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Reducing the calculation workload of the Green function for electromagnetic scattering in a Schwarzschild gravitational field*

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When the finite difference time domain (FDTD) method is used to solve electromagnetic scattering problems in Schwarzschild space-time, the Green functions linking source/observer to every surface element on connection/output boundary must be calculated. When the scatterer is electrically extended, a huge amount of calculation is required due to a large number of surface elements on the connection/output boundary. In this paper, a method for reducing the calculation workload of Green function is proposed. The Taylor approximation is applied for the calculation of Green function. New transport equations are deduced. The numerical results verify the effectiveness of this method.

Keywords: Green function, Schwarzschild space-time, electromagnetic scattering, finite difference time domain

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1. Introduction

It is important to understand the property of the electromagnetic waves in gravitational filed[1–6] for electromagnetic-wave-based communication between spacecraft in deep space. Numerical simulations of electromagnetic waves in curved space-time have been exploited by researchers. The electromagnetic particle-in-cell (EMPIC) algorithm is used to study the physics of high-frequency electromagnetic waves in a general relativistic plasma with the Schwarzschild metric.[7] The Kerr–Schild metric is incorporated into the EMPIC code for the simulation of charged particles in the region of a spinning black hole.[8] Finite difference time domain (FDTD) method and Green function method (GFM) were extended to Schwarzschild space-time in Ref. [9]. They were applied to electromagnetic scattering problems (not by black holes) in Ref. [10].

When solving electromagnetic scattering problems (not by black holes) in Schwarzschild space-time by FDTD and GFM, which have been introduced in Refs. [9] and [10], all of the Green functions linking source/observer to every surface element on connection/output boundary should be calculated (Fig. 1(a)). For an electrically extended scatterer, a large amount of calculation is required due to the huge number of surface elements on the connection/output boundary. A method to reduce the amount of calculation of Green function is proposed in this paper. The Taylor approximation is applied for the calculation of Green function in Section 2.

The main computational cost of solving the Green function is to solve differential equations numerically. After using the Taylor approximation, only the value of Green function at the center of the region needs to be numerically solved, while the value of the Green function at other points can be obtained directly by Taylor approximation, without the need to numerically solve the differential equation, thereby greatly reducing the amount of computation.

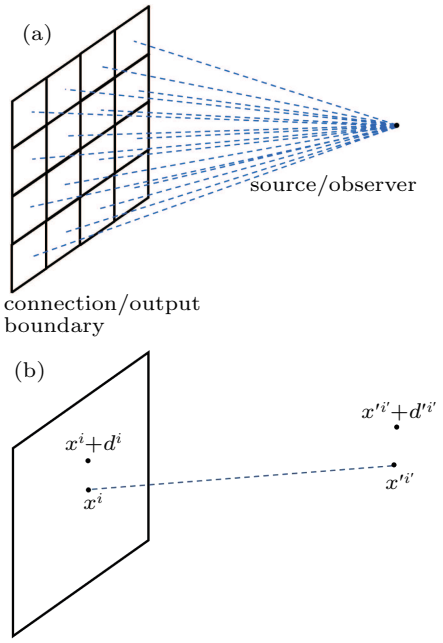


Fig. 1. Green function linking source/observer to connection/output boundary. (a) Before approximation; (b) after approximation.

The classical Runge–Kutta method is applied to solve differential equations numerically. The complexity is $O(n)$, where n is the number of grids. The method proposed in this paper does not reduces the complexity but also reduces the scale of the problem.

Green function's partial derivatives should be calculated

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before using the Taylor approximation. The partial derivatives can be expressed with covariant derivatives. These quantities cannot be calculated using the old version program. The transport equations for these quantities are deduced in Section 3. The numerical examples in Section 4 verify the effectiveness of this method.

The symbols in this paper are the same as those in Refs. [9] and [10].

2. Taylor approximation for the Green function

The Green function in curved space-time is[11–14]

$$G_{\alpha\beta'}(x,x') = U_{\alpha\beta'}(x,x')\delta(\sigma(x,x')) + V_{\alpha\beta'}(x,x')\theta(-\sigma(x,x')). \quad (1)$$

In the above expression, the bi-scalar $\sigma(x,x')$ is Synge's world function[15] which is defined as half the square geodesic distance between the points x and x', $U_{\alpha\beta'}$ and $V_{\alpha\beta'}$ are two bitensors, δ is Dirac function, and θ is the step function

$$\theta(\sigma) = \begin{cases} 1, & \sigma \geq 0, \\ 0, & \sigma < 0. \end{cases} \quad (2)$$

Generally, the source and observer are far away from the scatterer, so the Green function linking source/observer to a point on connection/output boundary changes only slightly spatially. Therefore, only the Green function linking source/observer to the midpoint on connection/output boundary needs to be calculated, and the Green function linking source/observer to the other points on connection/output boundary can be obtained with Taylor approximation. By taking $U_{\alpha\beta'}$ as an example, the following expression can be derived:

$$U_{\alpha\beta'}(x+d,x'+d') \approx U_{\alpha\beta'}(x,x') + U_{\alpha\beta',\gamma}(x,x')d^{\gamma} + U_{\alpha\beta',\gamma'}(x,x')d'^{\gamma'}, \quad (3)$$

where x' denotes the coordinate of the source/observer, $x'+d'$ denotes the coordinate of a point near x', x denotes the coordinate of the midpoint of connection/output boundary, and $x+d$ denotes the coordinate of a point near x (Fig. 1(b)). The spatial coordinates $d^i (i=1,2,3)$ and $d'^{i'} (i'=1,2,3)$ are known. However, the temporal coordinates $d'^{0'}$ and d^0 are unknown. The Schwarzschild space-time is a static space-time, in which time interval is irrelevant to the starting point of time. Accordingly, it can be set $d'^{0'}=0$. The temporal coordinate d^0 can be acquired in the following way.

By taking Synge's world function σ as a Taylor expansion, the following expression can be derived:

$$\sigma(x+d,x'+d') = \sigma(x,x') + \sigma_{\alpha}(x,x')d^{\alpha} + \sigma_{\alpha'}(x,x')d'^{\alpha'} + Q + \cdots, \quad (4)$$

where Q is quadratic term

$$Q = \frac{1}{2}\sigma_{\alpha,\beta}(x,x')d^{\alpha}d^{\beta} + \sigma_{\alpha,\beta'}(x,x')d^{\alpha}d'^{\beta'} + \frac{1}{2}\sigma_{\alpha',\beta'}(x,x')d'^{\alpha'}d'^{\beta'}. \quad (5)$$

The Synge's world function of the null geodesic is equal to zero; i.e., $\sigma(x+d,x'+d') = \sigma(x,x') = 0$. By ignoring higher order terms in Eq. (4), the following equation is acquired:

$$\sigma_{\alpha}(x,x')d^{\alpha} + \sigma_{\alpha'}(x,x')d'^{\alpha'} = 0. \quad (6)$$

By substituting $d'^{0'}=0$ into the above equation, the expression of d^0 is acquired

$$d^0 = -\frac{1}{\sigma_0(x,x')}[\sigma_i(x,x')d^i + \sigma_{i'}(x,x')d'^{i'}], \quad (7)$$

where the indices i and i' vary from 1 to 3.

The approximation error can be evaluated by the quadratic term Q in Eq. (4). Let us set $d'=0$ and $d^i = d^I\delta^i{}_I$, by substituting Eqs. (5) and (7) into inequation

$$|Q| < \varepsilon, \quad (8)$$

we obtain

$$|d^i| < d^i_{\max}(\varepsilon) = \sqrt{\frac{\varepsilon}{\left|\frac{\sigma_{0,0}}{2}\left(\frac{\sigma_i}{\sigma_0}\right)^2 + \frac{1}{2}\sigma_{i,i} - \frac{\sigma_i\sigma_{0,i}}{\sigma_0}\right|}}, \quad i=1,2,3. \quad (9)$$

For a specified approximation error ε, the permissible spatial range can be acquired by the above formula. The connection/output boundary can be divided into subareas based on $d^i_{\max}(i=1,2,3)$ as shown in Fig. 2. The Green function linking source/observer to the midpoint of these subareas can then be calculated, while the Green function linking source/observer to the other points on connection/output boundary can be obtained by Taylor approximation.

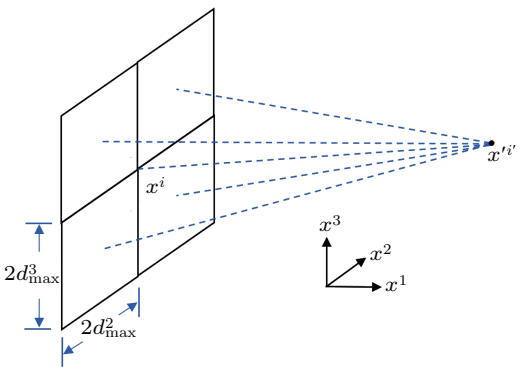


Fig. 2. Subareas on connection/output boundary.

3. Transport equations

Using Green function method, eight quantities should be calculated (Eq. (57) in Ref. [9])

$$\sigma_{\alpha},\sigma_{0'},\sigma_{\alpha,0'},\sigma_{0',0'},U_{\alpha\beta'},U_{\alpha\beta';\gamma},U_{\alpha\beta',0'},V^{0}_{\alpha\beta'}. \tag{10}$$

Their partial derivative on x and x' should be calculated before using Taylor approximation. The partial derivative can be expressed with covariant derivative. Taking $U_{\alpha\beta',0',\gamma}$ as an example, it can be calculated as

$$\begin{aligned} U_{\alpha\beta',0',\gamma} =\ & U_{\alpha\beta';0'\gamma}+(U_{\alpha\lambda';\gamma}+\Gamma^{\rho}{}_{\alpha\gamma}U_{\rho\lambda'})\Gamma^{\lambda'}{}_{\beta'0'} \\ & +\Gamma^{\lambda}{}_{\alpha\gamma}U_{\lambda\beta';0'}. \end{aligned} \tag{11}$$

In the above equation, $U_{\alpha\beta';0'\gamma}$ cannot be calculated using the old version program. The quantities that cannot be calculated with the old version program include

$$\begin{aligned} &\sigma^{\alpha}{}_{\beta'\gamma'},\sigma^{\alpha}{}_{\beta\gamma'},\sigma^{\alpha}{}_{\beta\gamma'\delta'},\sigma^{\alpha}{}_{\beta\gamma\delta'},\sigma^{\alpha}{}_{\beta\gamma\delta\varepsilon'},\sigma^{\alpha}{}_{\beta\gamma\delta\varepsilon}, \\ &U_{\alpha\beta';\gamma'\delta},U_{\alpha\beta';\gamma'\delta'},U_{\alpha\beta';\gamma\delta\varepsilon'},U_{\alpha\beta';\gamma\delta\varepsilon},V^{0}_{\alpha\beta';\gamma'},V^{0}_{\alpha\beta';\gamma'}. \end{aligned} \tag{12}$$

Their transport equations are deduced as below.

3.1. Transport equation for $\sigma^{\alpha}{}_{\beta'\gamma'}$

Equation (59) in Ref. [9] can be rewritten as

$$\sigma^{\alpha}{}_{\beta'\mu}\sigma^{\mu}+\left(\sigma^{\alpha}{}_{\mu}-\delta^{\alpha}{}_{\mu}\right)\sigma^{\mu}{}_{\beta'}=0. \tag{13}$$

Differentiating the above equation with respect to x', we obtain

$$\begin{aligned} &\sigma^{\alpha}{}_{\beta'\gamma\mu}\sigma^{\mu}+\sigma^{\alpha}{}_{\beta'\mu}\sigma^{\mu}{}_{\gamma}+\sigma^{\alpha}{}_{\mu\gamma}\sigma^{\mu}{}_{\beta'} \\ &+\left(\sigma^{\alpha}{}_{\mu}-\delta^{\alpha}{}_{\mu}\right)\sigma^{\mu}{}_{\beta'\gamma}=0. \end{aligned} \tag{14}$$

The initial value problem for $\sigma^{\alpha}{}_{\beta'\gamma'}$ can be acquired from the above equation

$$\begin{aligned} &\frac{\mathrm{d}\sigma^{\alpha}{}_{\beta'\gamma'}}{\mathrm{d}t}+\Gamma^{\alpha}{}_{\lambda\mu}\sigma^{\lambda}{}_{\beta'\gamma'}u^{\mu} \\ &+\frac{1}{t}\left[\sigma^{\alpha}{}_{\mu\beta'}\sigma^{\mu}{}_{\gamma'}+\sigma^{\alpha}{}_{\mu\gamma'}\sigma^{\mu}{}_{\beta'}+\left(\sigma^{\alpha}{}_{\mu}-\delta^{\alpha}{}_{\mu}\right)\sigma^{\mu}{}_{\beta'\gamma'}\right]=0, \\ &\sigma^{\alpha}{}_{\beta'\gamma'}\big|_{t=0}=0. \end{aligned} \tag{15}$$

We can obtain the limit formula

$$\begin{aligned} \left(\frac{\sigma^{\alpha}{}_{\beta\gamma'}}{t}\right)_{t\to0} &= \left[\sigma^{\alpha}{}_{\beta\gamma'\mu}\right]u^{\mu'} \\ &= \frac{1}{3}\left(R^{\alpha'}{}_{\gamma'\beta'\mu'}+R^{\alpha'}{}_{\mu'\beta'\gamma'}\right)u^{\mu'} \end{aligned} \tag{16}$$

from the covariant expansion of $\sigma^{\alpha}{}_{\beta\gamma'}$

$$\sigma^{\alpha}{}_{\beta\gamma'}=g^{\alpha}{}_{\alpha'}g^{\beta'}{}_{\beta}\left\{\left[\sigma^{\alpha}{}_{\beta\gamma'\mu}\right]tu^{\mu'}+O\left(t^{2}\right)\right\}, \tag{17}$$

where $[\cdot]$ denotes coincidence limit $x\to x'$.[11] We can obtain the limit formula

$$\begin{aligned} \left(\frac{\sigma^{\alpha}{}_{\beta'\gamma}}{t}\right)_{t\to0} &= \left[\sigma^{\alpha}{}_{\beta'\gamma\mu}\right]u^{\mu'} \\ &= -\frac{1}{3}\left(R^{\alpha'}{}_{\beta'\mu'\gamma}+R^{\alpha'}{}_{\gamma\mu'\beta'}\right)u^{\mu'} \end{aligned} \tag{18}$$

from the covariant expansion of $\sigma^{\alpha}{}_{\beta'\gamma}$

$$\sigma^{\alpha}{}_{\beta'\gamma}=g^{\alpha}{}_{\alpha'}\left\{\left[\sigma^{\alpha}{}_{\beta'\gamma\mu}\right]tu^{\mu'}+O\left(t^{2}\right)\right\}. \tag{19}$$

The expression of coincidence limit $[\sigma^{\alpha}{}_{\beta\gamma'\mu}]$ and $[\sigma^{\alpha}{}_{\beta'\gamma'\mu}]$ can be found in Ref. [11].

3.2. Transport equation for $\sigma^{\alpha}_{\beta\gamma'}$

Differentiating Eq. (28) in Ref. [9] with respect to x', we obtain

$$\begin{aligned} &\sigma^{\alpha}{}_{\beta\gamma'\mu}\sigma^{\mu}+\sigma^{\alpha}{}_{\beta\mu}\sigma^{\mu}{}_{\gamma'}-R^{\alpha}{}_{\rho\mu\beta}\sigma^{\rho}{}_{\gamma'}\sigma^{\mu}-R^{\alpha}{}_{\rho\mu\beta}\sigma^{\rho}\sigma^{\mu}{}_{\gamma'} \\ &+\sigma^{\alpha}{}_{\mu\gamma'}\sigma^{\mu}{}_{\beta}+\left(\sigma^{\alpha}{}_{\mu}-\delta^{\alpha}{}_{\mu}\right)\sigma^{\mu}{}_{\beta\gamma'}=0. \end{aligned} \tag{20}$$

The initial value problem for $\sigma^{\alpha}{}_{\beta\gamma'}$ is

$$\begin{aligned} &\frac{\mathrm{d}\sigma^{\alpha}{}_{\beta\gamma'}}{\mathrm{d}t}-\left(\sigma^{\alpha}{}_{\lambda\gamma'}\Gamma^{\lambda}{}_{\beta\mu}-\Gamma^{\alpha}{}_{\lambda\mu}\sigma^{\lambda}{}_{\beta\gamma'}\right. \\ &\left.+R^{\alpha}{}_{\rho\mu\beta}\sigma^{\rho}{}_{\gamma'}+R^{\alpha}{}_{\mu\rho\beta}\sigma^{\rho}{}_{\gamma'}\right)u^{\mu} \\ &+\frac{1}{t}\left[\sigma^{\alpha}{}_{\beta\mu}\sigma^{\mu}{}_{\gamma'}+\sigma^{\alpha}{}_{\mu\gamma'}\sigma^{\mu}{}_{\beta}+\left(\sigma^{\alpha}{}_{\mu}-\delta^{\alpha}{}_{\mu}\right)\sigma^{\mu}{}_{\beta\gamma'}\right]=0, \\ &\sigma^{\alpha}{}_{\beta\gamma'}\big|_{t=0}=0. \end{aligned} \tag{21}$$

3.3. Transport equation for $\sigma^{\alpha}{}_{\beta\gamma\delta'}$

Differentiating Eq. (20) with respect to x', we obtain

$$\begin{aligned} &\sigma^{\alpha}{}_{\beta\gamma\delta'\mu}\sigma^{\mu}+\sigma^{\alpha}{}_{\beta\mu\gamma'}\sigma^{\mu}{}_{\delta'}+\sigma^{\alpha}{}_{\beta\mu\delta'}\sigma^{\mu}{}_{\gamma'}+\sigma^{\alpha}{}_{\beta\mu}\sigma^{\mu}{}_{\gamma'\delta'} \\ &-R^{\alpha}{}_{\rho\mu\beta}\sigma^{\rho}{}_{\gamma'\delta'}\sigma^{\mu}-R^{\alpha}{}_{\rho\mu\beta}\sigma^{\rho}{}_{\gamma'}\sigma^{\mu}{}_{\delta'} \\ &-R^{\alpha}{}_{\rho\mu\beta}\sigma^{\rho}{}_{\delta'}\sigma^{\mu}{}_{\gamma'}-R^{\alpha}{}_{\rho\mu\beta}\sigma^{\rho}\sigma^{\mu}{}_{\gamma'\delta'} \\ &+\sigma^{\alpha}{}_{\mu\gamma'\delta'}\sigma^{\mu}{}_{\beta}+\sigma^{\alpha}{}_{\mu\gamma'}\sigma^{\mu}{}_{\beta\delta'}+\sigma^{\alpha}{}_{\mu\delta'}\sigma^{\mu}{}_{\beta\gamma'} \\ &+\left(\sigma^{\alpha}{}_{\mu}-\delta^{\alpha}{}_{\mu}\right)\sigma^{\mu}{}_{\beta\gamma'\delta'}=0. \end{aligned} \tag{22}$$

The initial value problem for $\sigma^{\alpha}{}_{\beta\gamma\delta'}$ is

$$\begin{aligned} &\frac{\mathrm{d}\sigma^{\alpha}{}_{\beta\gamma'\delta'}}{\mathrm{d}t}-\left(\sigma^{\alpha}{}_{\lambda\gamma'\delta'}\Gamma^{\lambda}{}_{\beta\mu}-\Gamma^{\alpha}{}_{\lambda\mu}\sigma^{\lambda}{}_{\beta\gamma'\delta'}\right. \\ &\left.+R^{\alpha}{}_{\rho\mu\beta}\sigma^{\rho}{}_{\gamma'\delta'}+R^{\alpha}{}_{\mu\rho\beta}\sigma^{\rho}{}_{\gamma'\delta'}\right)u^{\mu} \\ &+\frac{1}{t}\left(\sigma^{\alpha}{}_{\mu}-\delta^{\alpha}{}_{\mu}\right)\sigma^{\mu}{}_{\beta\gamma'\delta'} \\ &+\frac{1}{t}\left(\sigma^{\alpha}{}_{\beta\mu}\sigma^{\mu}{}_{\gamma'\delta'}+\sigma^{\alpha}{}_{\mu\gamma'}\sigma^{\mu}{}_{\beta\delta'}+\sigma^{\alpha}{}_{\mu\delta'}\sigma^{\mu}{}_{\beta\gamma'}\right) \\ &+\frac{1}{t}\left(\sigma^{\alpha}{}_{\mu\gamma'\delta'}\sigma^{\mu}{}_{\beta}+\sigma^{\alpha}{}_{\beta\mu\gamma'}\sigma^{\mu}{}_{\delta'}+\sigma^{\alpha}{}_{\beta\mu\delta'}\sigma^{\mu}{}_{\gamma'}\right. \\ &\left.+R^{\alpha}{}_{\rho\beta\mu}\sigma^{\rho}{}_{\gamma'}\sigma^{\mu}{}_{\delta'}+R^{\alpha}{}_{\rho\beta\mu}\sigma^{\rho}{}_{\delta'}\sigma^{\mu}{}_{\gamma'}\right)=0, \\ &\sigma^{\alpha_1}{}_{\alpha_2\alpha'_3\alpha'_4}\Big|_{t=0}=-\frac{1}{3}\left(R^{\alpha'_1}{}_{\alpha'_3\alpha'_2\alpha'_4}+R^{\alpha'_1}{}_{\alpha'_4\alpha'_2\alpha'_3}\right). \end{aligned} \tag{23}$$

We can write the following covariant expansion:

$$\begin{aligned} &\sigma^{\alpha}{}_{\mu\gamma'\delta'}\sigma^{\mu}{}_{\beta}+\sigma^{\alpha}{}_{\beta\mu\gamma'}\sigma^{\mu}{}_{\delta'}+\sigma^{\alpha}{}_{\beta\mu\delta'}\sigma^{\mu}{}_{\gamma'} \\ &+R^{\alpha}{}_{\rho\beta\mu}\sigma^{\rho}{}_{\gamma'}\sigma^{\mu}{}_{\delta'}+R^{\alpha}{}_{\rho\beta\mu}\sigma^{\rho}{}_{\delta'}\sigma^{\mu}{}_{\gamma'} \\ =\ & R^{\alpha'}{}_{\gamma'\beta'\delta'}+R^{\alpha'}{}_{\delta'\beta'\gamma'}+\left[\sigma^{\alpha}{}_{\beta\gamma'\delta'}\right]+\left[\sigma^{\alpha}{}_{\beta\gamma'\delta'\mu}\right]tu^{\mu'} \\ &-\left[\sigma^{\alpha}{}_{\beta\delta\gamma'}\right]-\left[\sigma^{\alpha}{}_{\beta\delta\gamma'\mu}\right]tu^{\mu'}-\left[\sigma^{\alpha}{}_{\beta\gamma\delta'}\right] \end{aligned}$$

$$-\left[\sigma^{\alpha}{}_{\beta\gamma\delta'\mu}\right]tu^{\mu'}+O\left(t^{2}\right).$$

Using Ricci identity,[15–17] we obtain the limit formula from the above equation

$$\begin{aligned}&\left\{\frac{1}{t}\left(\sigma^{\alpha}{}_{\mu\gamma'\delta'}\sigma^{\mu}{}_{\beta}+\sigma^{\alpha}{}_{\beta\mu\gamma'}\sigma^{\mu}{}_{\delta'}\right.\right.\\&\left.\left.+\sigma^{\alpha}{}_{\beta\mu\delta'}\sigma^{\mu}{}_{\gamma'}R^{\alpha}{}_{\rho\beta\mu}\sigma^{\rho}{}_{\gamma'}\sigma^{\mu}{}_{\delta'}+R^{\alpha}{}_{\rho\beta\mu}\sigma^{\rho}{}_{\delta'}\sigma^{\mu}{}_{\gamma'}\right)\right\}_{t\to0}\\&=\left(\left[\sigma^{\alpha}{}_{\beta\gamma'\delta'\mu}\right]-\left[\sigma^{\alpha}{}_{\beta\delta\gamma'\mu}\right]-\left[\sigma^{\alpha}{}_{\beta\gamma\delta'\mu}\right]\right)u^{\mu'}.\end{aligned}\tag{24}$$

The coincidence limits in the above equation can be acquired by applying Synge's rule[11] on Eq. (44) in Ref. [9].

3.4. Transport equation for $\sigma^{\alpha}{}_{\beta\gamma\delta'}$

Differentiating Eq. (34) in Ref. [9] with respect to x', we obtain

$$\begin{aligned}&\sigma^{\alpha}{}_{\beta\gamma\delta'\mu}\sigma^{\mu}+\sigma^{\alpha}{}_{\beta\gamma\mu}\sigma^{\mu}{}_{\delta'}+\sigma^{\alpha}{}_{\mu\beta\delta'}\sigma^{\mu}{}_{\gamma}+\sigma^{\alpha}{}_{\mu\beta}\sigma^{\mu}{}_{\gamma\delta'}\\&+\sigma^{\alpha}{}_{\mu\gamma\delta'}\sigma^{\mu}{}_{\beta}+\sigma^{\alpha}{}_{\mu\gamma}\sigma^{\mu}{}_{\beta\delta'}\\&+\sigma^{\alpha}{}_{\mu\delta'}\sigma^{\mu}{}_{\beta\gamma}+\left(\sigma^{\alpha}{}_{\mu}-\delta^{\alpha}{}_{\mu}\right)\sigma^{\mu}{}_{\beta\gamma\delta'}\\&-R^{\alpha}{}_{\rho\mu\beta;\gamma}\sigma^{\rho}{}_{\delta'}\sigma^{\mu}-R^{\alpha}{}_{\rho\mu\beta;\gamma}\sigma^{\rho}\sigma^{\mu}{}_{\delta'}\\&-R^{\alpha}{}_{\rho\mu\beta}\sigma^{\rho}{}_{\gamma\delta'}\sigma^{\mu}-R^{\alpha}{}_{\rho\mu\beta}\sigma^{\rho}{}_{\gamma}\sigma^{\mu}{}_{\delta'}\\&-R^{\alpha}{}_{\rho\mu\gamma}\sigma^{\rho}{}_{\beta\delta'}\sigma^{\mu}-R^{\alpha}{}_{\rho\mu\gamma}\sigma^{\rho}{}_{\beta}\sigma^{\mu}{}_{\delta'}\\&+R^{\rho}{}_{\beta\mu\gamma}\sigma^{\alpha}{}_{\rho\delta'}\sigma^{\mu}+R^{\rho}{}_{\beta\mu\gamma}\sigma^{\alpha}{}_{\rho}\sigma^{\mu}{}_{\delta'}=0.\end{aligned}\tag{25}$$

The initial value problem for $\sigma^{\alpha}{}_{\beta\gamma\delta'}$ is

$$\begin{aligned}&\frac{\mathrm{d}\sigma^{\alpha}{}_{\beta\gamma\delta'}}{\mathrm{d}t}-\left(\sigma^{\alpha}{}_{\lambda\gamma\delta'}\Gamma^{\lambda}{}_{\beta\mu}+\sigma^{\alpha}{}_{\beta\lambda\delta'}\Gamma^{\lambda}{}_{\gamma\mu}-\Gamma^{\alpha}{}_{\lambda\mu}\sigma^{\lambda}{}_{\beta\gamma\delta'}\right.\\&+R^{\alpha}{}_{\rho\mu\beta;\gamma}\sigma^{\rho}{}_{\delta'}+R^{\alpha}{}_{\mu\rho\beta;\gamma}\sigma^{\rho}{}_{\delta'}+R^{\alpha}{}_{\rho\mu\beta}\sigma^{\rho}{}_{\gamma\delta'}\\&\left.+R^{\alpha}{}_{\rho\mu\gamma}\sigma^{\rho}{}_{\beta\delta'}+\sigma^{\alpha}{}_{\rho\delta'}R^{\rho}{}_{\beta\gamma\mu}\right)u^{\mu}\\&+\frac{1}{t}\left[\sigma^{\alpha}{}_{\mu\beta}\sigma^{\mu}{}_{\gamma\delta'}+\sigma^{\alpha}{}_{\mu\gamma}\sigma^{\mu}{}_{\beta\delta'}\right.\\&\left.+\sigma^{\alpha}{}_{\mu\delta'}\sigma^{\mu}{}_{\beta\gamma}+\left(\sigma^{\alpha}{}_{\mu}-\delta^{\alpha}{}_{\mu}\right)\sigma^{\mu}{}_{\beta\gamma\delta'}\right]\\&+\frac{1}{t}\left(\sigma^{\alpha}{}_{\beta\gamma\mu}\sigma^{\mu}{}_{\delta'}+\sigma^{\alpha}{}_{\mu\beta\delta'}\sigma^{\mu}{}_{\gamma}+\sigma^{\alpha}{}_{\mu\gamma\delta'}\sigma^{\mu}{}_{\beta}\right.\\&+R^{\alpha}{}_{\rho\beta\mu}\sigma^{\rho}{}_{\gamma}\sigma^{\mu}{}_{\delta'}+R^{\alpha}{}_{\rho\gamma\mu}\sigma^{\rho}{}_{\beta}\sigma^{\mu}{}_{\delta'}\\&\left.+\sigma^{\alpha}{}_{\rho}R^{\rho}{}_{\beta\mu\gamma}\sigma^{\mu}{}_{\delta'}\right)=0,\\&\sigma^{\alpha}{}_{\beta\gamma\delta'}\big|_{t=0}=\frac{1}{3}\left(R^{\alpha'}{}_{\gamma'\beta'\delta'}+R^{\alpha'}{}_{\delta'\beta'\gamma'}\right).\end{aligned}\tag{26}$$

We can write the following covariant expansion:

$$\begin{aligned}&\sigma^{\alpha}{}_{\beta\gamma\mu}\sigma^{\mu}{}_{\delta'}+\sigma^{\alpha}{}_{\mu\beta\delta'}\sigma^{\mu}{}_{\gamma}+\sigma^{\alpha}{}_{\mu\gamma\delta'}\sigma^{\mu}{}_{\beta}\\&+R^{\alpha}{}_{\rho\beta\mu}\sigma^{\rho}{}_{\gamma}\sigma^{\mu}{}_{\delta'}+R^{\alpha}{}_{\rho\gamma\mu}\sigma^{\rho}{}_{\beta}\sigma^{\mu}{}_{\delta'}+R^{\rho}{}_{\beta\mu\gamma}\sigma^{\alpha}{}_{\rho}\sigma^{\mu}{}_{\delta'}\\=&-\left[\sigma^{\alpha}{}_{\beta\gamma\delta}\right]-\left[\sigma^{\alpha}{}_{\beta\gamma\delta\mu}\right]tu^{\mu'}+\left[\sigma^{\alpha}{}_{\gamma\beta\delta'}\right]+\left[\sigma^{\alpha}{}_{\gamma\beta\delta'\mu}\right]tu^{\mu'}\\&+\left[\sigma^{\alpha}{}_{\beta\gamma\delta'}\right]+\left[\sigma^{\alpha}{}_{\beta\gamma\delta'\mu}\right]tu^{\mu'}+R^{\alpha'}{}_{\gamma'\delta'\beta'}+O\left(t^{2}\right).\end{aligned}$$

Using Ricci identity, we obtain the limit formula

$$\begin{aligned}&\left\{\frac{1}{t}\left(\sigma^{\alpha}{}_{\beta\gamma\mu}\sigma^{\mu}{}_{\delta'}+\sigma^{\alpha}{}_{\mu\beta\delta'}\sigma^{\mu}{}_{\gamma}+\sigma^{\alpha}{}_{\mu\gamma\delta'}\sigma^{\mu}{}_{\beta}\right.\right.\\&+R^{\alpha}{}_{\rho\beta\mu}\sigma^{\rho}{}_{\gamma}\sigma^{\mu}{}_{\delta'}+R^{\alpha}{}_{\rho\gamma\mu}\sigma^{\rho}{}_{\beta}\sigma^{\mu}{}_{\delta'}\\&\left.\left.+R^{\rho}{}_{\beta\mu\gamma}\sigma^{\alpha}{}_{\rho}\sigma^{\mu}{}_{\delta'}\right)\right\}_{t\to0}\\=&\left(-\left[\sigma^{\alpha}{}_{\beta\gamma\delta\mu}\right]+\left[\sigma^{\alpha}{}_{\gamma\beta\delta'\mu}\right]+\left[\sigma^{\alpha}{}_{\beta\gamma\delta'\mu}\right]\right)u^{\mu'}.\end{aligned}\tag{27}$$

3.5. Transport equation for $\sigma^{\alpha}{}_{\beta\gamma\delta\varepsilon'}$

Differentiating Eq. (34) in Ref. [9] and commuting covariant derivatives, we obtain

$$\begin{aligned}&\sigma^{\alpha}{}_{\beta\gamma\delta\mu}\sigma^{\mu}+\sigma^{\alpha}{}_{\mu\beta\gamma}\sigma^{\mu}{}_{\delta}+\sigma^{\alpha}{}_{\mu\beta\delta}\sigma^{\mu}{}_{\gamma}+\sigma^{\alpha}{}_{\mu\gamma\delta}\sigma^{\mu}{}_{\beta}\\&+\sigma^{\alpha}{}_{\mu\beta}\sigma^{\mu}{}_{\gamma\delta}+\sigma^{\alpha}{}_{\mu\gamma}\sigma^{\mu}{}_{\beta\delta}+\sigma^{\alpha}{}_{\mu\delta}\sigma^{\mu}{}_{\beta\gamma}\\&+\left(\sigma^{\alpha}{}_{\mu}-\delta^{\alpha}{}_{\mu}\right)\sigma^{\mu}{}_{\beta\gamma\delta}-R^{\alpha}{}_{\rho\mu\beta;\gamma\delta}\sigma^{\rho}\sigma^{\mu}-R^{\alpha}{}_{\rho\mu\beta;\gamma}\sigma^{\rho}{}_{\delta}\sigma^{\mu}\\&-R^{\alpha}{}_{\rho\mu\beta;\delta}\sigma^{\rho}{}_{\gamma}\sigma^{\mu}-R^{\alpha}{}_{\rho\mu\gamma;\delta}\sigma^{\rho}{}_{\beta}\sigma^{\mu}+R^{\rho}{}_{\beta\mu\gamma;\delta}\sigma^{\alpha}{}_{\rho}\sigma^{\mu}\\&-R^{\alpha}{}_{\rho\mu\delta}\sigma^{\rho}{}_{\beta\gamma}\sigma^{\mu}+R^{\rho}{}_{\beta\mu\delta}\sigma^{\alpha}{}_{\rho\gamma}\sigma^{\mu}+R^{\rho}{}_{\gamma\mu\delta}\sigma^{\alpha}{}_{\beta\rho}\sigma^{\mu}\\&-R^{\alpha}{}_{\rho\mu\beta}\sigma^{\rho}{}_{\gamma\delta}\sigma^{\mu}-R^{\alpha}{}_{\rho\mu\gamma}\sigma^{\rho}{}_{\beta\delta}\sigma^{\mu}\\&+R^{\rho}{}_{\beta\mu\gamma}\sigma^{\alpha}{}_{\rho\delta}\sigma^{\mu}=0.\end{aligned}\tag{28}$$

Differentiating the above equation with respect to x', we obtain the initial value problem for $\sigma^{\alpha}_{\beta\gamma\delta\varepsilon'}$

$$\begin{aligned}&\frac{\mathrm{d}\sigma^{\alpha}{}_{\beta\gamma\delta\varepsilon'}}{\mathrm{d}t}-\left(\sigma^{\alpha}{}_{\lambda\gamma\delta\varepsilon'}\Gamma^{\lambda}{}_{\beta\mu}+\sigma^{\alpha}{}_{\beta\lambda\delta\varepsilon'}\Gamma^{\lambda}{}_{\gamma\mu}\right.\\&+\sigma^{\alpha}{}_{\beta\gamma\lambda\varepsilon'}\Gamma^{\lambda}{}_{\delta\mu}-\Gamma^{\alpha}{}_{\lambda\mu}\sigma^{\lambda}{}_{\beta\gamma\delta\varepsilon'}\\&+R^{\alpha}{}_{\rho\mu\beta;\gamma\delta}\sigma^{\rho}{}_{\varepsilon'}+R^{\alpha}{}_{\mu\rho\beta;\gamma\delta}\sigma^{\rho}{}_{\varepsilon'}\\&+R^{\alpha}{}_{\rho\mu\beta;\gamma}\sigma^{\rho}{}_{\delta\varepsilon'}+R^{\alpha}{}_{\rho\mu\beta;\delta}\sigma^{\rho}{}_{\gamma\varepsilon'}\\&+R^{\alpha}{}_{\rho\mu\gamma;\delta}\sigma^{\rho}{}_{\beta\varepsilon'}+\sigma^{\alpha}{}_{\rho\varepsilon'}R^{\rho}{}_{\beta\gamma\mu;\delta}\\&+R^{\alpha}{}_{\rho\mu\delta}\sigma^{\rho}{}_{\beta\gamma\varepsilon'}+R^{\alpha}{}_{\rho\mu\beta}\sigma^{\rho}{}_{\gamma\delta\varepsilon'}+R^{\alpha}{}_{\rho\mu\gamma}\sigma^{\rho}{}_{\beta\delta\varepsilon'}\\&\left.+\sigma^{\alpha}{}_{\rho\gamma\varepsilon'}R^{\rho}{}_{\beta\delta\mu}+\sigma^{\alpha}{}_{\rho\delta\varepsilon'}R^{\rho}{}_{\beta\gamma\mu}+\sigma^{\alpha}{}_{\beta\rho\varepsilon'}R^{\rho}{}_{\gamma\delta\mu}\right)u^{\mu}\\&+\frac{1}{t}\left(\sigma^{\alpha}{}_{\mu}-\delta^{\alpha}{}_{\mu}\right)\sigma^{\mu}{}_{\beta\gamma\delta\varepsilon'}\\&+\frac{1}{t}\left(\sigma^{\alpha}{}_{\mu\beta\gamma}\sigma^{\mu}{}_{\delta\varepsilon'}+\sigma^{\alpha}{}_{\mu\beta\delta}\sigma^{\mu}{}_{\gamma\varepsilon'}+\sigma^{\alpha}{}_{\mu\gamma\delta}\sigma^{\mu}{}_{\beta\varepsilon'}\right.\\&+\sigma^{\alpha}{}_{\mu\beta\varepsilon'}\sigma^{\mu}{}_{\gamma\delta}+\sigma^{\alpha}{}_{\mu\gamma\varepsilon'}\sigma^{\mu}{}_{\beta\delta}+\sigma^{\alpha}{}_{\mu\delta\varepsilon'}\sigma^{\mu}{}_{\beta\gamma}\\&+\sigma^{\alpha}{}_{\mu\beta}\sigma^{\mu}{}_{\gamma\delta\varepsilon'}+\sigma^{\alpha}{}_{\mu\gamma}\sigma^{\mu}{}_{\beta\delta\varepsilon'}+\sigma^{\alpha}{}_{\mu\delta}\sigma^{\mu}{}_{\beta\gamma\varepsilon'}+\sigma^{\alpha}{}_{\mu\varepsilon'}\sigma^{\mu}{}_{\beta\gamma\delta}\\&+R^{\alpha}{}_{\rho\beta\mu}\sigma^{\rho}{}_{\gamma\delta}\sigma^{\mu}{}_{\varepsilon'}+R^{\alpha}{}_{\rho\gamma\mu}\sigma^{\rho}{}_{\beta\delta}\sigma^{\mu}{}_{\varepsilon'}+R^{\alpha}{}_{\rho\delta\mu}\sigma^{\rho}{}_{\beta\gamma}\sigma^{\mu}{}_{\varepsilon'}\\&\left.+\sigma^{\alpha}{}_{\rho\delta}R^{\rho}{}_{\beta\mu\gamma}\sigma^{\mu}{}_{\varepsilon'}+\sigma^{\alpha}{}_{\rho\gamma}R^{\rho}{}_{\beta\mu\delta}\sigma^{\mu}{}_{\varepsilon'}+\sigma^{\alpha}{}_{\beta\rho}R^{\rho}{}_{\gamma\mu\delta}\sigma^{\mu}{}_{\varepsilon'}\right)\\&+\frac{1}{t}\left(\sigma^{\alpha}{}_{\mu\beta\varepsilon'}\sigma^{\mu}{}_{\delta}+\sigma^{\alpha}{}_{\mu\beta\delta\varepsilon'}\sigma^{\mu}{}_{\gamma}\right.\\&+\sigma^{\alpha}{}_{\mu\gamma\delta\varepsilon'}\sigma^{\mu}{}_{\beta}+\sigma^{\alpha}{}_{\beta\gamma\delta\mu}\sigma^{\mu}{}_{\varepsilon'}\\&+R^{\alpha}{}_{\rho\beta\mu;\gamma}\sigma^{\rho}{}_{\delta}\sigma^{\mu}{}_{\varepsilon'}+R^{\alpha}{}_{\rho\beta\mu;\delta}\sigma^{\rho}{}_{\gamma}\sigma^{\mu}{}_{\varepsilon'}\\&\left.+R^{\alpha}{}_{\rho\gamma\mu;\delta}\sigma^{\rho}{}_{\beta}\sigma^{\mu}{}_{\varepsilon'}+\sigma^{\alpha}{}_{\rho}R^{\rho}{}_{\beta\mu\gamma;\delta}\sigma^{\mu}{}_{\varepsilon'}\right)=0,\\&\sigma_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha'_5}\Big|_{t=0}\\=&-\frac{1}{12}(R_{\alpha'_1\alpha'_3\alpha'_2\alpha'_4;\alpha'_5}+R_{\alpha'_1\alpha'_4\alpha'_2\alpha'_3;\alpha'_5})\\&-\frac{1}{4}(R_{\alpha'_1\alpha'_5\alpha'_3\alpha'_2;\alpha'_4}+R_{\alpha'_1\alpha'_5\alpha'_4\alpha'_2;\alpha'_3}\\&+R_{\alpha'_1\alpha'_4\alpha'_5\alpha'_2;\alpha'_3}+R_{\alpha'_1\alpha'_3\alpha'_5\alpha'_2;\alpha'_4}).\end{aligned}\tag{29}$$

We can write the following covariant expansion:

$$\sigma^{\alpha}{}_{\beta\gamma\delta\mu}\sigma^{\mu}{}_{\varepsilon'}+\sigma^{\alpha}{}_{\mu\beta\gamma\varepsilon'}\sigma^{\mu}{}_{\delta}+\sigma^{\alpha}{}_{\mu\beta\delta\varepsilon'}\sigma^{\mu}{}_{\gamma}+\sigma^{\alpha}{}_{\mu\gamma\delta\varepsilon'}\sigma^{\mu}{}_{\beta}$$

$$
\begin{aligned}
&-R^{\alpha}{}_{\rho\mu\beta;\gamma}\sigma^{\rho}{}_{\delta}\sigma^{\mu}{}_{\varepsilon'}-R^{\alpha}{}_{\rho\mu\beta;\delta}\sigma^{\rho}{}_{\gamma}\sigma^{\mu}{}_{\varepsilon'}\\
&-R^{\alpha}{}_{\rho\mu\gamma;\delta}\sigma^{\rho}{}_{\beta}\sigma^{\mu}{}_{\varepsilon'}+R^{\rho}{}_{\beta\mu\gamma;\delta}\sigma^{\alpha}{}_{\rho}\sigma^{\mu}{}_{\varepsilon'}\\
=&-\left[\sigma^{\alpha}{}_{\beta\gamma\delta\varepsilon}\right]-\left[\sigma^{\alpha}{}_{\beta\gamma\delta\varepsilon\mu}\right]tu^{\mu'}+\left[\sigma^{\alpha}{}_{\delta\beta\gamma\varepsilon'}\right]\\
&+\left[\sigma^{\alpha}{}_{\delta\beta\gamma\varepsilon'\mu}\right]tu^{\mu'}+\left[\sigma^{\alpha}{}_{\gamma\beta\delta\varepsilon'}\right]+\left[\sigma^{\alpha}{}_{\gamma\beta\delta\varepsilon'\mu}\right]tu^{\mu'}\\
&+\left[\sigma^{\alpha}{}_{\beta\gamma\delta\varepsilon'}\right]+\left[\sigma^{\alpha}{}_{\beta\gamma\delta\varepsilon'\mu}\right]tu^{\mu'}\\
&+R^{\alpha'}{}_{\delta'\varepsilon'\beta';\gamma'}+R^{\alpha'}{}_{\gamma'\varepsilon'\beta';\delta'}+O\left(t^{2}\right).
\end{aligned}
$$

Using Ricci identity and Bianchi identity,[15] we obtain the limit formula

$$
\begin{aligned}
&\left\{\frac{1}{t}\left(\sigma^{\alpha}{}_{\beta\gamma\delta\mu}\sigma^{\mu}{}_{\varepsilon'}+\sigma^{\alpha}{}_{\mu\beta\gamma\varepsilon'}\sigma^{\mu}{}_{\delta}\right.\right.\\
&+\sigma^{\alpha}{}_{\mu\beta\delta\varepsilon'}\sigma^{\mu}{}_{\gamma}+\sigma^{\alpha}{}_{\mu\gamma\delta\varepsilon'}\sigma^{\mu}{}_{\beta}\\
&-R^{\alpha}{}_{\rho\mu\beta;\gamma}\sigma^{\rho}{}_{\delta}\sigma^{\mu}{}_{\varepsilon'}-R^{\alpha}{}_{\rho\mu\beta;\delta}\sigma^{\rho}{}_{\gamma}\sigma^{\mu}{}_{\varepsilon'}\\
&\left.\left.-R^{\alpha}{}_{\rho\mu\gamma;\delta}\sigma^{\rho}{}_{\beta}\sigma^{\mu}{}_{\varepsilon'}+R^{\rho}{}_{\beta\mu\gamma;\delta}\sigma^{\alpha}{}_{\rho}\sigma^{\mu}{}_{\varepsilon'}\right)\right\}_{t\to0}\\
=&\left(-\left[\sigma^{\alpha}{}_{\beta\gamma\delta\varepsilon\mu}\right]+\left[\sigma^{\alpha}{}_{\beta\gamma\delta\varepsilon'\mu}\right]+\left[\sigma^{\alpha}{}_{\gamma\beta\delta\varepsilon'\mu}\right]\right.\\
&\left.+\left[\sigma^{\alpha}{}_{\delta\beta\gamma\varepsilon'\mu}\right]\right)u^{\mu'}.
\end{aligned}
\tag{30}
$$

The coincidence limit $[\sigma^{\alpha}{}_{\beta\gamma\delta\varepsilon\mu}]$ can be derived by differentiating $\sigma^{\alpha}\sigma_{\alpha}=2\sigma$ for six times, and then taking coincidence limit. The coincidence limit $[\sigma^{\alpha}_{\beta\gamma\delta\varepsilon'\mu}]$ can be acquired by applying Synge's rule.

3.6. Transport equation for $\sigma^{\alpha}{}_{\beta\gamma\delta\varepsilon}$

Differentiating Eq. (28) with respect to x, we obtain the initial value problem for $\sigma^{\alpha}{}_{\beta\gamma\delta\varepsilon}$

$$
\begin{aligned}
&\frac{\mathrm{d}\sigma^{\alpha}{}_{\beta\gamma\delta\varepsilon}}{\mathrm{d}t}-\left(\sigma^{\alpha}{}_{\lambda\gamma\delta\varepsilon}\Gamma^{\lambda}{}_{\beta\mu}+\sigma^{\alpha}{}_{\beta\lambda\delta\varepsilon}\Gamma^{\lambda}{}_{\gamma\mu}\right.\\
&+\sigma^{\alpha}{}_{\beta\gamma\lambda\varepsilon}\Gamma^{\lambda}{}_{\delta\mu}+\sigma^{\alpha}{}_{\beta\gamma\delta\lambda}\Gamma^{\lambda}{}_{\varepsilon\mu}-\Gamma^{\alpha}{}_{\lambda\mu}\sigma^{\lambda}{}_{\beta\gamma\delta\varepsilon}\\
&+\sigma^{\alpha}{}_{\lambda\gamma\delta}R^{\lambda}{}_{\beta\varepsilon\mu}+\sigma^{\alpha}{}_{\beta\lambda\delta}R^{\lambda}{}_{\gamma\varepsilon\mu}+\sigma^{\alpha}{}_{\beta\gamma\lambda}R^{\lambda}{}_{\delta\varepsilon\mu}\\
&+R^{\alpha}{}_{\lambda\mu\varepsilon}\sigma^{\lambda}{}_{\beta\gamma\delta}+R^{\alpha}{}_{\rho\mu\beta;\gamma\delta\varepsilon}\sigma^{\rho}\\
&+R^{\alpha}{}_{\rho\mu\beta;\gamma\delta}\sigma^{\rho}{}_{\varepsilon}+R^{\alpha}{}_{\rho\mu\beta;\gamma\varepsilon}\sigma^{\rho}{}_{\delta}+R^{\alpha}{}_{\rho\mu\beta;\delta\varepsilon}\sigma^{\rho}{}_{\gamma}\\
&+R^{\alpha}{}_{\rho\mu\gamma;\delta\varepsilon}\sigma^{\rho}{}_{\beta}+R^{\alpha}{}_{\mu\rho\beta;\gamma\delta}\sigma^{\rho}{}_{\varepsilon}\\
&+R^{\alpha}{}_{\rho\mu\beta;\gamma}\sigma^{\rho}{}_{\delta\varepsilon}+R^{\alpha}{}_{\rho\mu\beta;\delta}\sigma^{\rho}{}_{\gamma\varepsilon}+R^{\alpha}{}_{\rho\mu\gamma;\delta}\sigma^{\rho}{}_{\beta\varepsilon}\\
&+R^{\alpha}{}_{\rho\mu\delta;\varepsilon}\sigma^{\rho}{}_{\beta\gamma}+R^{\alpha}{}_{\rho\mu\gamma;\varepsilon}\sigma^{\rho}{}_{\beta\delta}\\
&+\sigma^{\alpha}{}_{\rho\varepsilon}R^{\rho}{}_{\beta\gamma\mu;\delta}+\sigma^{\alpha}{}_{\rho\gamma}R^{\rho}{}_{\beta\delta\mu;\varepsilon}+\sigma^{\alpha}{}_{\rho\delta}R^{\rho}{}_{\beta\gamma\mu;\varepsilon}\\
&+\sigma^{\alpha}{}_{\beta\rho}R^{\rho}{}_{\gamma\delta\mu;\varepsilon}+\sigma^{\alpha}{}_{\rho}R^{\rho}{}_{\beta\gamma\mu;\delta\varepsilon}\\
&+R^{\alpha}{}_{\rho\mu\delta}\sigma^{\rho}{}_{\beta\gamma\varepsilon}+R^{\alpha}{}_{\rho\mu\beta}\sigma^{\rho}{}_{\gamma\delta\varepsilon}\\
&+R^{\alpha}{}_{\rho\mu\gamma}\sigma^{\rho}{}_{\beta\delta\varepsilon}+R^{\alpha}{}_{\rho\mu\beta;\varepsilon}\sigma^{\rho}{}_{\gamma\delta}\\
&\left.+\sigma^{\alpha}{}_{\rho\gamma\varepsilon}R^{\rho}{}_{\beta\delta\mu}+\sigma^{\alpha}{}_{\rho\delta\varepsilon}R^{\rho}{}_{\beta\gamma\mu}+\sigma^{\alpha}{}_{\beta\rho\varepsilon}R^{\rho}{}_{\gamma\delta\mu}\right)u^{\mu}\\
&+\frac{1}{t}\left[\left(\sigma^{\alpha}{}_{\mu}-\delta^{\alpha}{}_{\mu}\right)\sigma^{\mu}{}_{\beta\gamma\delta\varepsilon}\right.\\
&+\sigma^{\alpha}{}_{\mu\beta\gamma}\sigma^{\mu}{}_{\delta\varepsilon}+\sigma^{\alpha}{}_{\mu\beta\delta}\sigma^{\mu}{}_{\gamma\varepsilon}+\sigma^{\alpha}{}_{\mu\gamma\delta}\sigma^{\mu}{}_{\beta\varepsilon}\\
&+\sigma^{\alpha}{}_{\mu\beta\varepsilon}\sigma^{\mu}{}_{\gamma\delta}+\sigma^{\alpha}{}_{\mu\gamma\varepsilon}\sigma^{\mu}{}_{\beta\delta}+\sigma^{\alpha}{}_{\mu\delta\varepsilon}\sigma^{\mu}{}_{\beta\gamma}\\
&+\sigma^{\alpha}{}_{\mu\beta}\sigma^{\mu}{}_{\gamma\delta\varepsilon}+\sigma^{\alpha}{}_{\mu\gamma}\sigma^{\mu}{}_{\beta\delta\varepsilon}+\sigma^{\alpha}{}_{\mu\delta}\sigma^{\mu}{}_{\beta\gamma\varepsilon}+\sigma^{\alpha}{}_{\mu\varepsilon}\sigma^{\mu}{}_{\beta\gamma\delta}\\
&\left.+R^{\alpha}{}_{\rho\beta\mu}\sigma^{\rho}{}_{\gamma\delta}\sigma^{\mu}{}_{\varepsilon}+R^{\alpha}{}_{\rho\gamma\mu}\sigma^{\rho}{}_{\beta\delta}\sigma^{\mu}{}_{\varepsilon}+R^{\alpha}{}_{\rho\delta\mu}\sigma^{\rho}{}_{\beta\gamma}\sigma^{\mu}{}_{\varepsilon}\right.\\
&\left.+\sigma^{\alpha}{}_{\rho\delta}R^{\rho}{}_{\beta\mu\gamma}\sigma^{\mu}{}_{\varepsilon}+\sigma^{\alpha}{}_{\rho\gamma}R^{\rho}{}_{\beta\mu\delta}\sigma^{\mu}{}_{\varepsilon}+\sigma^{\alpha}{}_{\beta\rho}R^{\rho}{}_{\gamma\mu\delta}\sigma^{\mu}{}_{\varepsilon}\right]\\
&+\frac{1}{t}\left(\sigma^{\alpha}{}_{\mu\beta\gamma\varepsilon}\sigma^{\mu}{}_{\delta}+\sigma^{\alpha}{}_{\mu\beta\delta\varepsilon}\sigma^{\mu}{}_{\gamma}\right.\\
&+\sigma^{\alpha}{}_{\mu\gamma\delta\varepsilon}\sigma^{\mu}{}_{\beta}+\sigma^{\alpha}{}_{\beta\gamma\delta\mu}\sigma^{\mu}{}_{\varepsilon}\\
&+R^{\alpha}{}_{\rho\beta\mu;\gamma}\sigma^{\rho}{}_{\delta}\sigma^{\mu}{}_{\varepsilon}+R^{\alpha}{}_{\rho\beta\mu;\delta}\sigma^{\rho}{}_{\gamma}\sigma^{\mu}{}_{\varepsilon}\\
&\left.+R^{\alpha}{}_{\rho\gamma\mu;\delta}\sigma^{\rho}{}_{\beta}\sigma^{\mu}{}_{\varepsilon}+\sigma^{\alpha}{}_{\rho}R^{\rho}{}_{\beta\mu\gamma;\delta}\sigma^{\mu}{}_{\varepsilon}\right)=0,\\
&\sigma_{\alpha_1\cdots\alpha_5}\big|_{t=0}\\
=&\frac{1}{4}\left(R_{\alpha'_1\alpha'_5\alpha'_3\alpha'_2;\alpha'_4}+R_{\alpha'_1\alpha'_4\alpha'_3\alpha'_2;\alpha'_5}+R_{\alpha'_1\alpha'_5\alpha'_4\alpha'_2;\alpha'_3}\right.\\
&\left.+R_{\alpha'_1\alpha'_3\alpha'_4\alpha'_2;\alpha'_5}+R_{\alpha'_1\alpha'_4\alpha'_5\alpha'_2;\alpha'_3}+R_{\alpha'_1\alpha'_3\alpha'_5\alpha'_2;\alpha'_4}\right).
\end{aligned}
\tag{31}
$$

We can write the following covariant expansion:

$$
\begin{aligned}
&\sigma^{\alpha}{}_{\beta\gamma\delta\mu}\sigma^{\mu}{}_{\varepsilon}+\sigma^{\alpha}{}_{\mu\beta\gamma\varepsilon}\sigma^{\mu}{}_{\delta}+\sigma^{\alpha}{}_{\mu\beta\delta\varepsilon}\sigma^{\mu}{}_{\gamma}+\sigma^{\alpha}{}_{\mu\gamma\delta\varepsilon}\sigma^{\mu}{}_{\beta}\\
&-R^{\alpha}{}_{\rho\mu\beta;\gamma}\sigma^{\rho}{}_{\delta}\sigma^{\mu}{}_{\varepsilon}-R^{\alpha}{}_{\rho\mu\beta;\delta}\sigma^{\rho}{}_{\gamma}\sigma^{\mu}{}_{\varepsilon}\\
&-R^{\alpha}{}_{\rho\mu\gamma;\delta}\sigma^{\rho}{}_{\beta}\sigma^{\mu}{}_{\varepsilon}+R^{\rho}{}_{\beta\mu\gamma;\delta}\sigma^{\alpha}{}_{\rho}\sigma^{\mu}{}_{\varepsilon}\\
=&2\left(\left[\sigma^{\alpha}{}_{\beta\gamma\delta\varepsilon}\right]+\left[\sigma^{\alpha}{}_{\beta\gamma\delta\varepsilon\mu}\right]tu^{\mu'}\right)+\left[\sigma^{\alpha}{}_{\delta\beta\gamma\varepsilon}\right]\\
&+\left[\sigma^{\alpha}{}_{\delta\beta\gamma\varepsilon\mu}\right]tu^{\mu'}+\left[\sigma^{\alpha}{}_{\gamma\beta\delta\varepsilon}\right]\\
&+\left[\sigma^{\alpha}{}_{\gamma\beta\delta\varepsilon\mu}\right]tu^{\mu'}-R^{\alpha'}{}_{\delta'\varepsilon'\beta';\gamma'}-R^{\alpha'}{}_{\gamma'\varepsilon'\beta';\delta'}+O\left(t^{2}\right).
\end{aligned}
$$

Using Ricci identity and Bianchi identity, we obtain the limit formula

$$
\begin{aligned}
&\left\{\frac{1}{t}\left(\sigma^{\alpha}{}_{\beta\gamma\delta\mu}\sigma^{\mu}{}_{\varepsilon}+\sigma^{\alpha}{}_{\mu\beta\gamma\varepsilon}\sigma^{\mu}{}_{\delta}\right.\right.\\
&+\sigma^{\alpha}{}_{\mu\beta\delta\varepsilon}\sigma^{\mu}{}_{\gamma}+\sigma^{\alpha}{}_{\mu\gamma\delta\varepsilon}\sigma^{\mu}{}_{\beta}\\
&-R^{\alpha}{}_{\rho\mu\beta;\gamma}\sigma^{\rho}{}_{\delta}\sigma^{\mu}{}_{\varepsilon}-R^{\alpha}{}_{\rho\mu\beta;\delta}\sigma^{\rho}{}_{\gamma}\sigma^{\mu}{}_{\varepsilon}\\
&\left.\left.-R^{\alpha}{}_{\rho\mu\gamma;\delta}\sigma^{\rho}{}_{\beta}\sigma^{\mu}{}_{\varepsilon}+R^{\rho}{}_{\beta\mu\gamma;\delta}\sigma^{\alpha}{}_{\rho}\sigma^{\mu}{}_{\varepsilon}\right)\right\}_{t\to0}\\
=&\left(2\left[\sigma^{\alpha}{}_{\beta\gamma\delta\varepsilon\mu}\right]+\left[\sigma^{\alpha}{}_{\delta\beta\gamma\varepsilon\mu}\right]+\left[\sigma^{\alpha}{}_{\gamma\beta\delta\varepsilon\mu}\right]\right)u^{\mu'}.
\end{aligned}
\tag{32}
$$

3.7. Transport equation for $U_{\alpha\beta';\gamma'\delta'}$

By swapping x and x' in Eq. (21) in Ref. [9], we obtain

$$
U_{\alpha\beta';\mu'}\sigma^{\mu'}+\frac{1}{2}\left(\sigma^{\mu'}{}_{\mu'}-4\right)U_{\alpha\beta'}=0. \tag{33}
$$

Differentiating the above equation with respect to x' and commuting covariant derivatives, we obtain

$$
\begin{aligned}
&U_{\alpha\beta';\gamma'\mu'}\sigma^{\mu'}+R^{\rho'}{}_{\beta'\mu'\gamma'}U_{\alpha\rho'}\sigma^{\mu'}+U_{\alpha\beta';\mu'}\sigma^{\mu'}{}_{\gamma'}\\
&+\frac{1}{2}U_{\alpha\beta'}\sigma^{\mu'}{}_{\mu'\gamma'}+\frac{1}{2}\left(\sigma^{\mu'}{}_{\mu'}-4\right)U_{\alpha\beta';\gamma'}=0.
\end{aligned}
\tag{34}
$$

Differentiating the above equation with respect to x', we obtain the initial value problem for $U_{\alpha\beta';\gamma'\delta'}$

$$
\begin{aligned}
&\frac{\mathrm{d}U_{\alpha\beta';\gamma'\delta'}}{\mathrm{d}t}-\left(\Gamma^{\lambda'}{}_{\beta'\mu'}U_{\alpha\lambda';\gamma'\delta'}\right.\\
&+\Gamma^{\lambda'}{}_{\gamma'\mu'}U_{\alpha\beta';\lambda'\delta'}+\Gamma^{\lambda'}{}_{\delta'\mu'}U_{\alpha\beta';\gamma'\lambda'}\\
&+R^{\lambda'}{}_{\beta'\delta'\mu'}U_{\alpha\lambda';\gamma'}\sigma^{\mu'}+R^{\lambda'}{}_{\gamma'\delta'\mu'}U_{\alpha\beta';\lambda'}\sigma^{\mu'}\\
&\left.+R^{\rho'}{}_{\beta'\gamma'\mu';\delta'}U_{\alpha\rho'}\sigma^{\mu'}+R^{\rho'}{}_{\beta'\gamma'\mu'}U_{\alpha\rho';\delta'}\right)u^{\mu'}\\
&+\frac{1}{2t}\left[U_{\alpha\beta';\delta'}\sigma^{\mu'}{}_{\mu'\gamma'}+U_{\alpha\beta'}\sigma^{\mu'}{}_{\mu'\gamma'\delta'}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+U_{\alpha\beta';\gamma}\sigma^{\mu'}{}_{\mu'\delta'}+\left(\sigma^{\mu'}{}_{\mu'}-4\right)U_{\alpha\beta';\gamma\delta'}\big]\\
&+\frac{1}{t}U_{\alpha\beta';\mu'}\sigma^{\mu'}{}_{\gamma\delta'}+\frac{1}{t}\big(U_{\alpha\beta';\gamma\mu'}\sigma^{\mu'}{}_{\delta'}\\
&+U_{\alpha\beta';\mu'\delta'}\sigma^{\mu'}{}_{\gamma'}+R^{\rho'}{}_{\beta'\mu'\gamma}U_{\alpha\rho'}\sigma^{\mu'}{}_{\delta'}\big)=0,\\
&U_{\alpha\beta';\alpha'_1\alpha'_2}\big|_{t=0}=\frac{1}{2}R_{\alpha\beta\alpha_1\alpha_2}. \qquad (35)
\end{aligned}
$$

We can write the following covariant expansion:

$$
\begin{aligned}
&U_{\alpha\beta';\gamma\mu'}\sigma^{\mu'}{}_{\delta'}+U_{\alpha\beta';\mu'\delta'}\sigma^{\mu'}{}_{\gamma}+R^{\rho'}{}_{\beta'\mu'\gamma}U_{\alpha\rho'}\sigma^{\mu'}{}_{\delta'}\\
&=2\left[U_{\alpha\beta';\gamma\delta'}\right]+2\left[U_{\alpha\beta';\gamma\delta'\nu'}\right]tu^{\nu}+R_{\alpha\beta\delta\gamma}+O\left(t^2\right),
\end{aligned}
$$

from which we can obtain the limit formula

$$
\begin{aligned}
&\left\{\frac{1}{t}\left(U_{\alpha\beta';\gamma\mu'}\sigma^{\mu'}{}_{\delta'}+U_{\alpha\beta';\mu'\delta'}\sigma^{\mu'}{}_{\gamma'}\right.\right.\\
&\left.\left.+R^{\rho'}{}_{\beta'\mu'\gamma'}U_{\alpha\rho'}\sigma^{\mu'}{}_{\delta'}\right)\right\}_{t\to0}\\
&=2\left[U_{\alpha\beta';\gamma'\delta'\nu'}\right]u^{\nu}=\frac{2}{3}\left(R_{\alpha\beta\gamma\delta;\nu}+R_{\alpha\beta\gamma\nu;\delta}\right)u^{\nu}, \qquad (36)
\end{aligned}
$$

where $[\cdot]$ denotes limit $x'\to x$. $U_{\alpha\beta'}$ is the product of parallel propagator $g_{\alpha\beta'}$ and Van Vleck determinant[11] Δ

$$
U_{\alpha\beta'}=g_{\alpha\beta'}\Delta^{1/2}. \qquad (37)
$$

Differentiating the above equation and then taking coincidence limit, we can acquire the coincidence limit of $U_{\alpha\beta'}$'s covariant derivative of each order.

3.8. Transport equation for $U_{\alpha\beta';\gamma'\delta}$

Differentiating Eq. (30) in Ref. [9] with respect to x', we obtain the initial value problem for $U_{\alpha\beta';\gamma'\delta}$

$$
\begin{aligned}
&\frac{\mathrm{d}U_{\alpha\beta';\gamma'\delta}}{\mathrm{d}t}-\left(\Gamma^{\lambda}{}_{\alpha\mu}U_{\lambda\beta';\gamma'\delta}\right.\\
&\left.+U_{\alpha\beta';\gamma'\lambda}\Gamma^{\lambda}{}_{\delta\mu}+R^{\rho}{}_{\alpha\delta\mu}U_{\rho\beta';\gamma'}\right)u^{\mu}\\
&+\frac{1}{t}\left[U_{\alpha\beta';\mu}\sigma^{\mu}{}_{\delta\gamma'}+\frac{1}{2}U_{\alpha\beta';\gamma'}\sigma^{\mu}{}_{\mu\delta}+\frac{1}{2}U_{\alpha\beta'}\sigma^{\mu}{}_{\mu\delta\gamma'}\right.\\
&\left.+\frac{1}{2}U_{\alpha\beta';\delta}\sigma^{\mu}{}_{\mu\gamma'}+\frac{1}{2}\left(\sigma^{\mu}{}_{\mu}-4\right)U_{\alpha\beta';\gamma'\delta}\right]\\
&+\frac{1}{t}\left(U_{\alpha\beta';\delta\mu}\sigma^{\mu}{}_{\gamma'}+R^{\rho}{}_{\alpha\mu\delta}U_{\rho\beta'}\sigma^{\mu}{}_{\gamma'}+U_{\alpha\beta';\gamma'\mu}\sigma^{\mu}{}_{\delta}\right)=0,\\
&U_{\alpha\beta';\gamma'\delta}\big|_{t=0}=-\frac{1}{2}R_{\alpha'\beta'\gamma'\delta'}. \qquad (38)
\end{aligned}
$$

We can write the following covariant expansion:

$$
\begin{aligned}
&U_{\alpha\beta';\delta\mu}\sigma^{\mu}{}_{\gamma'}+R^{\rho}{}_{\alpha\mu\delta}U_{\rho\beta'}\sigma^{\mu}{}_{\gamma'}+U_{\alpha\beta';\gamma'\mu}\sigma^{\mu}{}_{\delta}\\
&=-\left[U_{\alpha\beta';\delta\gamma}\right]-\left[U_{\alpha\beta';\delta\gamma\mu}\right]tu^{\mu'}-R_{\beta'\alpha'\gamma'\delta'}\\
&\quad+\left[U_{\alpha\beta';\gamma'\delta}\right]+\left[U_{\alpha\beta';\gamma'\delta\mu}\right]tu^{\mu'}+O(t^2),
\end{aligned}
$$

from which we can obtain the limit formula

$$
\begin{aligned}
&\left\{\frac{1}{t}\left(U_{\alpha\beta';\delta\mu}\sigma^{\mu}{}_{\gamma'}+R^{\rho}{}_{\alpha\mu\delta}U_{\rho\beta'}\sigma^{\mu}{}_{\gamma'}+U_{\alpha\beta';\gamma'\mu}\sigma^{\mu}{}_{\delta}\right)\right\}_{t\to0}\\
&=\left(-\left[U_{\alpha\beta';\delta\gamma\mu}\right]+\left[U_{\alpha\beta';\gamma'\delta\mu}\right]\right)u^{\mu'}\\
&=\left(-\frac{2}{3}R_{\alpha'\beta'\gamma'\delta';\mu}+\frac{1}{6}R_{\alpha'\beta'\delta'\mu';\gamma'}\right)u^{\mu'}, \qquad (39)
\end{aligned}
$$

where $[\cdot]$ denotes limit $x\to x'$.

3.9. Transport equation for $U_{\alpha\beta';\gamma\delta\varepsilon'}$

Differentiating Eq. (30) in Ref. [9] with respect to x, we obtain

$$
\begin{aligned}
&U_{\alpha\beta';\gamma\delta\mu}\sigma^{\mu}+R^{\rho}{}_{\alpha\mu\gamma;\delta}U_{\rho\beta'}\sigma^{\mu}+R^{\rho}{}_{\alpha\mu\gamma}U_{\rho\beta';\delta}\sigma^{\mu}\\
&+R^{\rho}{}_{\alpha\mu\delta}U_{\rho\beta';\gamma}\sigma^{\mu}+R^{\rho}{}_{\gamma\mu\delta}U_{\alpha\beta';\rho}\sigma^{\mu}+U_{\alpha\beta';\mu\gamma}\sigma^{\mu}{}_{\delta}\\
&+U_{\alpha\beta';\mu\delta}\sigma^{\mu}{}_{\gamma}+\frac{1}{2}U_{\alpha\beta'}\sigma^{\mu}{}_{\mu\gamma\delta}+U_{\alpha\beta';\mu}\sigma^{\mu}{}_{\gamma\delta}+\frac{1}{2}U_{\alpha\beta';\delta}\sigma^{\mu}{}_{\mu\gamma}\\
&+\frac{1}{2}U_{\alpha\beta';\gamma}\sigma^{\mu}{}_{\mu\delta}+\frac{1}{2}U_{\alpha\beta';\gamma\delta}\left(\sigma^{\mu}{}_{\mu}-4\right)=0. \qquad (40)
\end{aligned}
$$

Differentiating the above equation with respect to x', we obtain the initial value problem for $U_{\alpha\beta';\gamma\delta\varepsilon'}$

$$
\begin{aligned}
&\frac{\mathrm{d}U_{\alpha\beta';\gamma\delta\varepsilon'}}{\mathrm{d}t}-\left(\Gamma^{\lambda}{}_{\alpha\mu}U_{\lambda\beta';\gamma\delta\varepsilon'}+U_{\alpha\beta';\lambda\delta\varepsilon'}\Gamma^{\lambda}{}_{\gamma\mu}\right.\\
&+U_{\alpha\beta';\gamma\lambda\varepsilon'}\Gamma^{\lambda}{}_{\delta\mu}+R^{\rho}{}_{\alpha\gamma\mu;\delta}U_{\rho\beta';\varepsilon'}+R^{\rho}{}_{\alpha\gamma\mu}U_{\rho\beta';\varepsilon'\delta}\\
&\left.+R^{\rho}{}_{\alpha\delta\mu}U_{\rho\beta';\varepsilon'\gamma}+U_{\alpha\beta';\varepsilon'\rho}R^{\rho}{}_{\gamma\delta\mu}\right)u^{\mu}\\
&+\frac{1}{2t}\left[U_{\alpha\beta';\gamma\delta}\sigma^{\mu}{}_{\mu\varepsilon'}+U_{\alpha\beta';\varepsilon'\delta}\sigma^{\mu}{}_{\mu\gamma}+U_{\alpha\beta';\varepsilon'\gamma}\sigma^{\mu}{}_{\mu\delta}\right.\\
&+U_{\alpha\beta';\varepsilon'}\sigma^{\mu}{}_{\mu\gamma\delta}+U_{\alpha\beta';\delta}\sigma^{\mu}{}_{\mu\gamma\varepsilon'}\\
&\left.+U_{\alpha\beta';\gamma}\sigma^{\mu}{}_{\mu\delta\varepsilon'}+U_{\alpha\beta';\gamma\delta\varepsilon'}\left(\sigma^{\mu}{}_{\mu}-4\right)\right]\\
&+\frac{1}{t}\left(R^{\rho}{}_{\alpha\mu\gamma}U_{\rho\beta';\delta}\sigma^{\mu}{}_{\varepsilon'}+R^{\rho}{}_{\alpha\mu\delta}U_{\rho\beta';\gamma}\sigma^{\mu}{}_{\varepsilon'}\right.\\
&+U_{\alpha\beta';\rho}R^{\rho}{}_{\gamma\mu\delta}\sigma^{\mu}{}_{\varepsilon'}+U_{\alpha\beta';\mu\gamma}\sigma^{\mu}{}_{\delta\varepsilon'}\\
&+U_{\alpha\beta';\mu\delta}\sigma^{\mu}{}_{\gamma\varepsilon'}+U_{\alpha\beta';\varepsilon'\mu}\sigma^{\mu}{}_{\gamma\delta}\\
&\left.+U_{\alpha\beta';\mu}\sigma^{\mu}{}_{\gamma\delta\varepsilon'}+\frac{1}{2}U_{\alpha\beta'}\sigma^{\mu}{}_{\mu\gamma\delta\varepsilon'}\right)\\
&+\frac{1}{t}\left(U_{\alpha\beta';\mu\gamma\varepsilon'}\sigma^{\mu}{}_{\delta}+U_{\alpha\beta';\mu\delta\varepsilon'}\sigma^{\mu}{}_{\gamma}\right.\\
&\left.+U_{\alpha\beta';\gamma\delta\mu}\sigma^{\mu}{}_{\varepsilon'}+R^{\rho}{}_{\alpha\mu\gamma;\delta}U_{\rho\beta'}\sigma^{\mu}{}_{\varepsilon'}\right)=0,\\
&U_{\alpha\beta';\alpha_1\alpha_2\alpha'_3}\big|_{t=0}=-\frac{1}{6}R_{\alpha'\beta'\alpha'_1\alpha'_2;\alpha'_3}+\frac{1}{3}R_{\alpha'\beta'\alpha'_1\alpha'_3;\alpha'_2}. \qquad (41)
\end{aligned}
$$

We can write the following covariant expansion:

$$
\begin{aligned}
&U_{\alpha\beta';\mu\gamma\varepsilon'}\sigma^{\mu}{}_{\delta}+U_{\alpha\beta';\mu\delta\varepsilon'}\sigma^{\mu}{}_{\gamma}\\
&+U_{\alpha\beta';\gamma\delta\mu}\sigma^{\mu}{}_{\varepsilon'}+R^{\rho}{}_{\alpha\mu\gamma;\delta}U_{\rho\beta'}\sigma^{\mu}{}_{\varepsilon'}\\
&=\left[U_{\alpha\beta';\delta\gamma\varepsilon'}\right]+\left[U_{\alpha\beta';\delta\gamma\varepsilon'\mu}\right]tu^{\mu'}+\left[U_{\alpha\beta';\gamma\delta\varepsilon'}\right]\\
&\quad+\left[U_{\alpha\beta';\gamma\delta\varepsilon'\mu}\right]tu^{\mu'}-\left[U_{\alpha\beta';\gamma\delta\varepsilon}\right]\\
&\quad-\left[U_{\alpha\beta';\gamma\delta\varepsilon\mu}\right]tu^{\mu'}+R_{\alpha'\beta'\varepsilon'\gamma';\delta'}+O(t^2),
\end{aligned}
$$

from which we can obtain the limit formula

$$
\begin{aligned}
&\left\{\frac{1}{t}\left(U_{\alpha\beta';\mu\gamma\varepsilon'}\sigma^{\mu}{}_{\delta}+U_{\alpha\beta';\mu\delta\varepsilon'}\sigma^{\mu}{}_{\gamma}\right.\right.\\
&\left.\left.+U_{\alpha\beta';\gamma\delta\mu}\sigma^{\mu}{}_{\varepsilon'}+R^{\rho}{}_{\alpha\mu\gamma;\delta}U_{\rho\beta'}\sigma^{\mu}{}_{\varepsilon'}\right)\right\}_{t\to0}\\
&=\left(\left[U_{\alpha\beta';\delta\gamma\varepsilon'\mu}\right]+\left[U_{\alpha\beta';\gamma\delta\varepsilon'\mu}\right]-\left[U_{\alpha\beta';\gamma\delta\varepsilon\mu}\right]\right)u^{\mu'}. \qquad (42)
\end{aligned}
$$

3.10. Transport equation for $U_{\alpha\beta';\gamma\delta\varepsilon}$

Differentiating Eq. (40) with respect to x, we obtain the initial value problem for $U_{\alpha\beta';\gamma\delta\varepsilon}$

$$\begin{aligned}
&\frac{\mathrm{d}U_{\alpha\beta';\gamma\delta\varepsilon}}{\mathrm{d}t}-\left(\Gamma^{\lambda}{}_{\alpha\mu}U_{\lambda\beta';\gamma\delta\varepsilon}+U_{\alpha\beta';\lambda\delta\varepsilon}\Gamma^{\lambda}{}_{\gamma\mu}\right.\\
&+U_{\alpha\beta';\gamma\lambda\varepsilon}\Gamma^{\lambda}{}_{\delta\mu}+U_{\alpha\beta';\gamma\delta\lambda}\Gamma^{\lambda}{}_{\varepsilon\mu}\\
&+R^{\lambda}{}_{\alpha\varepsilon\mu}U_{\lambda\beta';\gamma\delta}+R^{\rho}{}_{\alpha\delta\mu}U_{\rho\beta';\gamma\varepsilon}+R^{\rho}{}_{\alpha\gamma\mu}U_{\rho\beta';\delta\varepsilon}\\
&+U_{\alpha\beta';\lambda\delta}R^{\lambda}{}_{\gamma\varepsilon\mu}+U_{\alpha\beta';\rho\varepsilon}R^{\rho}{}_{\gamma\delta\mu}+U_{\alpha\beta';\gamma\lambda}R^{\lambda}{}_{\delta\varepsilon\mu}\\
&+R^{\rho}{}_{\alpha\gamma\mu;\delta}U_{\rho\beta';\varepsilon}+R^{\rho}{}_{\alpha\gamma\mu;\varepsilon}U_{\rho\beta';\delta}+R^{\rho}{}_{\alpha\delta\mu;\varepsilon}U_{\rho\beta';\gamma}\\
&\left.+U_{\alpha\beta';\rho}R^{\rho}{}_{\gamma\delta\mu;\varepsilon}+R^{\rho}{}_{\alpha\gamma\mu;\delta\varepsilon}U_{\rho\beta'}\right)u^{\mu}\\
&+\frac{1}{2t}\left[U_{\alpha\beta';\gamma\delta\varepsilon}\left(\sigma^{\mu}{}_{\mu}-4\right)+U_{\alpha\beta';\gamma\delta}\sigma^{\mu}{}_{\mu\varepsilon}\right.\\
&+U_{\alpha\beta';\delta\varepsilon}\sigma^{\mu}{}_{\mu\gamma}+U_{\alpha\beta';\gamma\varepsilon}\sigma^{\mu}{}_{\mu\delta}\\
&\left.+U_{\alpha\beta';\gamma}\sigma^{\mu}{}_{\mu\delta\varepsilon}+U_{\alpha\beta';\delta}\sigma^{\mu}{}_{\mu\gamma\varepsilon}+U_{\alpha\beta';\varepsilon}\sigma^{\mu}{}_{\mu\gamma\delta}\right]\\
&+\frac{1}{t}\left(U_{\alpha\beta';\mu\gamma}\sigma^{\mu}{}_{\delta\varepsilon}+U_{\alpha\beta';\mu\delta}\sigma^{\mu}{}_{\gamma\varepsilon}\right.\\
&+U_{\alpha\beta';\mu\varepsilon}\sigma^{\mu}{}_{\gamma\delta}+U_{\alpha\beta';\mu}\sigma^{\mu}{}_{\gamma\delta\varepsilon}\\
&+\frac{1}{2}U_{\alpha\beta'}\sigma^{\mu}{}_{\mu\gamma\delta\varepsilon}+R^{\rho}{}_{\alpha\mu\gamma}U_{\rho\beta';\delta}\sigma^{\mu}{}_{\varepsilon}\\
&+R^{\rho}{}_{\alpha\mu\delta}U_{\rho\beta';\gamma}\sigma^{\mu}{}_{\varepsilon}+U_{\alpha\beta';\rho}R^{\rho}{}_{\gamma\mu\delta}\sigma^{\mu}{}_{\varepsilon}\big)\\
&+\frac{1}{t}\left(U_{\alpha\beta';\mu\gamma\varepsilon}\sigma^{\mu}{}_{\delta}+U_{\alpha\beta';\mu\delta\varepsilon}\sigma^{\mu}{}_{\gamma}\right.\\
&\left.+U_{\alpha\beta';\gamma\delta\mu}\sigma^{\mu}{}_{\varepsilon}+R^{\rho}{}_{\alpha\mu\gamma;\delta}U_{\rho\beta'}\sigma^{\mu}{}_{\varepsilon}\right)=0,\\
&U_{\alpha\beta';\gamma\delta\varepsilon}\big|_{t=0}=-\frac{1}{3}\left(R_{\alpha'\beta'\gamma'\delta';\varepsilon'}+R_{\alpha'\beta'\gamma'\varepsilon';\delta'}\right). \qquad (43)
\end{aligned}$$

We can write the following covariant expansion:

$$\begin{aligned}
&U_{\alpha\beta';\gamma\delta\mu}\sigma^{\mu}{}_{\varepsilon}+U_{\alpha\beta';\mu\gamma\varepsilon}\sigma^{\mu}{}_{\delta}\\
&+U_{\alpha\beta';\mu\delta\varepsilon}\sigma^{\mu}{}_{\gamma}+R^{\rho}{}_{\alpha\mu\gamma;\delta}U_{\rho\beta'}\sigma^{\mu}{}_{\varepsilon}\\
&2\left[U_{\alpha\beta';\gamma\delta\varepsilon}\right]+2\left[U_{\alpha\beta';\gamma\delta\varepsilon\mu}\right]tu^{\mu'}+\left[U_{\alpha\beta';\delta\gamma\varepsilon}\right]\\
&+\left[U_{\alpha\beta';\delta\gamma\varepsilon\mu}\right]tu^{\mu'}+R_{\beta'\alpha'\varepsilon'\gamma';\delta'}+O(t^2),
\end{aligned}$$

from which we can obtain the limit formula

$$\begin{aligned}
&\left\{\frac{1}{t}\left(U_{\alpha\beta';\gamma\delta\mu}\sigma^{\mu}{}_{\varepsilon}+U_{\alpha\beta';\mu\gamma\varepsilon}\sigma^{\mu}{}_{\delta}\right.\right.\\
&\left.\left.+U_{\alpha\beta';\mu\delta\varepsilon}\sigma^{\mu}{}_{\gamma}+R^{\rho}{}_{\alpha\mu\gamma;\delta}U_{\rho\beta'}\sigma^{\mu}{}_{\varepsilon}\right)\right\}_{t\to0}\\
&=\left(2\left[U_{\alpha\beta';\gamma\delta\varepsilon\mu}\right]+\left[U_{\alpha\beta';\delta\gamma\varepsilon\mu}\right]\right)u^{\mu'}. \qquad (44)
\end{aligned}$$

3.11. Transport equation for $V^0_{\alpha\beta';\gamma'}$

Differentiating the second equation of Eq. (23) in Ref. [9] with respect to x', we obtain

$$\begin{aligned}
&V^0_{\alpha\beta';\gamma'\mu}\sigma^{\mu}+V^0_{\alpha\beta';\mu}\sigma^{\mu}{}_{\gamma'}+\frac{1}{2}V^0_{\alpha\beta'}\sigma^{\mu}{}_{\mu\gamma'}\\
&+\frac{1}{2}\left(\sigma^{\mu}{}_{\mu}-2\right)V^0_{\alpha\beta';\gamma'}=\frac{1}{2}\left(\Box U_{\alpha\beta'}\right)_{;\gamma'}. \qquad (45)
\end{aligned}$$

The initial value problem for $V^0_{\alpha\beta';\gamma'}$ is

$$\begin{aligned}
&\frac{\mathrm{d}V^0_{\alpha\beta';\gamma'}}{\mathrm{d}t}-\Gamma^{\lambda}{}_{\alpha\mu}V^0_{\lambda\beta';\gamma'}u^{\mu}+\frac{1}{t}\left[\frac{1}{2}V^0_{\alpha\beta'}\sigma^{\mu}{}_{\mu\gamma'}+V^0_{\alpha\beta';\mu}\sigma^{\mu}{}_{\gamma'}\right.\\
&\left.+\frac{1}{2}\left(\sigma^{\mu}{}_{\mu}-2\right)V^0_{\alpha\beta';\gamma'}-\frac{1}{2}\left(\Box U_{\alpha\beta'}\right)_{;\gamma'}\right]=0,\\
&V^0_{\alpha\beta';\gamma'}\big|_{t=0}=0. \qquad (46)
\end{aligned}$$

We have limit formula

$$\left(\frac{V^0_{\alpha\beta';\gamma'}}{t}\right)_{t\to0}=\left[V^0_{\alpha\beta';\gamma'\mu}\right]u^{\mu'}=-\frac{1}{6}\left[\left(\Box U_{\alpha\beta'}\right)_{;\mu\gamma}\right]u^{\mu'}, \qquad (47)$$

where $[V^0_{\alpha\beta';\gamma'\mu}]$ can be acquired by differentiating Eq. (45) and then taking the coincidence limit. The limit formula

$$\left(\frac{\left(\Box U_{\alpha\beta'}\right)_{;\gamma'}}{t}\right)_{t\to0}=\left[\left(\Box U_{\alpha\beta'}\right)_{;\gamma'\mu}\right]u^{\mu'} \qquad (48)$$

can be acquired by the covariant expansion of $\left(\Box U_{\alpha\beta'}\right)_{;\gamma'}$.

3.12. Transport equation for $V^0_{\alpha\beta';\gamma}$

Differentiating the second equation of Eq. (23) in Ref. [9] with respect to x, we obtain

$$\begin{aligned}
&V^0_{\alpha\beta';\gamma\mu}\sigma^{\mu}+R^{\lambda}{}_{\alpha\mu\gamma}V^0_{\lambda\beta'}\sigma^{\mu}\\
&+\frac{1}{2}\left[\sigma^{\mu}{}_{\mu\gamma}V^0_{\alpha\beta'}+\left(\sigma^{\mu}{}_{\mu}-2\right)V^0_{\alpha\beta';\gamma}\right]\\
&+V^0_{\alpha\beta';\mu}\sigma^{\mu}{}_{\gamma}-\frac{1}{2}\left(\Box U_{\alpha\beta'}\right)_{;\gamma}=0. \qquad (49)
\end{aligned}$$

The initial value problem for $V^0_{\alpha\beta';\gamma}$ is

$$\begin{aligned}
&\frac{\mathrm{d}V^0_{\alpha\beta';\gamma}}{\mathrm{d}t}-\left(\Gamma^{\lambda}{}_{\alpha\mu}V^0_{\lambda\beta';\gamma}+V^0_{\alpha\beta';\lambda}\Gamma^{\lambda}{}_{\gamma\mu}\right.\\
&\left.+R^{\lambda}{}_{\alpha\gamma\mu}V^0_{\lambda\beta'}\right)u^{\mu}+\frac{1}{2t}V^0_{\alpha\beta'}\sigma^{\mu}{}_{\mu\gamma}\\
&+\frac{1}{t}\left[\frac{1}{2}\left(\sigma^{\mu}{}_{\mu}-2\right)V^0_{\alpha\beta';\gamma}+V^0_{\alpha\beta';\mu}\sigma^{\mu}{}_{\gamma}-\frac{1}{2}\left(\Box U_{\alpha\beta'}\right)_{;\gamma}\right]=0,\\
&V^0_{\alpha\beta';\gamma}\big|_{t=0}=0. \qquad (50)
\end{aligned}$$

We have limit formulas which are similar to Eqs. (47) and (48)

$$\left(\frac{V^0_{\alpha\beta';\gamma}}{t}\right)_{t\to0}=\left[V^0_{\alpha\beta';\gamma\mu}\right]u^{\mu'}=\frac{1}{6}\left[\left(\Box U_{\alpha\beta'}\right)_{;\gamma\mu}\right]u^{\mu'}, \qquad (51)$$

$$\left(\frac{\left(\Box U_{\alpha\beta'}\right)_{;\gamma}}{t}\right)_{t\to0}=\left[\left(\Box U_{\alpha\beta'}\right)_{;\gamma\mu}\right]u^{\mu'}. \qquad (52)$$

4. Numerical results

The computing platform is configured as follows. CPU: Intel Xeon E3-1240 v3; memory: 32GB; operating system: Windows 10 professional; language: MATLAB.

The first example is to validate the connection boundary. The Schwarzschild radius is set to 1 m. There is no scatterer in FDTD domain (Fig. 3). The FDTD mesh size is set to 0.02 m. The FDTD domain is: 5 m to 6 m in the x direction, -0.5 m to 0.5 m in the y direction, and -0.5 m to 0.5 m in the z direction. The number of perfectly matched layer (PML) is set to

10. The connection boundary is: $x = 5.5 \pm 0.2$ m, $y = \pm 0.2$ m, and $z = \pm 0.2$ m. A z-directed electric dipole is placed at P_s (20 m, 0 m, 0 m) (the midpoint of the two charges). The waveform of charge is a Gaussian pulse

$$q(t) = A \cdot \exp\left[-\frac{1}{2}\left(\frac{t}{\sigma} - 4\right)^2\right], \tag{53}$$

where $A = 1 \times 10^{-5}$ C and $\sigma = 0.966$ ns. The z component of the electric field at P (5.5 m, 0 m, 0 m) is calculated by FDTD method with Taylor approximation and Green function method without approximation (calculate Green function linking P_s to P). The approximation error ε is set to 0.01. The results are shown in Fig. 4 in which the horizontal axis represents the light speed multiplying time (the unit is meter).

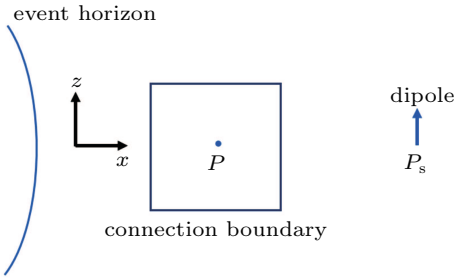


Fig. 3. Validating the connection boundary.

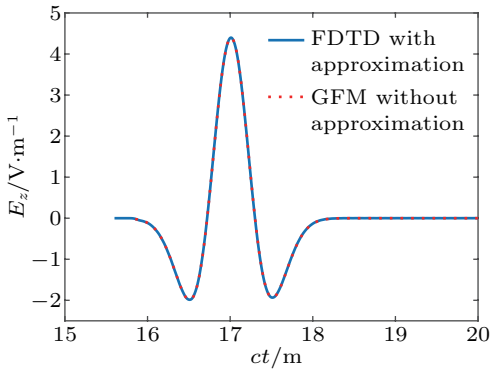


Fig. 4. E_z at P.

To calculate the single point Green function, the old program needs to calculate 8 quantities (see Eq. (10)). In the new version, it needs to calculate 20 quantities (see Eqs. (10) and (12)), which is 2.5 times as much as the old version. This is only a rough estimation because the time required to calculate different quantities is also different. Although the new version for the single point Green function is more computational, it needs to calculate a lot fewer points than the old version. In this example, there are $20 \times 20 \times 6 = 2400$ elements on the connection boundary, so the old version program needs to calculate the Green function at 2400 points. The new version program only needs to calculate the Green function at six central points on the connection boundary; i.e., $1/400$ of the old version. Therefore, the calculation amount of the new version is about $2.5/400 = 0.625\%$ of the old version. The time comparison between the new version and the old one are shown in Table 1.

Table 1. Time comparison for connection boundary.

	Old version	New version
CPU time	569 hr 21 min	6 hr 17 min

When changing the waveform of source into a differential Gaussian pulse

$$q(t) = A\sqrt{e}\left(\frac{t}{\sigma} - 4\right)\exp\left[-\frac{1}{2}\left(\frac{t}{\sigma} - 4\right)^2\right], \tag{54}$$

and the results are shown in Fig. 5.

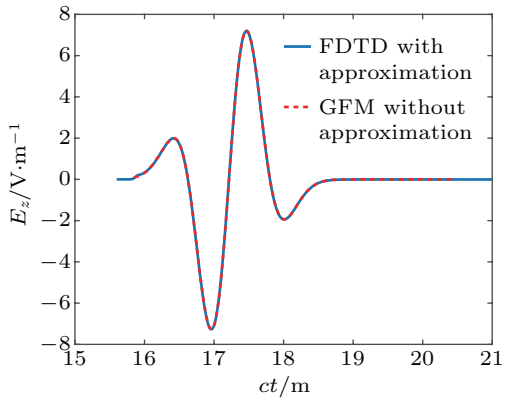


Fig. 5. E_z at P.

The second example is to validate the output boundary. The Schwarzschild radius is set to 1 m. A z-directed electric dipole is placed at P_s (5.5 m, 0 m, 0 m). The waveform of charge is a Gaussian pulse with $A = 1 \times 10^{-9}$ C and $\sigma = 0.966$ ns (Eq. (53)). The FDTD mesh size is set to 0.02 m and the number of PML layers is set to 10. The FDTD domain is: 5 m to 6 m in the x direction, -0.5 m to 0.5 m in the y direction, and -0.5 m to 0.5 m in the z direction. The output boundary is: $x = 5.5 \pm 0.02$ m, $y = \pm 0.02$ m, and $z = \pm 0.02$ m (Fig. 6). The z component of the electric field at P (20 m, 0 m, 0 m) is calculated with FDTD method with Taylor approximation and Green function method without approximation (calculate Green function linking P_s to P). The approximation error ε is set to 0.01. The results are shown in Fig. 7. The pulse amplitude of this example is several orders smaller than that of the first one. According to Eq. (12) in Ref. [9], the closer to the event horizon, the greater the equivalent permittivity and permeability, and the greater the energy density. Therefore, the magnitude of the response is the same order of magnitude as that of the first example. The number of surface element on output boundary is $20 \times 20 \times 6$. With the new version program, only the six midpoints are calculated. The time comparison between the new method and the old one is shown in Table 2.

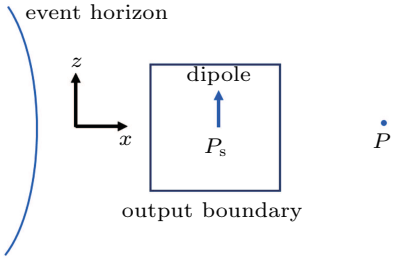


Fig. 6. Validating the output boundary.

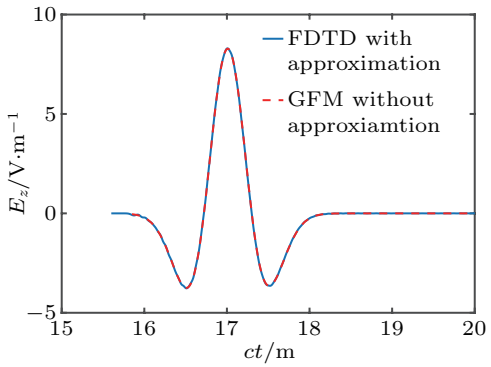


Fig. 7. E_z at P.

Table 2. Time comparison for output boundary.

	Old version	New version
CPU time	570 hr 43 min	4 hr 23 min

The third example is scattering by a thin plate. The size of the thin perfectly electric conductor (PEC) plate is 0.28 m×0.28 m. It spreads out in the plane $x = 5.5$ m (Fig. 8), and the center is located at (5.5 m, 0 m, 0 m). The Schwarzschild radius is set to 1 m. A z-directed electric dipole is placed at P (20 m, 0 m, 0 m). The waveform of charge is a Gaussian pulse with $A = 1$ C and $\sigma = 0.483$ ns (Eq. (53)). The FDTD mesh size is set to 0.01 m, and the number of PML layers is set to 10. The FDTD domain is: 5.25 m to 5.75 m in the x direction, −0.35 m to 0.35 m in the y direction, and −0.35 m to 0.35 m in the z direction. The connection boundary is: $x = 5.5 \pm 0.05$ m, $y = \pm 0.17$ m, and $z = \pm 0.17$ m. The output boundary is: $x = 5.5 \pm 0.1$ m, $y = \pm 0.2$ m, and $z = \pm 0.2$ m. The approximation error ε is set to 0.01. The number of surface element on connection boundary is $34 \times 34 \times 2 + 10 \times 34 \times 4$ and the number of surface element on output boundary is $40 \times 40 \times 2 + 20 \times 40 \times 4$. With the new version program, only the six midpoints of connection boundary and the six midpoints of output boundary are calculated for Green function. The time comparison between the new method and the old one is shown in Table 3. The z component of the scattered electric field (both in time domain and frequency domain) at P is shown in Fig. 9. The scattered electric field in flat space-time is also shown in Fig. 9. The effective light speed is smaller than that in flat space-time. This leads to time delay which is shown in Fig. 9(a). The inhomogeneity leads to pulse broading in time domain and red shift in frequency domain, which is shown in Fig. 9(b).

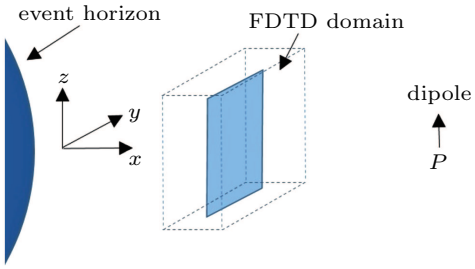


Fig. 8. Scattering by a thin PEC plate.

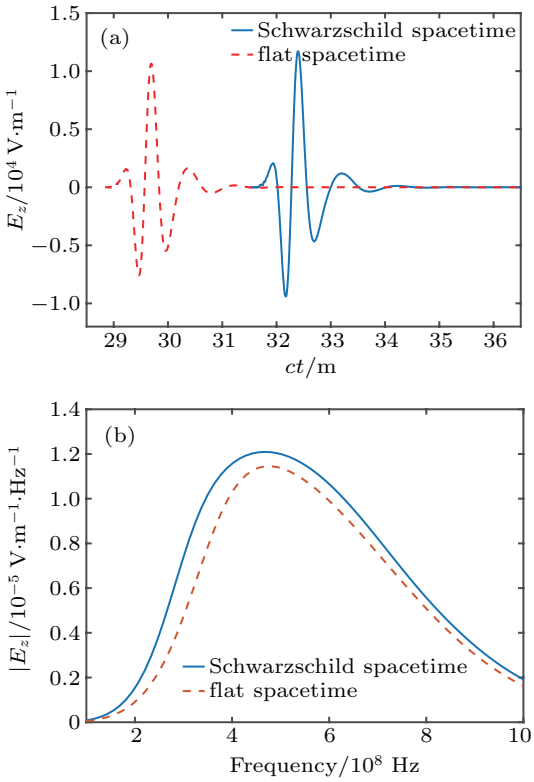


Fig. 9. E_z at P. (a) Time domain and (b) frequency domain.

Table 3. Time comparison for scattering.

	Old version	New version
CPU time	2378 hr 14 min	10 hr 23 min

5. Summary and discussion

To reduce the computation workload of the Green function, the Taylor approximation is applied. The transport equations for Green function's covariant derivatives are deduced. The connection/output boundary can be divided into subareas. The Green function linking source/observer to the midpoint of these subareas can then be calculated, while the Green

function linking source/observer to the other points on connection/output boundary can be obtained through Taylor approximation. The numerical results verify the effectiveness of this method. Although the amount of computation of single point's Green function is increased, the total computation workload is greatly reduced because the number of computation points for Green function is greatly reduced, which is helpful in calculating the scattering of electrically extended objects.

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