ISSN 1674-1056
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Citation: Chin. Phys. B . 2019, 28(7): 070304. **doi:** 10.1088/1674-1056/28/7/070304
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Robustness self-testing of states and measurements in the prepare-and-measure scenario with $3 \to 1$ random access code*

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(Received 23 December 2018; revised manuscript received 13 May 2019; published online 24 June 2019)

Recently, Tavakoli *et al.* proposed a self-testing scheme in the prepare-and-measure scenario, showing that self-testing is not necessarily based on entanglement and violation of a Bell inequality [*Phys. Rev. A* **98** 062307 (2018)]. They realized the self-testing of preparations and measurements in an $N \to 1$ ($N \geq 2$) random access code (RAC), and provided robustness bounds in a $2 \to 1$ RAC. Since all $N \to 1$ RACs with shared randomness are combinations of $2 \to 1$ and $3 \to 1$ RACs, the $3 \to 1$ RAC is just as important as the $2 \to 1$ RAC. In this paper, we find a set of preparations and measurements in the $3 \to 1$ RAC, and use them to complete the robustness self-testing analysis in the prepare-and-measure scenario. The method is robust to small but inevitable experimental errors.

Keywords: robustness self-testing, prepare-and-measure scenario, $3 \to 1$ random access code

PACS: 03.65.Ud, 03.67.–a, 03.67.Dd **DOI:** 10.1088/1674-1056/28/7/070304

1. Introduction

Quantum devices are used in all quantum information processing tasks. The credibility of quantum devices (whether it works as the quantum form claimed by the device provider) is crucial for the success of quantum information processing tasks. Without opening the device, it is difficult to judge whether a quantum device is credible. Self-testing is an effective way to solve this problem and it has successfully been employed in many protocols, such as quantum key distribution,[1,2] randomness expansion,[3,4] and verification of quantum computations.[5–8] Under the sole assumptions of no-signaling and the validity of quantum theory, self-testing refers to determining the nature of a physical system or device, without knowing any detail of the inner mechanism of the system. The only information required is the statistics from the experiments.

The possibility of self-testing has been known since 1992.[9] The maximal violation of the Clauser–Horne–Shimony–Holt inequality[10] indicates that the state in the device is the maximum entangled state of two particles. In 2004, Mayers and Yao[11] first came up with the term "self-testing," and proposed a self-testing scheme under a device-independent framework to certify the presence of a quantum state and the structure of a set of measurement operators. This inspired further works by McKague *et al.*,[12] who simplified the proof of Mayers and Yao and made robustness analysis to small fluctuations. There has also been growing interest in designing self-testing methods.[13–20] One of the main issues in self-testing is that it should give us a robust tolerance to experimental fluctuations (such as imperfect experimental equipment, statistical errors, etc.). Some works[13–16] not only gave ways to self-test states in the ideal situation, but also provided robustness bounds for some experimental imperfections. In 2016, Kaniewski[21] presented a nearly optimal robustness bound by using a new technique with the help of extraction channels and operator inequalities.

Recently, Tavakoli *et al.*[22] presented a self-testing method for quantum prepare-and-measure experiments that was not just based on entanglement and nonlocality. They discussed how to self-test the preparations and measurements in an $N \to 1$ random access code (RAC), and provided robustness bounds in a $2 \to 1$ RAC in detail. From Ref. [23], we know that the $2 \xrightarrow{0.85} 1$ and $3 \xrightarrow{0.79} 1$ RAC can be implemented, while an $N \xrightarrow{p} 1$ ($n \geq 4$) RAC with $p > \frac{1}{2}$ is not possible without shared randomness. Reference [23] also tells us that all $N \xrightarrow{p} 1$ RACs with shared randomness exist, and can be implemented by some combinations of $2 \to 1$ and $3 \to 1$ RACs. Therefore, it is necessary to analyze the robustness of self-testing in the $3 \to 1$ RAC. In this paper, we find a set of preparations and measurements and complete the robustness analysis. The method allows one to assess the lower bound of average fidelity between the unknown preparations (measurements) and a set of ideal quantum states (measurements).

2. The $3 \to 1$ RAC

We first illustrate the quantum prepare-and-measure experiments in which two black boxes in the safe area should be considered. For convenience, we call one of the black boxes Alice and the other Bob. Alice features N buttons that label the prepared states. When button x is pressed, Alice emits the state

*Project supported by the National Natural Science Foundation of China (Grant Nos. 61572081, 61672110, and 61671082).
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ρ_x ($x \in \{0,1,\ldots,N-1\}$). The prepared state is then sent to Bob. Bob performs a measurement M_y ($y \in \{0,1,\ldots,m-1\}$) on the state and obtains the outcome $b \in \{0,1,\ldots,k-1\}$. Note that both the specific states ρ_x and measurements M_y^b are priori unknown to the observer. The experiment statistics is thus described by the probability distribution $P(b|x,y) = \mathrm{tr}(\rho_x M_y^b)$, given the probability of obtaining the outcome b when the measurement M_y is performed on the prepared state ρ_x. A function f is then termed a quantum dimension witness if

$$f[P(b|x,y)] \leq Q_d, \tag{1}$$

for all experiments involving quantum systems of Hilbert space with dimension less than or equal to d, and there exists a set of data $P^*(b|x,y)$ involving systems in higher dimensions such that $f[P^*(b|x,y)] > Q_d$. A classical dimension witness is defined in a similar way, letting the upper bound C_d replace Q_d. From Refs. [24]–[26], we know that for a given system dimension d, quantum systems outperform classical systems because certain quantum distributions cannot be reproduced by classical ones. This quantum advantage is seen by Tavakoli *et al.* as the origin of the possibility of developing self-testing methods for the prepare-and-measure scenario.[22] This is similar to Bell inequality violation being the root for self-testing entangled states. Here we will consider the case $d = 2$.

In Ref. [22], it was shown how to self-test the preparations and measurements in an $N \to 1$ RAC, and the authors provided robustness bounds in a $2 \to 1$ RAC. In this paper, we study the robustness self-testing of preparations and measurements in a $3 \to 1$ RAC. Let us first review the $3 \to 1$ RAC in the prepare-and-measure experiment. There are eight possible preparations, denoted by $x = x_0x_1x_2$ ($x_0, x_1, x_2 \in \{0,1\}$), and three possible binary measurements M_y ($y \in \{0,1,2\}$) and $b \in \{0,1\}$, as shown in Fig. 1.

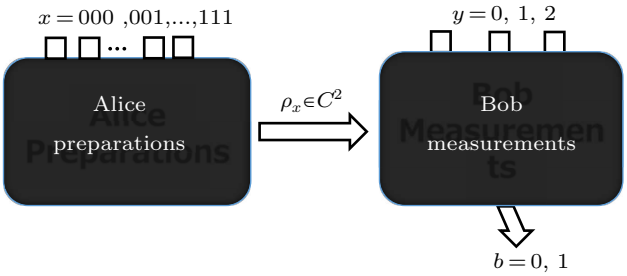


Fig. 1. Sketch of the $3 \to 1$ RAC. Alice prepares a qubit ρ_x according to her three classical bits $x \in \{000, 001, \cdots, 111\}$ and sends it to Bob. Bob performs his measurement on the received qubit depending on his input $y \in \{0,1,2\}$ and gets the measurement result denoted as $b \in \{0,1\}$.

The score is given by

$$\mathcal{A}_3 = \frac{1}{24} \sum_{x_0,x_1,x_2,y} p(b = x_y|x_0,x_1,x_2,y). \tag{2}$$

This means that the measurement device will return the output b when receiving input y. We choose the b that equals x_y to obtain $\mathcal{A}_3$, where x_y is the y-th bit of the input bit-string x received by the preparation device. Note that all inputs are assumed to be chosen uniformly at random. The upper bound of $\mathcal{A}_3$ corresponding to the quantum system with $d = 2$ is $Q_2^{3\to1} = \frac{1}{2} + \frac{1}{2\sqrt{3}} \approx 0.79$ according to Chuang (see also Ref. [27] and Ref. [28] for more details). The maximum value of $\mathcal{A}_3$ corresponding to the classical system is $C_2^{3\to1} = \frac{3}{4}$. For the convenience of robustness analysis in the following article, especially in constructing a general form that covers all possible choices of three observables on the Bloch sphere, the quantum bound $Q_2^{3\to1}$ can be obtained via the following "ideal" strategy. The eight preparations are chosen as

$$\rho_{x_0x_1x_2}^{\mathrm{ideal}} = \frac{\mathbb{I} + \boldsymbol{r}_{x_0x_1x_2} \cdot \boldsymbol{\sigma}}{2}, \tag{3}$$

where

$$\begin{aligned}
&\boldsymbol{r}_{000} = \left(\frac{\sqrt{6}}{3}, \frac{1}{\sqrt{3}}, 0\right), \quad \boldsymbol{r}_{001} = \left(\frac{\sqrt{6}}{3}, -\frac{1}{\sqrt{3}}, 0\right),\\
&\boldsymbol{r}_{010} = \left(0, \frac{1}{\sqrt{3}}, \frac{\sqrt{6}}{3}\right), \quad \boldsymbol{r}_{100} = \left(0, \frac{1}{\sqrt{3}}, -\frac{\sqrt{6}}{3}\right),\\
&\boldsymbol{r}_{111} = \left(-\frac{\sqrt{6}}{3}, -\frac{1}{\sqrt{3}}, 0\right), \quad \boldsymbol{r}_{110} = \left(-\frac{\sqrt{6}}{3}, \frac{1}{\sqrt{3}}, 0\right),\\
&\boldsymbol{r}_{101} = \left(0, -\frac{1}{\sqrt{3}}, -\frac{\sqrt{6}}{3}\right), \quad \boldsymbol{r}_{011} = \left(0, -\frac{1}{\sqrt{3}}, \frac{\sqrt{6}}{3}\right).
\end{aligned}$$

We can derive the measurements as follows:

$$\begin{aligned}
&M_0^{\mathrm{ideal}} = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z), \quad M_1^{\mathrm{ideal}} = \frac{1}{\sqrt{2}}(\sigma_x - \sigma_z),\\
&M_2^{\mathrm{ideal}} = \sigma_y.
\end{aligned} \tag{4}$$

Next, we will discuss in detail the robustness self-testing of quantum states and measurements in the prepare-and-measure scenario with the $3 \to 1$ RAC.

3. Robustness self-testing of preparations

The idea of robustness in self-testing is to reasonably estimate the distance between the realized preparations and the ideal ones when experimental errors or any kind of imperfections are presented in the statistics. We discuss the robustness of the above self-testing based on $\mathcal{A}_3$. The average fidelity of the preparations with respect to the ideal ones is

$$G(\{\rho_{x_0x_1x_2}\}) = \frac{1}{8} \max_{\Lambda} \sum_{x_0,x_1,x_2} F\left(\rho_{x_0x_1x_2}^{\mathrm{ideal}}, \Lambda[\rho_{x_0x_1x_2}]\right), \tag{5}$$

where Λ is a qubit and completely positive trace-preserving map, and the maximum is taken over all quantum channels. Here the fidelities $F(\rho_{x_0x_1x_2}^{\mathrm{ideal}}, \Lambda[\rho_{x_0x_1x_2}]) = \mathrm{tr}(\Lambda[\rho_{x_0x_1x_2}]\rho_{x_0x_1x_2}^{\mathrm{ideal}})$, and $\rho_{x_0x_1x_2}^{\mathrm{ideal}}$ are pure states. We derive the lower bound of G given a value of $\mathcal{A}_3$, i.e.,

$$\mathcal{F}(\mathcal{A}_3) = \min_{\{\rho_{x_0x_1x_2}\} \in R(\mathcal{A}_3)} G(\{\rho_{x_0x_1x_2}\}), \tag{6}$$

where $R(\mathcal{A}_3)$ represents the set of preparations compatible with a certain value of $\mathcal{A}_3$.

We first rewrite $\mathcal{A}_3 = \frac{1}{2} + \sum_{x_0,x_1,x_2} \mathrm{tr}(\rho_x W_{x_0x_1x_2})$, where $W_{x_0x_1x_2} = \frac{1}{48}\sum_y (-1)^{x_y} M_y$. We define operators corresponding to the chosen channels and the ideal preparations

$$K_{x_0x_1x_2}(M_0,M_1,M_2) = \Lambda^{\dagger}(M_0,M_1,M_2)[\rho^{\mathrm{ideal}}_{x_0x_1x_2}], \tag{7}$$

where $\Lambda^{\dagger}$ refers to the dual channel of the quantum channel Λ. We will try to find real parameters s and $t_{x_0x_1x_2}(M_0,M_1,M_2)$ to make operator inequality Eq. (8) valid for all inputs $x_0x_1x_2$ and the given measurements:

$$K_{x_0x_1x_2}(M_0,M_1,M_2) \geq sW_{x_0x_1x_2} + t_{x_0x_1x_2}(M_0,M_1,M_2)\mathbb{I}. \tag{8}$$

Thus, the lower bound G can be re-expressed as

$$\begin{aligned} G &\geq \frac{1}{8}\sum_{x_0,x_1,x_2} \mathrm{tr}(K_{x_0x_1x_2}\rho_{x_0x_1x_2}) \\ &\geq \frac{s}{8}\sum_{x_0,x_1,x_2} \mathrm{tr}(W_{x_0x_1x_2}\rho_{x_0x_1x_2}) + \frac{1}{8}\sum_{x_0,x_1,x_2} t_{x_0x_1x_2} \\ &= \frac{s}{8}(\mathcal{A}_3 - \frac{1}{2}) + \frac{1}{8}\sum_{x_0,x_1,x_2} t_{x_0x_1x_2}. \end{aligned} \tag{9}$$

Finally, we find that Eq. (6) reduces to

$$\mathcal{F}(\mathcal{A}_3) \geq \frac{s}{8}(\mathcal{A}_3 - \frac{1}{2}) + t, \tag{10}$$

where $t = \frac{1}{8}\min_{M_0,M_1,M_2}\sum_{x_0,x_1,x_2} t_{x_0x_1x_2}(M_0,M_1,M_2)$.

Next, let us make a detailed calculation. We first construct a dephasing channel of the form

$$\begin{aligned} \Lambda_{\theta,\alpha,\beta}(\rho) = &\frac{1+c(\theta,\alpha,\beta)}{2}\rho + \frac{1-c(\theta,\alpha,\beta)}{2} \\ &\times\Gamma(\theta,\alpha,\beta)\rho\Gamma(\theta,\alpha,\beta), \end{aligned} \tag{11}$$

where for $\theta \in [0,\frac{\pi}{4}], \alpha \in [0,\frac{\pi}{2}], \beta \in [0,\frac{\pi}{2})$ we use $\Gamma = \sigma_x$ for $\theta \in (\frac{\pi}{4},\frac{\pi}{2}], \alpha \in [0,\frac{\pi}{2}), \beta \in [0,\frac{\pi}{2}]$ we use $\Gamma = \sigma_z$; and for $\theta \in [0,\frac{\pi}{2}], \alpha = \frac{\pi}{2}, \beta = \frac{\pi}{2}$ we use $\Gamma = \sigma_y$. Since $c(\theta,\alpha,\beta) \in [-1,1]$, we choose $c(\theta,\alpha,\beta) = \min\{s\sin\theta\sin\alpha\sin\beta\cos\theta\cos\alpha\cos\beta, 1\}$.

More precisely, if $\theta \in [0,\frac{\pi}{4}], \alpha \in [0,\frac{\pi}{2}], \beta \in [0,\frac{\pi}{2})$, the action of the channel leads to

$$\begin{aligned} K_{000} &= \frac{\mathbb{I} + \frac{\sqrt{6}}{3}\sigma_x + c(\theta,\alpha,\beta)\frac{1}{\sqrt{3}}\sigma_y}{2}, \\ K_{001} &= \frac{\mathbb{I} + \frac{\sqrt{6}}{3}\sigma_x - c(\theta,\alpha,\beta)\frac{1}{\sqrt{3}}\sigma_y}{2}, \\ K_{010} &= \frac{\mathbb{I} + c(\theta,\alpha,\beta)(\frac{1}{\sqrt{3}}\sigma_y + \frac{\sqrt{6}}{3}\sigma_z)}{2}, \\ K_{100} &= \frac{\mathbb{I} + c(\theta,\alpha,\beta)(\frac{1}{\sqrt{3}}\sigma_y - \frac{\sqrt{6}}{3}\sigma_z)}{2}, \\ K_{111} &= \frac{\mathbb{I} - \frac{\sqrt{6}}{3}\sigma_x - c(\theta,\alpha,\beta)\frac{1}{\sqrt{3}}\sigma_y}{2}, \\ K_{110} &= \frac{\mathbb{I} - frac\sqrt{6}3\sigma_x + c(\theta,\alpha,\beta)\frac{1}{\sqrt{3}}\sigma_y}{2}, \\ K_{101} &= \frac{\mathbb{I} - c(\theta,\alpha,\beta)(\frac{1}{\sqrt{3}}\sigma_y + \frac{\sqrt{6}}{3}\sigma_z)}{2}, \\ K_{011} &= \frac{\mathbb{I} + c(\theta,\alpha,\beta)(-\frac{1}{\sqrt{3}}\sigma_y + \frac{\sqrt{6}}{3}\sigma_z)}{2}. \end{aligned} \tag{12}$$

If $\theta \in (\frac{\pi}{4},\frac{\pi}{2}], \alpha \in [0,\frac{\pi}{2}), \beta \in [0,\frac{\pi}{2}]$, we have

$$\begin{aligned} K_{000} &= \frac{\mathbb{I} + c(\theta,\alpha,\beta)(\frac{\sqrt{6}}{3}\sigma_x + \frac{1}{\sqrt{3}}\sigma_y)}{2}, \\ K_{001} &= \frac{\mathbb{I} + c(\theta,\alpha,\beta)(\frac{\sqrt{6}}{3}\sigma_x - \frac{1}{\sqrt{3}}\sigma_y)}{2}, \\ K_{010} &= \frac{\mathbb{I} + c(\theta,\alpha,\beta)\frac{1}{\sqrt{3}}\sigma_y + \frac{\sqrt{6}}{3}\sigma_z}{2}, \\ K_{100} &= \frac{\mathbb{I} + c(\theta,\alpha,\beta)\frac{1}{\sqrt{3}}\sigma_y - \frac{\sqrt{6}}{3}\sigma_z}{2}, \\ K_{111} &= \frac{\mathbb{I} - c(\theta,\alpha,\beta)(\frac{\sqrt{6}}{3}\sigma_x + \frac{1}{\sqrt{3}}\sigma_y)}{2}, \\ K_{110} &= \frac{\mathbb{I} + c(\theta,\alpha,\beta)(-\frac{\sqrt{6}}{3}\sigma_x + \frac{1}{\sqrt{3}}\sigma_y)}{2}, \\ K_{101} &= \frac{\mathbb{I} - c(\theta,\alpha,\beta)\frac{1}{\sqrt{3}}\sigma_y + \frac{\sqrt{6}}{3}\sigma_z}{2}, \\ K_{011} &= \frac{\mathbb{I} - c(\theta,\alpha,\beta)\frac{1}{\sqrt{3}}\sigma_y + \frac{\sqrt{6}}{3}\sigma_z}{2}. \end{aligned} \tag{13}$$

If $\theta \in [0,\frac{\pi}{2}], \alpha = \frac{\pi}{2}, \beta = \frac{\pi}{2}$, we have

$$\begin{aligned} K_{000} &= \frac{\mathbb{I} + c(\theta,\alpha,\beta)\frac{\sqrt{6}}{3}\sigma_x + \frac{1}{\sqrt{3}}\sigma_y}{2}, \\ K_{001} &= \frac{\mathbb{I} + c(\theta,\alpha,\beta)\frac{\sqrt{6}}{3}\sigma_x - \frac{1}{\sqrt{3}}\sigma_y}{2}, \\ K_{010} &= \frac{\mathbb{I} + \frac{1}{\sqrt{3}}\sigma_y + c(\theta,\alpha,\beta)\frac{\sqrt{6}}{3}\sigma_z}{2}, \\ K_{100} &= \frac{\mathbb{I} + \frac{1}{\sqrt{3}}\sigma_y - c(\theta,\alpha,\beta)\frac{\sqrt{6}}{3}\sigma_z}{2}, \\ K_{111} &= \frac{\mathbb{I} - c(\theta,\alpha,\beta)\frac{\sqrt{6}}{3}\sigma_x - \frac{1}{\sqrt{3}}\sigma_y}{2}, \\ K_{110} &= \frac{\mathbb{I} - c(\theta,\alpha,\beta)\frac{\sqrt{6}}{3}\sigma_x + \frac{1}{\sqrt{3}}\sigma_y}{2}, \\ K_{101} &= \frac{\mathbb{I} - \frac{1}{\sqrt{3}}\sigma_y - c(\theta,\alpha,\beta)\frac{\sqrt{6}}{3}\sigma_z}{2}, \\ K_{011} &= \frac{\mathbb{I} - \frac{1}{\sqrt{3}}\sigma_y + c(\theta,\alpha,\beta)\frac{\sqrt{6}}{3}\sigma_z}{2}. \end{aligned} \tag{14}$$

Secondly, we can assume that the optimal measurements are projective and rank-one. Furthermore, any three such measurements can be represented in a three-dimensional space. Due to the freedom of setting the reference frame, without loss

of generality, we can express the measurements as

$$M_0=\cos\theta\sigma_x+\sin\theta\sigma_z,\ \ M_1=\cos\theta\sigma_x-\sin\theta\sigma_z,$$
$$M_2=\sin\alpha\cos\beta\sigma_x+\sin\alpha\sin\beta\sigma_y+\cos\alpha\sigma_z, \tag{15}$$

where $\theta,\alpha,\beta\in[0,\frac{\pi}{2}]$. It is crucial to realize that this covers all possible choices of three observables on the Bloch sphere (up to a unitary rotation).

Therefore, we can write $W_{x_0x_1x_2}$ as

$$\begin{aligned}
W_{000}&=\frac{1}{48}\big[(2\cos\theta+\sin\alpha\cos\beta)\sigma_x+\sin\alpha\sin\beta\sigma_y+\cos\alpha\sigma_z\big],\\
W_{001}&=\frac{1}{48}\big[(2\cos\theta-\sin\alpha\cos\beta)\sigma_x-\sin\alpha\sin\beta\sigma_y-\cos\alpha\sigma_z\big],\\
W_{010}&=\frac{1}{48}\big[\sin\alpha\cos\beta\sigma_x+\sin\alpha\sin\beta\sigma_y+(2\sin\theta+\cos\alpha)\sigma_z\big],\\
W_{100}&=\frac{1}{48}\big[\sin\alpha\cos\beta\sigma_x+\sin\alpha\sin\beta\sigma_y+(-2\sin\theta+\cos\alpha)\sigma_z\big],\\
W_{111}&=-\frac{1}{48}\big[(2\cos\theta+\sin\alpha\cos\beta)\sigma_x+\sin\alpha\sin\beta\sigma_y+\cos\alpha\sigma_z\big],\\
W_{110}&=-\frac{1}{48}\big[(2\cos\theta-\sin\alpha\cos\beta)\sigma_x-\sin\alpha\sin\beta\sigma_y-\cos\alpha\sigma_z\big],\\
W_{101}&=-\frac{1}{48}\big[\sin\alpha\cos\beta\sigma_x+\sin\alpha\sin\beta\sigma_y+(2\sin\theta+\cos\alpha)\sigma_z\big],\\
W_{011}&=-\frac{1}{48}\big[\sin\alpha\cos\beta\sigma_x+\sin\alpha\sin\beta\sigma_y\\
&\quad+(-2\sin\theta+\cos\alpha)\sigma_z\big].
\end{aligned}\tag{16}$$

According to inequality Eq. (8), here, we denote the operator $Z(M_0,M_1,M_2)$ as

$$\begin{aligned}
&K_{x_0x_1x_2}(M_0,M_1,M_2)-sW_{x_0x_1x_2}-t_{x_0x_1x_2}(M_0,M_1,M_2)\mathbb{I}\\
&\equiv Z(M_0,M_1,M_2).
\end{aligned}\tag{17}$$

Our goal is to show that for some specific values of s and $t_{x_0x_1x_2}(M_0,M_1,M_2)$, the operator $Z(M_0,M_1,M_2)$ is positive semidefinite for all angles between the local observables. Due to symmetry, we can simplify operator inequality Eq. (17) by $t_1=t_{000}=t_{111},t_2=t_{001}=t_{110},t_3=t_{010}=t_{101},t_4=t_{100}=t_{011}$. Therefore, there are four operator inequalities in each interval.

More precisely, if $\theta\in[0,\frac{\pi}{4}],\alpha\in[0,\frac{\pi}{2}],\beta\in[0,\frac{\pi}{2})$, we have

$$\begin{aligned}
&\frac{\mathbb{I}+\frac{\sqrt{6}}{3}\sigma_x+c(\theta,\alpha,\beta)\frac{1}{\sqrt{3}}\sigma_y}{2}-\frac{s}{48}\big[(2\cos\theta+\sin\alpha\cos\beta)\sigma_x\\
&+\sin\alpha\sin\beta\sigma_y+\cos\alpha\sigma_z\big]-t_1\mathbb{I}\geq 0,\\
&\frac{\mathbb{I}+\frac{\sqrt{6}}{3}\sigma_x-c(\theta,\alpha,\beta)\frac{1}{\sqrt{3}}\sigma_y}{2}-\frac{s}{48}\big[(2\cos\theta-\sin\alpha\cos\beta)\sigma_x\\
&-\sin\alpha\sin\beta\sigma_y-\cos\alpha\sigma_z\big]-t_2\mathbb{I}\geq 0,\\
&\frac{\mathbb{I}+c(\theta,\alpha,\beta)(\frac{1}{\sqrt{3}}\sigma_y+\frac{\sqrt{6}}{3}\sigma_z)}{2}-\frac{s}{48}\big[\sin\alpha\cos\beta\sigma_x\\
&+\sin\alpha\sin\beta\sigma_y+(2\sin\theta+\cos\alpha)\sigma_z\big]-t_3\mathbb{I}\geq 0,\\
&\frac{\mathbb{I}+c(\theta,\alpha,\beta)(\frac{1}{\sqrt{3}}\sigma_y-\frac{\sqrt{6}}{3}\sigma_z)}{2}-\frac{s}{48}\big[\sin\alpha\cos\beta\sigma_x\\
&+\sin\alpha\sin\beta\sigma_y+(-2\sin\theta+\cos\alpha)\sigma_z\big]-t_4\mathbb{I}\geq 0.
\end{aligned}\tag{18}$$

Correspondingly, we can obtain t_1,t_2,t_3 and t_4 satisfying inequality Eq. (22):

$$t_1=\min\left\{\frac{1}{2}-\frac{s}{48}\cos\alpha,\frac{1}{2}+\frac{s}{48}\cos\alpha,\frac{1-\sqrt{\left(\frac{s}{24}\cos\alpha\right)^2+\left[\frac{\sqrt{6}}{3}-\frac{s}{24}(2\cos\theta+\sin\alpha\cos\beta)\right]^2+\left(\frac{c(\theta,\alpha,\beta)}{\sqrt{3}}-\frac{s}{24}\sin\alpha\sin\beta\right)^2}}{2}\right\},$$

$$t_2=\min\left\{\frac{1}{2}-\frac{s}{48}\cos\alpha,\frac{1}{2}+\frac{s}{48}\cos\alpha,\frac{1-\sqrt{\left(\frac{s}{24}\cos\alpha\right)^2+\left[\frac{\sqrt{6}}{3}-\frac{s}{24}(2\cos\theta-\sin\alpha\cos\beta)\right]^2+\left(-\frac{c(\theta,\alpha,\beta)}{\sqrt{3}}+\frac{s}{24}\sin\alpha\sin\beta\right)^2}}{2}\right\},$$

$$t_3=\min\left\{\frac{1}{2}+\frac{\sqrt{6}c(\theta,\alpha,\beta)}{6}-\frac{s}{48}(2\sin\theta+\cos\alpha),\frac{1}{2}-\frac{\sqrt{6}c(\theta,\alpha,\beta)}{6}+\frac{s}{48}(2\sin\theta+\cos\alpha),\right.$$
$$\left.\frac{1-\sqrt{\left[\frac{\sqrt{6}c(\theta,\alpha,\beta)}{3}-\frac{s}{24}(2\sin\theta+\cos\alpha)\right]^2+\left(\frac{s}{24}\sin\alpha\cos\beta\right)^2+\left(\frac{c(\theta,\alpha,\beta)}{\sqrt{3}}-\frac{s}{24}\sin\alpha\sin\beta\right)^2}}{2}\right\},$$

$$t_4=\min\left\{\frac{1}{2}-\frac{\sqrt{6}c(\theta,\alpha,\beta)}{6}-\frac{s}{48}(2\sin\theta+\cos\alpha),\frac{1}{2}+\frac{\sqrt{6}c(\theta,\alpha,\beta)}{6}+\frac{s}{48}(2\sin\theta+\cos\alpha),\right.$$
$$\left.\frac{1-\sqrt{\left[\frac{\sqrt{6}c(\theta,\alpha,\beta)}{3}+\frac{s}{24}(-2\sin\theta+\cos\alpha)\right]^2+\left(\frac{s}{24}\sin\alpha\cos\beta\right)^2+\left(\frac{c(\theta,\alpha,\beta)}{\sqrt{3}}-\frac{s}{24}\sin\alpha\sin\beta\right)^2}}{2}\right\}.\tag{19}$$

If $\theta\in(\frac{\pi}{4},\frac{\pi}{2}],\alpha\in[0,\frac{\pi}{2}),\beta\in[0,\frac{\pi}{2}]$, we have

$$\frac{\mathbb{I}+c(\theta,\alpha,\beta)(\frac{\sqrt{6}}{3}\sigma_x+\frac{1}{\sqrt{3}}\sigma_y)}{2}-\frac{s}{48}[(2\cos\theta+\sin\alpha\cos\beta)\sigma_x+\sin\alpha\sin\beta\sigma_y+\cos\alpha\sigma_z]-t_1\mathbb{I}\geq 0,$$

$$\frac{\mathbb{I}+c(\theta,\alpha,\beta)(\frac{\sqrt{6}}{3}\sigma_x-\frac{1}{\sqrt{3}}\sigma_y)}{2}-\frac{s}{48}[(2\cos\theta-\sin\alpha\cos\beta)\sigma_x-\sin\alpha\sin\beta\sigma_y-\cos\alpha\sigma_z]-t_2\mathbb{I}\geq 0,$$
$$\frac{\mathbb{I}+c(\theta,\alpha,\beta)\frac{1}{\sqrt{3}}\sigma_y+\frac{\sqrt{6}}{3}\sigma_z}{2}-\frac{s}{48}[\sin\alpha\cos\beta\sigma_x+\sin\alpha\sin\beta\sigma_y+(2\sin\theta+\cos\alpha)\sigma_z]-t_3\mathbb{I}\geq 0,$$
$$\frac{\mathbb{I}+c(\theta,\alpha,\beta)\frac{1}{\sqrt{3}}\sigma_y-\frac{\sqrt{6}}{3}\sigma_z}{2}-\frac{s}{48}[\sin\alpha\cos\beta\sigma_x+\sin\alpha\sin\beta\sigma_y+(-2\sin\theta+\cos\alpha)\sigma_z]-t_4\mathbb{I}\geq 0. \tag{20}$$

Correspondingly, we can obtain

$$t_1=\min\left\{\frac{1}{2}-\frac{s}{48}\cos\alpha,\frac{1}{2}+\frac{s}{48}\cos\alpha,\right.$$
$$\left.\frac{1-\sqrt{\left(\frac{s}{24}\cos\alpha\right)^2+\left[\frac{\sqrt{6}c(\theta,\alpha,\beta)}{3}-\frac{s}{24}(2\cos\theta+\sin\alpha\cos\beta)\right]^2+\left(\frac{c(\theta,\alpha,\beta)}{\sqrt{3}}-\frac{s}{24}\sin\alpha\sin\beta\right)^2}}{2}\right\},$$
$$t_2=\min\left\{\frac{1}{2}-\frac{s}{48}\cos\alpha,\frac{1}{2}+\frac{s}{48}\cos\alpha,\right.$$
$$\left.\frac{1-\sqrt{\left(\frac{s}{24}\cos\alpha\right)^2+\left[\frac{\sqrt{6}c(\theta,\alpha,\beta)}{3}-\frac{s}{24}(2\cos\theta-\sin\alpha\cos\beta)\right]^2+\left(-\frac{c(\theta,\alpha,\beta)}{\sqrt{3}}+\frac{s}{24}\sin\alpha\sin\beta\right)^2}}{2}\right\},$$
$$t_3=\min\left\{\frac{1}{2}+\frac{\sqrt{6}}{6}-\frac{s}{48}(2\sin\theta+\cos\alpha),\frac{1}{2}-\frac{\sqrt{6}}{6}+\frac{s}{48}(2\sin\theta+\cos\alpha),\right.$$
$$\left.\frac{1-\sqrt{\left[\frac{\sqrt{6}}{3}-\frac{s}{24}(2\sin\theta+\cos\alpha)\right]^2+\left(\frac{s}{24}\sin\alpha\cos\beta\right)^2+\left(\frac{c(\theta,\alpha,\beta)}{\sqrt{3}}-\frac{s}{24}\sin\alpha\sin\beta\right)^2}}{2}\right\},$$
$$t_4=\min\left\{\frac{1}{2}-\frac{\sqrt{6}}{6}-\frac{s}{48}(2\sin\theta+\cos\alpha),\frac{1}{2}+\frac{\sqrt{6}}{6}+\frac{s}{48}(2\sin\theta+\cos\alpha),\right.$$
$$\left.\frac{1-\sqrt{\left[\frac{\sqrt{6}}{3}+\frac{s}{24}(-2\sin\theta+\cos\alpha)\right]^2+\left(\frac{s}{24}\sin\alpha\cos\beta\right)^2+\left(\frac{c(\theta,\alpha,\beta)}{\sqrt{3}}-\frac{s}{24}\sin\alpha\sin\beta\right)^2}}{2}\right\}. \tag{21}$$

If $\theta\in[0,\frac{\pi}{2}],\alpha=\frac{\pi}{2},\beta=\frac{\pi}{2}$, we have

$$\frac{\mathbb{I}+c(\theta,\alpha,\beta)\frac{\sqrt{6}}{3}\sigma_x+\frac{1}{\sqrt{3}}\sigma_y}{2}-\frac{s}{48}[(2\cos\theta+\sin\alpha\cos\beta)\sigma_x+\sin\alpha\sin\beta\sigma_y+\cos\alpha\sigma_z]-t_1\mathbb{I}\geq 0,$$
$$\frac{\mathbb{I}+c(\theta,\alpha,\beta)\frac{\sqrt{6}}{3}\sigma_x-\frac{1}{\sqrt{3}}\sigma_y}{2}-\frac{s}{48}[(2\cos\theta-\sin\alpha\cos\beta)\sigma_x-\sin\alpha\sin\beta\sigma_y-\cos\alpha\sigma_z]-t_2\mathbb{I}\geq 0,$$
$$\frac{\mathbb{I}+\frac{1}{\sqrt{3}}\sigma_y+c(\theta,\alpha,\beta)\frac{\sqrt{6}}{3}\sigma_z}{2}-\frac{s}{48}[\sin\alpha\cos\beta\sigma_x+\sin\alpha\sin\beta\sigma_y+(2\sin\theta+\cos\alpha)\sigma_z]-t_3\mathbb{I}\geq 0,$$
$$\frac{\mathbb{I}+\frac{1}{\sqrt{3}}\sigma_y-c(\theta,\alpha,\beta)\frac{\sqrt{6}}{3}\sigma_z}{2}-\frac{s}{48}[\sin\alpha\cos\beta\sigma_x+\sin\alpha\sin\beta\sigma_y+(-2\sin\theta+\cos\alpha)\sigma_z]-t_4\mathbb{I}\geq 0. \tag{22}$$

Correspondingly, we can obtain

$$t_1=\min\left\{\frac{1}{2}-\frac{s}{48}\cos\alpha,\frac{1}{2}+\frac{s}{48}\cos\alpha,\right.$$
$$\left.\frac{1-\sqrt{\left(\frac{s}{24}\cos\alpha\right)^2+\left[\frac{\sqrt{6}c(\theta,\alpha,\beta)}{3}-\frac{s}{24}(2\cos\theta+\sin\alpha\cos\beta)\right]^2+\left(\frac{1}{\sqrt{3}}-\frac{s}{24}\sin\alpha\sin\beta\right)^2}}{2}\right\},$$
$$t_2=\min\left\{\frac{1}{2}-\frac{s}{48}\cos\alpha,\frac{1}{2}+\frac{s}{48}\cos\alpha,\right.$$

$$\frac{1-\sqrt{\left(\frac{s}{24}\cos\alpha\right)^2+\left[\frac{\sqrt{6}c(\theta,\alpha,\beta)}{3}-\frac{s}{24}(2\cos\theta-\sin\alpha\cos\beta)\right]^2+\left(-\frac{1}{\sqrt{3}}+\frac{s}{24}\sin\alpha\sin\beta\right)^2}}{2}\Bigg\},$$

$$t_3=\min\Bigg\{\frac{1}{2}+\frac{\sqrt{6}c(\theta,\alpha,\beta)}{6}-\frac{s}{48}(2\sin\theta+\cos\alpha),\frac{1}{2}-\frac{\sqrt{6}c(\theta,\alpha,\beta)}{6}+\frac{s}{48}(2\sin\theta+\cos\alpha),$$
$$\frac{1-\sqrt{\left[\frac{\sqrt{6}c(\theta,\alpha,\beta)}{3}-\frac{s}{24}(2\sin\theta+\cos\alpha)\right]^2+\left(\frac{s}{24}\sin\alpha\cos\beta\right)^2+\left(\frac{1}{\sqrt{3}}-\frac{s}{24}\sin\alpha\sin\beta\right)^2}}{2}\Bigg\},$$

$$t_4=\min\Bigg\{\frac{1}{2}-\frac{\sqrt{6}c(\theta,\alpha,\beta)}{6}-\frac{s}{48}(2\sin\theta+\cos\alpha),\frac{1}{2}+\frac{\sqrt{6}c(\theta,\alpha,\beta)}{6}+\frac{s}{48}(2\sin\theta+\cos\alpha),$$
$$\frac{1-\sqrt{\left[\frac{\sqrt{6}c(\theta,\alpha,\beta)}{3}+\frac{s}{24}(-2\sin\theta+\cos\alpha)\right]^2+\left(\frac{s}{24}\sin\alpha\cos\beta\right)^2+\left(\frac{1}{\sqrt{3}}-\frac{s}{24}\sin\alpha\sin\beta\right)^2}}{2}\Bigg\}. \tag{23}$$

According to inequality Eq. (10), we can obtain our lower bound on the average fidelity

$$\mathcal{F}(\mathcal{A}_3)\geq\frac{s}{8}(\mathcal{A}_3-\frac{1}{2})+\min_{M_0,M_1,M_2}t(M_0,M_1,M_2), \tag{24}$$

where $t(M_0,M_1,M_2)=\frac{t_1+t_2+t_3+t_4}{4}$. To compute this quantity, we can fix the value of s to be

$$s=8(2\sqrt{3}+3).$$

After the above calculations, it is not difficult to get

$$t=-\frac{\sqrt{3}}{2}.$$

Finally, we obtain the lower bound

$$\mathcal{F}(\mathcal{A}_3)\geq(2\sqrt{3}+3)\mathcal{A}_3-\frac{3\sqrt{3}+3}{2}. \tag{25}$$

One can check that choosing distinct values of s will not lead to improved lower bounds. A maximum value $\mathcal{A}_3=Q_2^{3\to1}$ implies $\mathcal{F}=1$ (see Fig. 2.), *i.e.* the preparations must be the ideal ones (up to a unitary rotation).

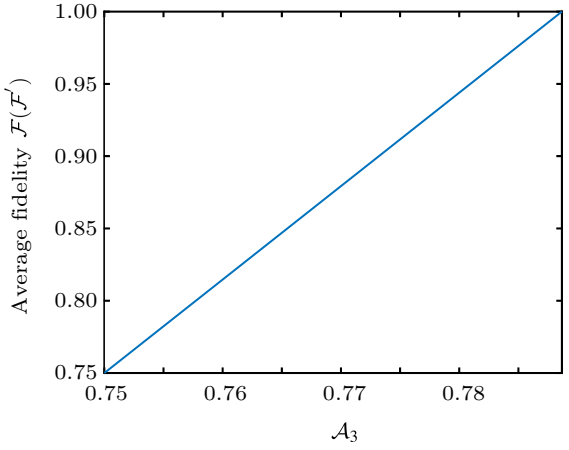


Fig. 2. The analytic lower bound on the average fidelity $\mathcal{F}$ ($\mathcal{F}'$) for prepared states (measurements), as a function of the observed value of $\mathcal{A}_3$.

4. Robustness self-testing of measurements

Similarly, we can quantify the average fidelity of the measurements with respect to the ideal ones:

$$G'(\{M_y^b\})=\frac{1}{6}\max_{\Lambda}\sum_{y,b}F((M_y^b)^{\text{ideal}},\Lambda[M_y^b]), \tag{26}$$

where Λ is a completely positive trace-preserving map, and the maximum is taken over all quantum channels. Our goal is to give the lower bound of G', *i.e.*

$$\mathcal{F}'(\mathcal{A}_3)=\min_{\{M_y^b\}\in R'(\mathcal{A}_3)}G'(\{M_y^b\}), \tag{27}$$

where $R'(\mathcal{A}_3)$ represents all sets of measurements compatible with a certain value of $\mathcal{A}_3$.

From Eq. (2), we can rewrite $\mathcal{A}_3=\sum_{y,b}\mathrm{tr}(M_y^bZ_{yb})$, where $Z_{yb}=\frac{1}{24}\Sigma_{x_0,x_1,x_2}\rho_{x_0x_1x_2}\delta_{b,x_y}$. We also define operators corresponding to the chosen channels and the ideal measurements

$$K_{yb}(\{\rho_{x_0x_1x_2}\})=\Lambda^\dagger[(M_y^b)^{\text{ideal}}]. \tag{28}$$

We aim to construct operator inequalities of the form

$$K_{yb}(\{\rho_{x_0x_1x_2}\})\geq sZ_{yb}+t_{yb}(\{\rho_{x_0x_1x_2}\})\mathbb{I} \tag{29}$$

for all inputs (x_0,x_1,x_2) and for any given preparations. For the sake of simplicity, we first apply a unitary channel to $(M_y^b)^{\text{ideal}}$ eqnarray these operators with the eigenstates of $\sigma_x,\sigma_y,\sigma_z$, *i.e.*

$$\begin{aligned}(M_0^0)^{\text{ideal}}&=\frac{\mathbb{I}+\sigma_x}{2},\quad(M_0^1)^{\text{ideal}}=\frac{\mathbb{I}-\sigma_x}{2},\\(M_1^0)^{\text{ideal}}&=\frac{\mathbb{I}+\sigma_y}{2},\quad(M_1^1)^{\text{ideal}}=\frac{\mathbb{I}-\sigma_y}{2},\\(M_2^0)^{\text{ideal}}&=\frac{\mathbb{I}+\sigma_z}{2},\quad(M_2^1)^{\text{ideal}}=\frac{\mathbb{I}-\sigma_z}{2}.\end{aligned} \tag{30}$$

Then, we use a similar channel Λ, but it is slightly different from the previous one. Next, let us make a detailed calculation. We firstly construct a dephasing channel of the form

$$\Lambda_{\theta,\varphi}(\rho)=\frac{1+c(\theta,\varphi)}{2}\rho+\frac{1-c(\theta,\varphi)}{2}\Gamma(\theta,\varphi)\rho\Gamma(\theta,\varphi) \tag{31}$$

where for $\theta \in [0, \arccos\frac{1}{\sqrt{3}}], \varphi \in [0, \frac{\pi}{2}]$, we use $\Gamma = \sigma_z$; for $\theta \in (\arccos\frac{1}{\sqrt{3}}, \frac{\pi}{2}], \varphi \in [0, \frac{\pi}{4}]$, we use $\Gamma = \sigma_x$; for $\theta \in (\arccos\frac{1}{\sqrt{3}}, \frac{\pi}{2}], \varphi \in (\frac{\pi}{4}, \frac{\pi}{2}]$, we use $\Gamma = \sigma_y$. $c(\theta,\varphi) \in [-1,1]$, and $c(\theta,\varphi) = \min\{s\sin\theta\sin\varphi\cos\theta\cos\varphi, 1\}$.

More precisely, if $\theta \in [0, \arccos\frac{1}{\sqrt{3}}], \varphi \in [0, \frac{\pi}{2}]$, we have

$$\begin{aligned} K_{00} &= \frac{\mathbb{I}+c(\theta,\varphi)\sigma_x}{2}, \quad K_{01} = \frac{\mathbb{I}-c(\theta,\varphi)\sigma_x}{2}, \\ K_{10} &= \frac{\mathbb{I}+c(\theta,\varphi)\sigma_y}{2}, \quad K_{11} = \frac{\mathbb{I}+c(\theta,\varphi)\sigma_y}{2}, \\ K_{20} &= \frac{\mathbb{I}+\sigma_z}{2}, K_{21} = \frac{\mathbb{I}+\sigma_z}{2}. \end{aligned} \tag{32}$$

A similar procedure is used for the other two intervals.

Secondly, it is straightforward to see that, for any given pair of measurements, the optimal choice of preparations are eight pure qubit states on the Bloch sphere. Due to the freedom of setting the reference frame, without loss of generality, we can express the preparations as $\rho_{x_0x_1x_2} = \frac{\mathbb{I}+\boldsymbol{n}_{x_0,x_1,x_2}\cdot\boldsymbol{\sigma}}{2}$ by Bloch vectors

$$\boldsymbol{n}_{x_0,x_1,x_2} = ((-1)^{x_0}\sin\theta\cos\varphi, (-1)^{x_1}\sin\theta\sin\varphi, (-1)^{x_2}\cos\theta). \tag{33}$$

Therefore, we can write Z_{yb} as

$$\begin{aligned} Z_{00} &= \frac{\mathbb{I}+\sin\theta\cos\varphi\sigma_x}{12}, \quad Z_{01} = \frac{\mathbb{I}-\sin\theta\cos\varphi\sigma_x}{12}, \\ Z_{10} &= \frac{\mathbb{I}+\sin\theta\sin\varphi\sigma_x}{12}, \quad Z_{11} = \frac{\mathbb{I}-\sin\theta\sin\varphi\sigma_x}{12}, \\ Z_{20} &= \frac{\mathbb{I}+\cos\theta\sigma_z}{12}, \quad Z_{21} = \frac{\mathbb{I}-\cos\theta\sigma_z}{12}. \end{aligned} \tag{34}$$

Due to symmetry, we can simplify operator inequality Eq. (29) by $t_1 = t_{00} = t_{01}, t_2 = t_{10} = t_{11}, t_3 = t_{20} = t_{21}$. Therefore, there are three operator inequalities in each interval. Due to symmetries, we restrict ourselves so that $t_1 = t_{00} = t_{01}, t_2 = t_{10} = t_{11}, t_3 = t_{20} = t_{21}$. Thus, we have to consider three operator inequalities in each interval.

More precisely, if $\theta \in [0, \arccos\frac{1}{\sqrt{3}}], \varphi \in [0, \frac{\pi}{2}]$, we have

$$\begin{aligned} &\frac{\mathbb{I}+c(\theta,\varphi)\sigma_x}{2} - \frac{s}{12}(\mathbb{I}+\sin\theta\cos\varphi\sigma_x) - t_1\mathbb{I} \geq 0, \\ &\frac{\mathbb{I}+c(\theta,\varphi)\sigma_y}{2} - \frac{s}{12}(\mathbb{I}+\sin\theta\sin\varphi\sigma_y) - t_2\mathbb{I} \geq 0, \\ &\frac{\mathbb{I}+\sigma_z}{2} - \frac{s}{12}(\mathbb{I}+\cos\theta\sigma_z) - t_3\mathbb{I} \geq 0. \end{aligned} \tag{35}$$

When making any choice of t_1, t_2, t_3, satisfying these aforementioned constraints gives rise to valid operator inequality Eq. (35). Correspondingly, we can obtain

$$\begin{aligned} t_1 = \min\Big\{ &\frac{1}{2} - \frac{s}{12}, \frac{1}{2} - \frac{s}{12} - \frac{c(\theta,\varphi)}{2} + \frac{s}{12}\sin\theta\cos\varphi, \\ &\frac{1}{2} - \frac{s}{12} + \frac{c(\theta,\varphi)}{2} - \frac{s}{12}\sin\theta\cos\varphi\Big\}, \\ t_2 = \min\Big\{ &\frac{1}{2} - \frac{s}{12}, \frac{1}{2} - \frac{s}{12} - \frac{c(\theta,\varphi)}{2} + \frac{s}{12}\sin\theta\cos\varphi, \\ &\frac{1}{2} - \frac{s}{12} + \frac{c(\theta,\varphi)}{2} - \frac{s}{12}\sin\theta\sin\varphi\Big\}, \\ t_3 = \min\Big\{ &1 - \frac{s}{12}(1+\cos\theta), -\frac{s}{12}(1-\cos\theta)\Big\}. \end{aligned} \tag{36}$$

Correspondingly, if $\theta \in (\arccos\frac{1}{\sqrt{3}}, \frac{\pi}{2}], \varphi \in [0, \frac{\pi}{4}]$, we have

$$\begin{aligned} t_1 = \min\Big\{ &\frac{1}{2} - \frac{s}{12}, 1 - \frac{s}{12} - \frac{s}{12}\sin\theta\cos\varphi, \\ &-\frac{s}{12} + \frac{s}{12}\sin\theta\cos\varphi\Big\}, \\ t_2 = \min\Big\{ &\frac{1}{2} - \frac{s}{12}, \frac{1}{2} - \frac{s}{12} - \frac{c(\theta,\varphi)}{2} + \frac{s}{12}\sin\theta\sin\varphi, \\ &\frac{1}{2} - \frac{s}{12} + \frac{c(\theta,\varphi)}{2} - \frac{s}{12}\sin\theta\sin\varphi\Big\}, \\ t_3 = \min\Big\{ &\frac{1}{2} + \frac{c(\theta,\varphi)}{2} - \frac{s}{12}(1+\cos\theta), \\ &\frac{1}{2} - \frac{c(\theta,\varphi)}{2} - \frac{s}{12}(1-\cos\theta)\Big\}. \end{aligned} \tag{37}$$

If $\theta \in (\arccos\frac{1}{\sqrt{3}}, \frac{\pi}{2}], \varphi \in (\frac{\pi}{4}, \frac{\pi}{2}]$, we have

$$\begin{aligned} t_1 = \min\Big\{ &\frac{1}{2} - \frac{s}{12}, \frac{1}{2} - \frac{s}{12} - \frac{c(\theta,\varphi)}{2} + \frac{s}{12}\sin\theta\cos\varphi, \\ &\frac{1}{2} - \frac{s}{12} + \frac{c(\theta,\varphi)}{2} - \frac{s}{12}\sin\theta\cos\varphi\Big\}, \\ t_2 = \min\Big\{ &\frac{1}{2} - \frac{s}{12}, 1 - \frac{s}{12} - \frac{s}{12}\sin\theta\sin\varphi, \\ &-\frac{s}{12} + \frac{s}{12}\sin\theta\sin\varphi\Big\}, \\ t_3 = \min\Big\{ &\frac{1}{2} + \frac{c(\theta,\varphi)}{2} - \frac{s}{12}(1+\cos\theta), \\ &\frac{1}{2} - \frac{c(\theta,\varphi)}{2} - \frac{s}{12}(1-\cos\theta)\Big\}. \end{aligned} \tag{38}$$

In order to obtain our lower bound on $\mathcal{F}'$, we must minimize the quantity $t(\theta,\varphi) = \frac{t_1+t_2+t_3}{3}$ for a specific choice of s. We can fix the value of s to be

$$s = 6(2\sqrt{3}+3),$$

which returns the minimum value of t

$$t = -\frac{3\sqrt{3}+3}{2}.$$

Finally, we obtain the lower bound

$$\mathcal{F}'(\mathcal{A}_3) \geq (2\sqrt{3}+3)\mathcal{A}_3 - \frac{3\sqrt{3}+3}{2}. \tag{39}$$

5. Conclusion

In conclusion, we give a set of preparations and deduce the corresponding measurements suitable for robustness self-testing analysis in the prepare-and measure scenario with a

$3 \to 1$ RAC. We give the lower bound of average fidelity between the actual preparations (measurements) and the ideal quantum states (measurements), and then complete the robustness analysis. Our bound is robust against inevitable experimental errors.

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