\psi_i\rangle = |\varphi_i\rangle \otimes |\xi_i\rangle \longrightarrow |\psi_i\rangle = \sum_{l_i=0}^{N} c_{l_i} \left| \frac{N}{2}, l_i - \frac{N}{2} \right\rangle \otimes \sum_{k_i=0}^{N'} c_{k_i} \left| \frac{N'}{2}, k_i - \frac{N'}{2} \right\rangle,
\end{aligned}$$

which allows us to use a new concept of separability, known as biseparability[13] in order to discuss criteria of separability in the case of bipartite qudit mixed states.

To exemplify this construction, we consider the qutrit-qubit matrix density and its representation in 3-partite qubit state as $(2\times 2)\times 2$. Thus using Eq. (13), we have

$$\rho_{3\times 2} = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} & \rho_{15} & \rho_{16} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} & \rho_{25} & \rho_{26} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} & \rho_{35} & \rho_{36} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} & \rho_{45} & \rho_{46} \\ \rho_{51} & \rho_{52} & \rho_{53} & \rho_{54} & \rho_{55} & \rho_{56} \\ \rho_{61} & \rho_{62} & \rho_{63} & \rho_{64} & \rho_{65} & \rho_{66} \end{pmatrix} \longrightarrow \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13}/\sqrt{2} & \rho_{13}/\sqrt{2} & \rho_{14}/\sqrt{2} & \rho_{14}/\sqrt{2} & \rho_{15} & \rho_{16} \\ \rho_{21} & \rho_{22} & \rho_{23}/\sqrt{2} & \rho_{23}/\sqrt{2} & \rho_{24}/\sqrt{2} & \rho_{24}/\sqrt{2} & \rho_{25} & \rho_{26} \\ \rho_{31}/\sqrt{2} & \rho_{32}/\sqrt{2} & \rho_{33}/2 & \rho_{33}/2 & \rho_{34}/2 & \rho_{34}/2 & \rho_{35}/\sqrt{2} & \rho_{36}/\sqrt{2} \\ \rho_{31}/\sqrt{2} & \rho_{32}/\sqrt{2} & \rho_{33}/2 & \rho_{33}/2 & \rho_{34}/2 & \rho_{34}/2 & \rho_{35}/\sqrt{2} & \rho_{36}/\sqrt{2} \\ \rho_{41}/\sqrt{2} & \rho_{42}/\sqrt{2} & \rho_{43}/2 & \rho_{43}/2 & \rho_{44}/2 & \rho_{44}/2 & \rho_{45}/\sqrt{2} & \rho_{46}/\sqrt{2} \\ \rho_{41}/\sqrt{2} & \rho_{42}/\sqrt{2} & \rho_{43}/2 & \rho_{43}/2 & \rho_{44}/2 & \rho_{44}/2 & \rho_{45}/\sqrt{2} & \rho_{46}/\sqrt{2} \\ \rho_{51} & \rho_{52} & \rho_{53}/\sqrt{2} & \rho_{53}/\sqrt{2} & \rho_{54}/\sqrt{2} & \rho_{54}/\sqrt{2} & \rho_{55} & \rho_{56} \\ \rho_{61} & \rho_{62} & \rho_{63}/\sqrt{2} & \rho_{63}/\sqrt{2} & \rho_{64}/\sqrt{2} & \rho_{64}/\sqrt{2} & \rho_{65} & \rho_{66} \end{pmatrix}. \tag{14}$$

We notice that although the size of the density matrix is increased, this does not affect the separability of the state. From a state $|\psi_{AB}\rangle$ in the form 3×2, we introduce a tripartite qubit state $|\psi_{A'B'C'}\rangle$ in the form $2\times 2\times 2$, where the two first qubits $(A'B')$ represent our qutrit (A) and the third one (C') is the qubit (B). The investigation of the separability in the multipartite qubit state leads us to introduce the concept of partial separability,[13] which will be useful to have a new criteria of separability in bipartite qudit states.

3. Partial separability of multi-partite qubit mixed-states

It is known that the characterization of entangled multi-qubit states is one of the challenging issues in quantum mechanics. In this respect, one of the important tasks is to find the criteria of separability in mixed-states and multipartite systems. Many studies have been done, but, many efforts still needed to provide fully satisfactory answers.[14–19] In general, by separability, we mean the full separability. In this section, we will consider a concept of partial separability in multipar-

tite systems that is weaker than full separability. For instance, for tripartite system ABC, the partial separability concerns the separability between the components $AC|B$, $BC|A$ and all possible partitions.[11,12,20,21]

We will start by discussing the idea of biseparability in multipartite systems. We will also discuss the case of Dicke states separability. This is based on the decomposition of Dicke states in multiqubit states possessing the exchange symmetry. By this method a density matrix of qudit systems is converted into a density matrix describing a set of qubits with permutation invariance. Such transformations will allow us to find a new criterion of separability in bipartite qudit mixed states and a new way to describe such systems.

Suppose that $\rho_{1,2,3}$ is a mixed state for 3-partite qubit state of which the standard basis is $\{\otimes_{s=1}^{3}|i_s\rangle\}$ $(i_s=0,1)$. There are three ways to have two partitions in this system, $(1||2,3)$, $(1,2||3)$, and $(i2||1,3)$. For a pure state:

$$|\psi\rangle = a|000\rangle + b|001\rangle + c|010\rangle + d|011\rangle + e|100\rangle + f|101\rangle + g|110\rangle + h|111\rangle. \tag{15}$$

The bi-separability associated to the partition $1||2,3$ can be expressed as follows:

$$|\psi\rangle = \underbrace{\left[a|0\rangle\left(|00\rangle + \frac{d}{a}|11\rangle\right) + e|1\rangle\left(|00\rangle + \frac{h}{e}|11\rangle\right)\right]}_{A} + \underbrace{\left[b|0\rangle\left(|01\rangle + \frac{c}{b}|10\rangle\right) + f|1\rangle\left(|01\rangle + \frac{g}{f}|10\rangle\right)\right]}_{B}. \tag{16}$$

Through this method of factorization, we decompose the system in two parts A and B as can be seen above. In part A, the states $|00\rangle$ and $|11\rangle$ can be considered as a basis states for one qubit. The same concept can be applied to the states $|01\rangle$ and $|10\rangle$ in part B, which allows us to consider the two parts A and B as a system of $2\otimes 2$ dimensions. If the state $|\psi\rangle$ is $(1||2,3)$-separable, the two parts should be separable. So we can then define a necessary criterion of bi-separability, which imposes the separability on all parts of dimension 4 that can be factorizable into a system of $2\otimes 2$ dimension. In the case of 4-partite qubit state, the subsystems that can be $(1,2||3,4)$-factorizable are:

$$\begin{aligned}
A &= \{|00,00\rangle, |00,11\rangle, |11,00\rangle, |11,11\rangle\},\\
B &= \{|00,01\rangle, |00,10\rangle, |11,01\rangle, |11,10\rangle\}\\
C &= \{|01,00\rangle, |01,11\rangle, |10,00\rangle, |10,11\rangle\},\\
D &= \{|01,01\rangle, |01,10\rangle, |10,01\rangle, |10,10\rangle\}.
\end{aligned}$$

And for $(1||2,3,4)$-partition, the subsystems are

$$\begin{aligned}
A &= \{|0,111\rangle, |0,000\rangle, |1,000\rangle, |1,111\rangle\},\\
B &= \{|0,001\rangle, |0,110\rangle, |1,001\rangle, |1,110\rangle\},\\
C &= \{|0,101\rangle, |0,010\rangle, |1,101\rangle, |1,010\rangle\},\\
D &= \{|0,011\rangle, |0,100\rangle, |1,011\rangle, |1,100\rangle\}.
\end{aligned}$$

Each part is associated with a reduced density matrix which can be extracted from the density matrix of the whole system. There is a simple way to find the reduced density matrix for all parts.[13] For example, in the case of a 4-partite qubit. The reduced density matrices associated to the $(1,2||3,4)$-partition can be defined as follows:

$$\begin{aligned}
&\{(uu,vv;\mu\mu,\nu\nu)\},\\
&\{(u(1-u),vv;\mu(1-\mu),\nu\nu)\},\\
&\{(uu,v(1-v);\mu\mu,\nu(1-\nu))\},\\
&\{(u(1-u),v(1-v);\mu(1-\mu),\nu(1-\nu))\}
\end{aligned}$$

with u, v, μ, $\nu = 0,1$. For example, the set of elements $\{(uu,vv;\mu\mu,\nu\nu)\}$ are the matrix elements

$$\{|uu,vv\rangle\langle\mu\mu,\nu\nu|\}.$$

To have all the reduced density matrix associated with each part, we shall browse all the cases. Elements with equal index (uu) and different index $(u(1-u))$, as illustrated below

$$\begin{aligned}
A &= \rho^{uu,vv}_{\mu\mu,\nu\nu}, \qquad & B &= \rho^{u(1-u),vv}_{\mu(1-\mu),\nu\nu}m,\\
C &= \rho^{uu,v(1-v)}_{\mu\mu,\nu(1-\nu)}, & D &= \rho^{u(1-u),v(1-v)}_{\mu(1-\mu),\nu(1-\nu)}.
\end{aligned}$$

To further clarify this method we will take another example: the case of $(1||2,3,4)$ partition. The reduced density matrix associated with each part will be defined as follows: $\rho^{u,vvv}_{\mu,\nu\nu\nu}$, $\rho^{u,(1-v)vv}_{\mu,(1-\nu)\nu\nu}$, $\rho^{u,v(1-v)v}_{\mu,\nu(1-\nu)\nu}$, $\rho^{u,vv(1-v)}_{\mu,\nu\nu(1-\nu)}$. The system can be reduced into several density matrices of bipartite qubit state that should be separable as a necessary condition in order to have the biseparability of a given partition. We can use several criteria to have separability on each reduced density matrices. In the first time, we will use the PPT criterion for its simplicity.

4. Separability of bipartite qudit mixed-states

The PPT criterion is a necessary condition for a given density matrix ρ_{AB} describing two quantum systems A and B, to be separable. This criterion is a sufficient condition only in

the 2×2 and 2×3 bipartite systems. The PPT consists to apply the operator $\mathbb{1}_A\otimes T_B$, where T_B is the transposition on the second subsystem of the compound system $S_A\otimes S_B$, therefore if the eigenvalues of $\rho_{AB}^{T_B}=(\mathbb{1}_A\otimes T_B)\rho_{AB}$ is nonpositive, the state ρ_{AB} is entangled.

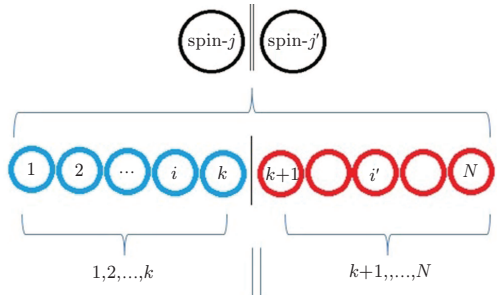


Fig. 2. The representation of a bipartite qudit state as a N-symmetric state.

The main idea of the present paper is to establish a criterion of separability in bipartite qudit states by decomposing each qudit into multipartite qubit state, as we can see in Fig. 2, and to verify the separability criterion for each qubit $(i=1,...,k)$ of the first qudit with all the qubits $(i'=k+1,...,N)$ of the second qudit, which can be a necessary and sufficient criterion.

In this section we will apply the criterion of the biseparability on a bipartite qudit mixed state on its Majorana representation (an N-partite qubit mixed state) as we have seen before, it can preserve the separability. This kind of representation will allow us to describe a quantum system by the basic element of information (qubit), even if such a representation which converts a bipartite qudit state into a multipartite qubit state can include entanglement which affects the entanglement measure of the whole system, but in our case we consider a bipartite separable state $|\psi\rangle=|\varphi\rangle\otimes|\xi\rangle$ which can be converted into

$$|\psi\rangle=\sum_{i=0}^{d-1}c_i\frac{1}{\sqrt{C_i^d}}(|\underbrace{0,0\ldots,0}_{i},\underbrace{1,1\ldots,1}_{(d-1)-i}\rangle+\text{permutations})\otimes\sum_{i=0}^{d'-1}c_i'\frac{1}{\sqrt{C_i^{d'}}}(|\underbrace{0,0\ldots,0}_{i},\underbrace{1,1\ldots,1}_{(d'-1)-i}\rangle+\text{permutations}),$$

each subsystems has been rewritten in the Dicke basis. Without using any unitary transformations $\underbrace{SU(2)\otimes SU(2)\otimes\cdots\otimes SU(2)}_{d-1}$ the state still is biseparable as we showed in Section 3. Now if the separability is preserved after this transformation, we can apply some criteria of biseparability on our state as a necessary condition to have the separability in a bipartite qudit mixed state.

For a given bipartite qudit mixed state of dimension $d\times d'$, if the bipartite state is non-entangled, its representation in a multi-partite qubit state must be biseparable between the set of $d-1$ first qubits and the rest of qubits. The reduced bipartite qubit state $\rho_{1,2,\ldots,d-1||1',2',\ldots,d'-1}$ in particular satisfies the PPT condition, and if this reduced density matrix ρ violates the PPT condition then it is $(1,2,\ldots,d-1||1',2',\ldots,d'-1)$-inseparable and the bipartite qudit mixed state is entangled.

Example 1 To illustrate this method, we consider the case of qutrit–qubit system in mixed X-state, the case where the PPT is a necessary and sufficient condition. The matrix density ρ describing this system is 6×6 matrix which will become an 8×8 matrix in the representation by multi-partite qubit state

$$\rho_{3\times 2}=\begin{pmatrix}\rho_{11}&0&0&0&0&\rho_{16}\\0&\rho_{22}&0&0&\rho_{25}&0\\0&0&\rho_{33}&\rho_{34}&0&0\\0&0&\rho_{43}&\rho_{44}&0&0\\0&\rho_{52}&0&0&\rho_{55}&0\\\rho_{61}&0&0&0&0&\rho_{66}\end{pmatrix}\longrightarrow\begin{pmatrix}\rho_{11}&0&0&0&0&0&0&\rho_{16}\\0&\rho_{22}&0&0&0&0&\rho_{25}&0\\0&0&\rho_{33}/2&\rho_{33}/2&\rho_{34}/2&\rho_{34}/2&0&0\\0&0&\rho_{33}/2&\rho_{33}/2&\rho_{34}/2&\rho_{34}/2&0&0\\0&0&\rho_{43}/2&\rho_{43}/2&\rho_{44}/2&\rho_{44}/2&0&0\\0&0&\rho_{43}/2&\rho_{43}/2&\rho_{44}/2&\rho_{44}/2&0&0\\0&\rho_{52}&0&0&0&0&\rho_{55}&0\\\rho_{61}&0&0&0&0&0&0&\rho_{66}\end{pmatrix}.\quad(17)$$

The two first qubits in the multi-partite qubit state describe the qutrit, and the third qubit describes the second qubit in the bipartite qudit state. The partition is defined as $\rho_{(1,2,3)}\longrightarrow\rho_{(1,2||3)}$.

The sub-matrix associated with these conditions are: $\rho_{\mu\mu,\nu}^{uu,v}$ and $\rho_{\mu(1-\mu),\nu}^{u(1-u),v}$ avec $\mu,\nu,u,v=\{0,1\}$

$$\rho_{\mu\mu,\nu}^{uu,v}=\begin{pmatrix}[\rho]_{000,000}&[\rho]_{001,000}&[\rho]_{110,000}&[\rho]_{111,000}\\[\rho]_{000,001}&[\rho]_{001,001}&[\rho]_{110,001}&[\rho]_{111,001}\\[\rho]_{000,110}&[\rho]_{001,110}&[\rho]_{110,110}&[\rho]_{111,110}\\[\rho]_{000,111}&[\rho]_{001,111}&[\rho]_{110,111}&[\rho]_{111,111}\end{pmatrix}=\begin{pmatrix}\rho_{11}&0&0&\rho_{16}\\0&\rho_{22}&\rho_{25}&0\\0&\rho_{52}&\rho_{55}&0\\\rho_{61}&0&0&\rho_{66}\end{pmatrix},\quad(18)$$

$$\rho^{u(1-u),v}_{\mu(1-\mu),\nu} = \begin{pmatrix} [\rho]_{010,010} & [\rho]_{011,010} & [\rho]_{100,010} & [\rho]_{101,010} \\ [\rho]_{010,011} & [\rho]_{011,001} & [\rho]_{100,011} & [\rho]_{101,011} \\ [\rho]_{010,100} & [\rho]_{011,100} & [\rho]_{100,100} & [\rho]_{101,100} \\ [\rho]_{010,101} & [\rho]_{011,101} & [\rho]_{100,101} & [\rho]_{101,101} \end{pmatrix} = \begin{pmatrix} \rho_{33}/2 & \rho_{33}/2 & \rho_{34}/2 & \rho_{34}/2 \\ \rho_{33}/2 & \rho_{33}/2 & \rho_{34}/2 & \rho_{34}/2 \\ \rho_{43}/2 & \rho_{43}/2 & \rho_{44}/2 & \rho_{44}/2 \\ \rho_{43}/2 & \rho_{43}/2 & \rho_{44}/2 & \rho_{44}/2 \end{pmatrix}. \tag{19}$$

The reduced density matrix associated with this partition $(1,2||3)$

$$\rho_{(1,2||3)} = \left(\begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{16} \\ 0 & \rho_{22} & \rho_{25} & 0 \\ 0 & \rho_{52} & \rho_{55} & 0 \\ \rho_{61} & 0 & 0 & \rho_{66} \end{pmatrix}, \begin{pmatrix} \rho_{33}/2 & \rho_{33}/2 & \rho_{34}/2 & \rho_{34}/2 \\ \rho_{33}/2 & \rho_{33}/2 & \rho_{34}/2 & \rho_{34}/2 \\ \rho_{43}/2 & \rho_{43}/2 & \rho_{44}/2 & \rho_{44}/2 \\ \rho_{43}/2 & \rho_{43}/2 & \rho_{44}/2 & \rho_{44}/2 \end{pmatrix} \right). \tag{20}$$

If the density matrix is bi-separable under the $(1,2||3)$-partition, the reduced matrices associated should satisfy the PPT condition.

The eigenvalues of $(1\!\!1_A \otimes T_B)[\rho]_{i(i-1)j,u(u-1)v}$ and $(1\!\!1_A \otimes T_B)\rho^{uu,v}_{\mu\mu,\nu}$ must still positive under this map.

The eigenvalues associated to the sub-matrix $\rho^{uu,v}_{\mu\mu,\nu}$ under the partial transpose:

$$\begin{aligned} \lambda_1 &= \frac{1}{2}(\rho_{55}+\rho_{22}-\sqrt{\rho_{55}^2+4\rho_{61}\rho_{16}-2\rho_{55}\rho_{22}+\rho_{22}^2}, \\ \lambda_2 &= \frac{1}{2}(\rho_{55}+\rho_{22}+\sqrt{\rho_{55}^2+4\rho_{61}\rho_{16}-2\rho_{55}\rho_{22}+\rho_{22}^2}), \\ \lambda_3 &= \frac{1}{2}(\rho_{11}+\rho_{66}-\sqrt{\rho_{11}^2-2\rho_{11}\rho_{66}+\rho_{66}^2+4\rho_{25}\rho_{52}}), \\ \lambda_4 &= \frac{1}{2}(\rho_{11}+\rho_{66}+\sqrt{\rho_{11}^2-2\rho_{11}\rho_{66}+\rho_{66}^2+4\rho_{25}\rho_{52}}). \end{aligned}$$

The eigenvalues associated to the sub-matrix $\rho^{u(1-u),v}_{\mu(1-\mu),\nu}$ under the partial transpose:

$$\begin{aligned} &\lambda'_1 = 0, \quad \lambda'_2 = 0, \\ &\lambda'_3 = \frac{1}{2}\left(\frac{\rho_{33}}{2}+\frac{\rho_{44}}{2}-\sqrt{\frac{\rho_{33}}{2}^2-2\frac{\rho_{34}}{2}\frac{\rho_{43}}{2}+\frac{\rho_{44}}{2}^2+4\frac{\rho_{33}}{2}\frac{\rho_{44}}{2}}\right), \\ &\lambda'_4 = \frac{1}{2}\left(\frac{\rho_{33}}{2}+\frac{\rho_{44}}{2}+\sqrt{\frac{\rho_{33}}{2}^2-2\frac{\rho_{34}}{2}\frac{\rho_{43}}{2}+\frac{\rho_{44}}{2}^2+4\frac{\rho_{33}}{2}\frac{\rho_{44}}{2}}\right), \end{aligned}$$

all the eigenvalues must be non-negative $\lambda_i, \lambda'_i \geq 0$, $i = \{1,2,3,4\}$, in particular λ_1, λ_3, and λ'_3 must be positive. If we compare it with the PPT criterion on the density matrix of the bipartite qudit state $\rho_{3\times 2}$. The eigenvalues associated with this matrix under the partial transpose.

$$\begin{aligned} \lambda''_1 &= \frac{1}{2}\left(\frac{\rho_{33}}{2}+\frac{\rho_{44}}{2}-\sqrt{\frac{\rho_{33}}{2}^2-2\frac{\rho_{34}}{2}\frac{\rho_{43}}{2}+\frac{\rho_{44}}{2}^2+4\frac{\rho_{33}}{2}\frac{\rho_{44}}{2}}\right), \\ \lambda''_2 &= \frac{1}{2}\left(\rho_{55}+\rho_{22}-\sqrt{\rho_{55}^2+4\rho_{61}\rho_{16}-2\rho_{55}\rho_{22}+\rho_{22}^2}\right), \\ \lambda''_3 &= \frac{1}{2}\left(\rho_{11}+\rho_{66}-\sqrt{\rho_{11}^2-2\rho_{11}\rho_{66}+\rho_{66}^2+4\rho_{25}\rho_{52}}\right), \\ \lambda''_4 &= \frac{1}{2}\left(\frac{\rho_{33}}{2}+\frac{\rho_{44}}{2}+\sqrt{\frac{\rho_{33}}{2}^2-2\frac{\rho_{34}}{2}\frac{\rho_{43}}{2}+\frac{\rho_{44}}{2}^2+4\frac{\rho_{33}}{2}\frac{\rho_{44}}{2}}\right), \\ \lambda''_5 &= \frac{1}{2}\left(\rho_{55}+\rho_{22}+\sqrt{\rho_{55}^2+4\rho_{61}\rho_{16}-2\rho_{55}\rho_{22}+\rho_{22}^2}\right), \\ \lambda''_6 &= \frac{1}{2}\left(\rho_{11}+\rho_{66}+\sqrt{\rho_{11}^2-2\rho_{11}\rho_{66}+\rho_{66}^2+4\rho_{25}\rho_{52}}\right). \end{aligned}$$

If the state is separable, λ''_1, λ''_2, and λ''_3 must be positive, we note that we have the same conditions for separability in the two cases.

5. Criteria of separability using concurrence

In this section we will try to define a necessary criterion for separability of pure and mixed bipartite state by using the

notion of concurrence[22] instead of PPT criterion. In order to have the separability in the case of pure qubit–qutrit state, we will factorize our state after its decomposition as a product of qubits $|\psi\rangle_{1,2,3} = c_1|000\rangle + c_2|001\rangle + c_3|010\rangle + c_4|011\rangle + c_5|100\rangle + c_6|101\rangle + c_7|110\rangle + c_8|111\rangle$, the first qubit and the two second ones are respectively the qubit and the qutrit of the bipartite qudit state. The factorization can be done as follows:

$$|\psi\rangle_{1||2,3} = c_1|0\rangle\left(|00\rangle + \frac{c_2}{c_1}|01\rangle + \frac{c_3}{c_1}|10\rangle + \frac{c_4}{c_1}|11\rangle\right) + c_5|1\rangle\left(|00\rangle + \frac{c_6}{c_5}|01\rangle + \frac{c_7}{c_5}|10\rangle + \frac{c_8}{c_5}|11\rangle\right).$$

To have the separability, the coefficients should satisfy these conditions,

$$\frac{c_2}{c_1} = \frac{c_6}{c_5},\ \frac{c_3}{c_1} = \frac{c_7}{c_5},\ \frac{c_4}{c_1} = \frac{c_8}{c_5}. \tag{21}$$

There are different ways to factorize this state, however, if the state is separable it should be factorizable in all factorization ways. The conditions (21) are a necessary criteria of separability which can be rewritten as the concurrence measure associated to all parts of the form qubit–qubit in the system, $c_1c_6 - c_2c_5 = 0$, $c_3c_5 - c_1c_7 = 0$, $c_4c_5 - c_1c_8 = 0$. This method decompose the tripartite qubit state as a set of three subsystems of dimension 2×2. In the case of pure states, the separability of the three subsystems can be seen as a necessary criterion of the biseparability $(1||2,3)$ of the whole system.

Using the formalism of density matrices and in correspondence to bipartite qubit states we can have the concurrence as $C(\rho) = \max\{0, \lambda_1^2 - \lambda_2^2 - \lambda_3^2 - \lambda_4^2\}$ in which $\lambda_1, \ldots, \lambda_4$ are the eigenvalues in decreasing order, of $R = \rho\tilde{\rho}$ with $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$. In the case of pure states $|\psi\rangle = c_1|00\rangle + c_2|01\rangle + c_3|10\rangle + c_4|11\rangle$, with $\rho = |\psi\rangle\langle\psi|$, the concurrence becomes $C(\rho) = 2|c_1c_4 - c_2c_3|$ which can be seen as a projection of the spin flipped state $|\tilde{\psi}\rangle$ on the state $|\psi\rangle$

$$|c_1c_4 - c_2c_3| = \left|\begin{pmatrix} c_1 & c_2 & c_3 & c_4 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}\right|. \tag{22}$$

In correspondence of bipartite qubit states, we can do a partial spin flip on a qubit in a multipartite state in order to have the conditions in Eq. (21)

$$|c_1c_8 - c_4c_5| = \left|\begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix}\right|. \tag{23}$$

We can do a spin flip with a specific matrices to have all the conditions in Eq. (21) using for example these two matrices (23) as a spin flip on the state $|\psi\rangle$ we can reproduce respectively the conditions $|c_1c_6 - c_2c_5|$ and $|c_1c_7 - c_3c_5|$. If the state is separable all conditions must be equal to zero as a necessary criterion of separability

$$\left(\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right). \tag{24}$$

Now if we try to find all the ways to factorize our state, we can be sure to have a sufficient and necessary criterion of separability which will be very useful in the sense to find a new measure of entanglement in bipartite qudit systems. Let us call the matrices which can do a partial spin flip by ξ_i, each one corresponds to a condition as in Eq. (21) for pure states. In correspondence to the concurrence in the mixed case of bipartite qubit state, we can rewrite all these conditions for mixed states as follows:

$$C(\rho_{\xi_i}) = \max\{0, \lambda_{1_i}^2 - \lambda_{2_i}^2 - \lambda_{3_i}^2 - \lambda_{4_i}^2\} \tag{25}$$

with $\lambda_{1_i}, \ldots, \lambda_{4_i}$ being the eigenvalues associated to the matrix $R = \rho\tilde{\rho}$ and $\tilde{\rho}$ being the spin-flipped density matrix using ξ_i. The set of all conditions in Eq. (25) $\{C(\rho_{\xi_i})\}$ forms a criteria of separability, if the state is separable implies that $C(\rho_{\xi_i}) = 0$. There are other ways to factorize $|\psi\rangle$ which can be reached using other paths of factorization as follows:

$$|\psi\rangle = c_2|0\rangle\left(\frac{c_1}{c_2}|00\rangle + |01\rangle + \frac{c_3}{c_2}|10\rangle + \frac{c_4}{c_2}|11\rangle\right) + c_6|1\rangle\left(\frac{c_5}{c_6}|00\rangle + |01\rangle + \frac{c_7}{c_6}|10\rangle + \frac{c_8}{c_6}|11\rangle\right), \tag{26}$$

we find other conditions using the above factorization

$$\frac{c_1}{c_2} = \frac{c_5}{c_6},\quad \frac{c_3}{c_2} = \frac{c_7}{c_6},\quad \frac{c_4}{c_2} = \frac{c_8}{c_6},$$

and *vice versa*, we can look over all the ways of factorization in order to have all the conditions which will form a necessary and sufficient criterion of separability. In this case, (2×3)-dimensional states, all the conditions of separability can be presented as follows:

$$\begin{aligned} &c_1c_6-c_2c_5=0,\ \ c_3c_5-c_1c_7=0,\ \ c_4c_5-c_1c_8=0,\\ &c_3c_6-c_2c_7=0,\ \ c_4c_6-c_2c_8=0,\ \ c_4c_7-c_3c_8=0. \end{aligned} \tag{27}$$

In the (2×3)-dimensional case, the set of conditions in Eq. (27) represent a sufficient and necessary criteria of separability, on which we can find in literature other necessary and sufficient criteria as PPT.[7] Now if we want to have a sufficient criteria for all dimensional cases, we must find all factorization ways. Let us take the example of bipartit qutrit state 3×3 which can be seen as a four-partite qubit state $|\psi_{ABCD}\rangle = c_1|0000\rangle + c_2|0001\rangle + \cdots + c_{16}|1111\rangle$. The state can be factorized in different ways

$$\begin{aligned} |\psi_{AB-CD}\rangle = c_1|00\rangle&\left(|00\rangle+\frac{c_2}{c_1}|01\rangle+\frac{c_3}{c_1}|10\rangle+\frac{c_4}{c_1}|11\rangle\right)\\ &+c_5|01\rangle\left(|00\rangle+\frac{c_6}{c_5}|01\rangle+\frac{c_7}{c_5}|10\rangle+\frac{c_8}{c_5}|11\rangle\right)\\ &+c_9|10\rangle\left(|00\rangle+\frac{c_{10}}{c_9}|01\rangle+\frac{c_{11}}{c_9}|10\rangle+\frac{c_{12}}{c_9}|11\rangle\right)\\ &+c_{13}|11\rangle\left(|00\rangle+\frac{c_{14}}{c_{13}}|01\rangle+\frac{c_{15}}{c_{13}}|10\rangle+\frac{c_{16}}{c_{13}}|11\rangle\right). \end{aligned}$$

To have a complete factorization, the coefficients must satisfy some conditions.

$$\begin{aligned} &\frac{c_2}{c_1}=\frac{c_6}{c_5}=\frac{c_{10}}{c_9}=\frac{c_{14}}{c_{13}},\\ &\frac{c_3}{c_1}=\frac{c_7}{c_5}=\frac{c_{11}}{c_9}=\frac{c_{15}}{c_{13}},\\ &\frac{c_4}{c_1}=\frac{c_8}{c_5}=\frac{c_{12}}{c_9}=\frac{c_{16}}{c_{13}}. \end{aligned} \tag{28}$$

The set of conditions in Eq. (28) forms a necessary criterion of separability, and *vice versa*, we can browse all the factorization ways in order to have a necessary and sufficient criterion.

6. Conclusion

This work gives a detailed description of the Majorana representation in case of bipartite qudit state that we can write it in the form of symmetric multiqubit state. This representation allows us to have a new approach to discriminate the existence of the entanglement in our case and have a new criterion of separability exploiting the notion of partial separability. Especially, the biseparability which can work as a good criterion in a qudit–qudit state using PPT and Wootters concurrence as we have seen in the example of qutrit–qubit X-state. This method still has a problem when we use the unitary transformation because the Majorana representation may create entanglement either in one of the subsystem or both. In the general case of any non-unitary transformation, this method works without any issue.

The separability criteria are one of the missed pieces of the entanglement quantification problem. Investigation in this sense can lead us to a good understanding of entanglement and a general formulation to quantify such kind of quantum correlations.

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