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Quantifying quantum non-Markovianity via max-relative entropy*

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We investigate the non-Markovian behavior in open quantum systems from an information-theoretic perspective. Our main tool is the max-relative entropy, which quantifies the maximum probability with which a state ρ can appear in a convex decomposition of a state σ. This operational interpretation provides a new view for the non-Markovian process. We also find that max-relative entropy can be the witness and measure of non-Markovian processes. As applications, some examples are also given and compared with other measures in this paper.

Keywords: non-Markovian, max-relative entropy, open systems

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1. Introduction

The dynamical behavior of open quantum systems plays a core role in quantum mechanics and quantum information.[1–11] In the open system, the prototype of a Markovian process is given by the solution of a master equation for the reduced density matrix with Lindblad structure.[12,13] However, in most realistic physical systems, Markovian time evolution cannot predict many dynamical processes.[1–4,12,14–17] This is general because the relevant environmental correlation times are not smaller than the systems' relaxation or decoherence time, making the Markov process impossible;[1] for example, in the cases of strong system–environment couplings, structured or finite reservoirs, low temperatures, or large initial system environment correlations. For these results, a lot of research has focused on the development of analysis and quantification of the non-Markovian process. In fact, both Markovian and non-Markovian processes have been well studied in the classical stochastic process. While the quantum non-Markovian process cannot be directly analogous to classical stochastic processes, quantum extensions of non-Markovianity remain elusive and subtle.

Based on the different motivations, several witnesses and measures of the non-Markovian process have been proposed. Riva *et al*. proposed two non-Markovian measures based on the divisibility of the dynamical map and entanglement of environment.[18] Based on the trace distance norm, Breuer *et al*. studied a non-Markovian measure in terms of the increasing of distinguishability between different evolving states.[2] Luo *et al*. introduced a non-Markovian measure in terms of mutual information.[19] Lu *et al*. studied non-Markovianity by the quantum Fisher information flow.[20] He *et al*. introduced an alternative method based on local quantum uncertainty to measure non-Markovianity.[21] All these measures above do not coincide exactly in quantifying non-Markovian processes, although there are some coincided instances. At present, exploring the hierarchy of these measures and finding a universal definition of non-Markovianity for open quantum processes is still an important context of research.

In this paper, we investigate the quantification of non-Markovian processes via max-relative entropy.[22] Max-relative entropy has been investigated in Refs. [22]– [25]. The well-known (smooth) conditional and unconditional max-entropy[26,27] can be obtained from this quantity. It has been shown that max-relative entropy is of operational significance in applications ranging from data compression[28,29] to state merging[30] and security of keys.[26,31] In addition, max-relative entropy has been used to define entanglement and coherence monotones and their operational significance in the manipulation of entanglement and coherence.[22,32] Here, we find that max-relative entropy can be a witness of non-Markovian processes. The max-relative entropy quantifies the maximum probability with which a state ρ can appear in a convex decomposition of a state σ,[33] which can be seen as a "distinguishability". The non-Markovian process implies that the increasing of distinguishability — a certain flow of information from environment to system as time increases. We also combine the results with other kinds of quantum non-Markovianity, *i.e.*, based on mutual information, Fisher information, trace norm, and entanglement).

This paper is organized as follows. In Section 2, the witness and measure of non-Markovian processes based on max-relative entropy are introduced. In Section 3, we illustrate

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some examples of the new witness. We summarize our results in Section 4.

2. Quantifying the non-Markovian process via max-relative entropy

We first introduce the necessary notations. We consider a Hilbert space $\mathscr{H}$ of finite dimension d. Throughout this paper, we take the logarithm to base 2 and all Hilbert spaces considered are finite dimensional. Let ρ and σ be the density matrixes in the Hilbert space $\mathscr{H}$. The support set $\operatorname{supp}\rho$ of a density operator ρ is the vector space spanned by eigenvectors of the operator with non-zero eigenvalues.

The max-relative entropy is given by[22]

$$D_{\max}(\rho||\sigma) = \log\min\{r : \rho \le r\sigma\}. \tag{1}$$

Note that $D_{\max}(\rho||\sigma)$ is well defined if $\operatorname{supp}\rho \subset \operatorname{supp}\sigma$. An equivalent definition of max-relative entropy is

$$D_{\max}(\rho||\sigma) = \log\mu_{\max}\{\sigma^{-1/2}\rho\sigma^{-1/2}\}, \tag{2}$$

where $\mu_{\max}(X)$ is the maximum eigenvalue of the operator X.

It is well known that the max-relative entropy is an upper bound of relative entropy $S(\rho||\sigma) = \operatorname{Tr}(\rho\log\rho) - \operatorname{Tr}(\rho\log\sigma)$:[34]

$$S(\rho||\sigma) \le D_{\max}(\rho||\sigma). \tag{3}$$

The max-relative entropy is of operational significance in applications ranging from data compression[28,29] to state merging[30] and security of keys.[26,31]

A remarkable feature of max-relative entropy is that for a completely positive and trace preserving (CPTP) map Λ, we have

$$D_{\max}(\Lambda\rho||\Lambda\sigma) \le D_{\max}(\rho||\sigma). \tag{4}$$

Now, we will introduce the witness and measure of the non-Markovian process. We recall here that the CPTP map $\Lambda = \{\Lambda_t\}$ is Markovian in an open system in the sense that

$$\Lambda_t = \Lambda_{t,s}\Lambda_s, \quad s \le t \tag{5}$$

for some operations $\Lambda_{t,s}$, then we have

$$\begin{aligned} D_{\max}(\rho(t)||\sigma(t)) &= D_{\max}(\Lambda_t\rho(0)||\Lambda_t\sigma(0)) \\ &= D_{\max}(\Lambda_{t,s}\rho(s)||\Lambda_{t,s}\sigma(s)) \\ &\le D_{\max}(\rho(s)||\sigma(s)). \end{aligned} \tag{6}$$

Therefore, the max-relative entropy is a decreasing function for $t \ge 0$, which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}D_{\max}(\rho(t)||\sigma(t)) \le 0$$

for any Markovian processes. Any violation of this monotonicity, *i.e.*,

$$\frac{\mathrm{d}}{\mathrm{d}t}D_{\max}(\rho(t)||\sigma(t)) > 0$$

is a witness of the non-Markovian process.

We also note an interesting operational meaning for max-relative entropy, which quantifies the maximum probability with which a state ρ can appear in a convex decomposition of a state σ.[33] A non-Markovian process means the maximum probability with which ρ can appear in a convex decomposition of σ could be increased under such dynamics.

Now, we can introduce a measure for the non-Markovian process via the max-relative entropy as

$$\mathscr{N}(\Lambda) = \sup_{\rho,\sigma}\int_{(\mathrm{d}/\mathrm{d}t)D_{\max}(\rho(t)||\sigma(t))>0}\frac{\mathrm{d}}{\mathrm{d}t}D_{\max}(\rho(t)||\sigma(t))\,\mathrm{d}t, \tag{7}$$

where sup is taken over all the states. In particular, if $\operatorname{supp}\sigma \subset \operatorname{supp}\rho$, *i.e.*, the max-relative entropy is not well defined), then we could replace the two density operators such that the max-relative entropy is well defined.

3. Some applications

In this section, we explore some examples as applications for the quantification of non-Markovian processes based on the max-relative entropy.

Example 1 [Phase-damping channel]

Consider the dynamical evolution $\rho(t) = \Lambda_t\rho(0)$ in the case that a qubit undergoes phase-damping noises, which can be represented as the master equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = \gamma(t)[\sigma_z\rho(t)\sigma_z - \rho(t)], \tag{8}$$

where $\int_{t_0}^{t}\gamma(s)\,\mathrm{d}s \ge 0$ for completely positive dynamics. For the initial state

$$\rho(0) = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix},$$

then the dynamics can be expressed as

$$\rho(t) = \Lambda_t\rho(0) = \begin{pmatrix}\alpha & \beta R(t)\\ \gamma R(t) & \delta\end{pmatrix}, \tag{9}$$

with $R(t) = \exp\big(-2\int_0^t\gamma(s)\,\mathrm{d}s\big)$ and note that $0 \le R(t) \le 1$. Let the initial states be superposition states

$$\rho_1(0) = |+\rangle\langle+| = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$$

and

$$\rho_2(0) = |-\rangle\langle-| = \frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$$

in this dynamical evolution, respectively. We have

$$\rho_1(t)=\frac{1}{2}\begin{pmatrix}1 & R(t)\\ R(t) & 1\end{pmatrix}$$

and

$$\rho_2(t)=\frac{1}{2}\begin{pmatrix}1 & -R(t)\\ -R(t) & 1\end{pmatrix}.$$

The max-relative entropy of $\rho_1(t)$ and $\rho_2(t)$ is

$$D_{\max}(\rho_1(t),\rho_2(t))=\log\frac{1+R(t)}{1-R(t)}. \tag{10}$$

Therefore, the witness for the non-Markovian process based on the max-relative entropy is

$$\frac{\mathrm{d}}{\mathrm{d}t}D_{\max}(\rho_1(t),\rho_2(t))=-4\gamma(t)\frac{R(t)}{1-R(t)^2}. \tag{11}$$

From the equation given above, we note that

$$\frac{\mathrm{d}}{\mathrm{d}t}D_{\max}(\rho_1(t),\rho_2(t))>0$$

is equivalent to $\gamma(t)<0$. Thus, the condition $\gamma(t)<0$ can be a witness for the non-Markovian process.

First, from Ref. [19], we know that the witness for the non-Markovian process based on mutual information is

$$-\gamma(t)R(t)\frac{1+R(t)}{1-R(t)}. \tag{12}$$

Second, from Ref. [35], the witness for the non-Markovian process based on Fisher information is

$$-\frac{8}{3}\gamma(t)R^2(t). \tag{13}$$

Third, from Ref. [2], the witness for the non-Markovian process based on the trace norm is

$$-2\gamma(t)R(t). \tag{14}$$

Finally, from Ref. [18], the witness for the non-Markovian process based on entanglement is

$$-2\gamma(t). \tag{15}$$

To compare the witness based on the max-relative entropy with other witnesses, as suggested in Ref. [36], we can set the time-dependent rate

$$\gamma(t)=2\gamma\frac{1-\gamma t}{\mathrm{e}^{\gamma t}-2\gamma t},$$

where γ is the free scaling parameter with

$$\lim_{t\to 0}\gamma(t)=2\gamma>0.$$

The corresponding evolution is completely positive and non-Markovian. As shown in Fig. 1, we have plotted the five kinds of witness in the quantum non-Markovian process. All of them coincide when they are witnesses for this evolution.

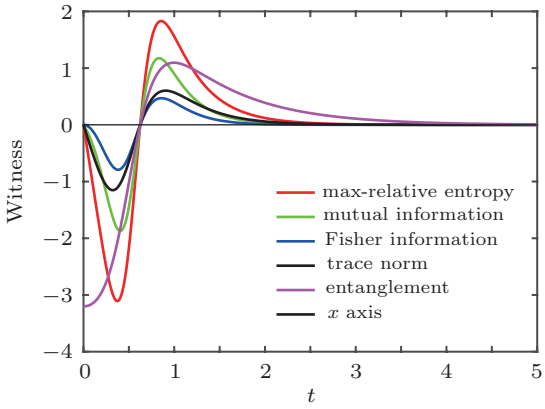


Fig. 1. The five kinds of witness in the quantum non-Markovian process (based on max-relative entropy, mutual information, Fisher information, trace norm, and entanglement).

Example 2 [Amplitude-damping channel]

Consider the dynamical evolution $\rho(t)=\Lambda_t\rho(0)$ in the case that a qubit undergoes the amplitude-damping channel, which can be represented as the master equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t)=-\frac{\mathrm{i}}{2}s(t)[\sigma_+\sigma_-,\rho(t)]+\gamma(t)\cdot\left(\sigma_-\rho(t)\sigma_+-\frac{1}{2}\{\sigma_+\sigma_-,\rho(t)\}\right), \tag{16}$$

where

$$s(t)=-2\,\mathrm{Im}\frac{\dot{p}(t)}{p(t)},\quad \gamma(t)=-2\,\mathrm{Re}\frac{\dot{p}(t)}{p(t)}$$

and $p(t)$ is described by the following intergro–differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}p(t)=-\int_0^t \mathrm{d}t_1 f(t-t_1)p(t_1) \tag{17}$$

with condition $p(0)=1$. The kernel is related to the spectral density $J(\omega)$ of the reservoir via $f(t-t_1)=\int \mathrm{d}\omega\exp(\mathrm{i}(\omega_0-\omega)(t-t_1))$.

For the initial state

$$\rho(0)=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix},$$

then the dynamics can be expressed as

$$\rho(t)=\Lambda_t\rho(0)=\begin{pmatrix}\alpha+(1-|p(t)|^2)\delta & p^*(t)\beta\\ p(t)\gamma & |p(t)|^2\delta\end{pmatrix}, \tag{18}$$

Let the initial states be superposition states $\rho_1(0)=|+\rangle\langle+|$ and $\rho_2(0)=|-\rangle\langle-|$ in this dynamical evolution, respectively. We have

$$\rho_1(t)=\frac{1}{2}\begin{pmatrix}2-|p(t)|^2 & p^*(t)\\ p(t) & |p(t)|^2\end{pmatrix}$$

and

$$\rho_2(t)=\frac{1}{2}\begin{pmatrix}2-|p(t)|^2 & -p^*(t)\\ -p(t) & |p(t)|^2\end{pmatrix}.$$

Therefore

$$D_{\max}(\rho_1(t),\rho_2(t))=\log\frac{|p(t)|^2+2\sqrt{2-|p(t)|^2}-3}{4(|p(t)|^2-1)}, \tag{19}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}D_{\max}(\rho_1(t),\rho_2(t)) = \frac{8|p(t)|}{\ln 2\sqrt{2-|p(t)|^2}(|p(t)|^2-1)}\frac{\mathrm{d}}{\mathrm{d}t}|p(t)|. \tag{20}$$

From the above equation, we note that

$$\frac{\mathrm{d}}{\mathrm{d}t}D_{\max}(\rho_1(t),\rho_2(t)) > 0$$

is equivalent to $\gamma(t) < 0$ with

$$\gamma(t) = \frac{1}{|p(t)|^2-1}\frac{\mathrm{d}}{\mathrm{d}t}|p(t)|.$$

Thus, the condition $\gamma(t) < 0$ can be a witness for the non-Markovian process.

Example 3 [Random unitary channel]

Consider the dynamical evolution $\rho(t) = \Lambda_t\rho(0)$ in the case that a qubit undergoes a stochastic unitary channel, which can be represented as the master equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = \sum_{i=1}^{3}\gamma_i(t)[\sigma_i\rho(t)\sigma_i - \rho(t)], \tag{21}$$

where $\gamma_i(t)$ are time-dependent decaying rates and the Pauli matrix.

For a single-qubit state

$$\rho(0) = \frac{1}{2}(I + \boldsymbol{r}\boldsymbol{\sigma}) = \frac{1}{2}\begin{pmatrix} 1+r_3 & r_1 - \mathrm{i}r_2 \\ r_1 + \mathrm{i}r_2 & 1-r_3 \end{pmatrix},$$

with $\boldsymbol{r} = (r_1, r_2, r_3)$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$. After the stochastic unitary channel, we have

$$\rho(t) = \frac{1}{2}I + \frac{1}{2}\mathrm{e}^{-2(\Gamma_1(t)+\Gamma_2(t))}\begin{pmatrix} r_3 & \omega(t) \\ \omega^*(t) & -r_3 \end{pmatrix} \tag{22}$$

with $\omega(t) = \mathrm{e}^{-2\Gamma_3(t)}(\mathrm{e}^{2\Gamma_1(t)}r_1 - \mathrm{i}\,\mathrm{e}^{2\Gamma_2(t)}r_2)$ and $\Gamma_i(t) = \int_0^t \gamma_i(\tau)\mathrm{d}\tau$ for $i = 1,2,3$.

Now, consider that the initial states are superposition states $\rho_1(0) = |+\rangle\langle+|$ and $\rho_2(0) = |-\rangle\langle-|$ in this dynamical evolution, respectively. We have

$$\rho_1(t) = \frac{1}{2}\begin{pmatrix} 1 & \mathrm{e}^{-2(\Gamma_2(t)+\Gamma_3(t))} \\ \mathrm{e}^{-2(\Gamma_2(t)+\Gamma_3(t))} & 1 \end{pmatrix}$$

and

$$\rho_2(t) = \frac{1}{2}\begin{pmatrix} 1 & -\mathrm{e}^{-2(\Gamma_2(t)+\Gamma_3(t))} \\ -\mathrm{e}^{-2(\Gamma_2(t)+\Gamma_3(t))} & 1 \end{pmatrix}.$$

The max-relative entropy of $\rho_1(t)$ and $\rho_2(t)$ is

$$D_{\max}(\rho_1(t),\rho_2(t)) = \log\frac{1+\mathrm{e}^{-2(\Gamma_2(t)+\Gamma_3(t))}}{1-\mathrm{e}^{-2(\Gamma_2(t)+\Gamma_3(t))}}. \tag{23}$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}D_{\max}(\rho_1(t),\rho_2(t)) = -4\gamma(t)\frac{\mathrm{e}^{-2(\Gamma_2(t)+\Gamma_3(t))}}{1-(\mathrm{e}^{-2(\Gamma_2(t)+\Gamma_3(t))})^2}, \tag{24}$$

where $\gamma(t) = \gamma_2(t) + \gamma_3(t)$.

From the equation given above, we note that

$$\frac{\mathrm{d}}{\mathrm{d}t}D_{\max}(\rho_1(t),\rho_2(t)) > 0$$

is equivalent to $\gamma(t) < 0$. Thus, the condition $\gamma(t) < 0$ can be a witness for the non-Markovian process.

4. Discussion and conclusion

We have investigated the quantum non-Markovian process based on max-relative entropy in open quantum systems. The max-relative entropy quantifies the maximum probability with which a state ρ can appear in a convex decomposition of a state σ. This operational interpretation provides a new view for non-Markovian processes. We also find that the max-relative entropy can be a witness of non-Markovian processes. As applications, some examples are also given and the results are combined with other kinds of quantum non-Markovianity, *i.e.*, the witnesses based on trace norm, mutual information, entanglement, and Fisher information). For the phase-damping channel and stochastic unitary channel, the proposed non-Markovian witness is consistent with the other non-Markovian witnesses. But for the random unitary channel, not all the non-Markovian witnesses are consistent. The relation between those non-Markovian witnesses above is a worthwhile subject for study in the future.

Meanwhile, we note that the min-relative entropy of two operators ρ and σ is given[22]

$$D_{\min}(\rho||\sigma) = -\log\mathrm{Tr}\,\pi_\rho\sigma, \tag{25}$$

where π_ρ is the projector onto $\mathrm{supp}\,\rho$. For a CPTP map Λ, we also have the monotonicity for the min-relative entropy

$$D_{\min}(\Lambda\rho||\Lambda\sigma) \le D_{\min}(\rho||\sigma). \tag{26}$$

As in the analysis of max-relative entropy, the min-relative entropy is a decreasing function for $t \ge 0$, which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}D_{\min}(\rho(t)||\sigma(t)) \le 0$$

for any Markovian process. Any violation of this monotonicity, *i.e.*,

$$\frac{\mathrm{d}}{\mathrm{d}t}D_{\min}(\rho(t)||\sigma(t)) > 0$$

is a witness of the non-Markovian process. However, as in the examples shown in Section 3, the witness of non-Markovian

processes based on the min-relative entropy will expire. This is because the excited state and the ground state we considered satisfy $\mathrm{supp}\,\rho_1(0) = \mathrm{supp}\,\rho_2(0)$. From Eq. (25), we know that if $\pi_{\rho_1(0)} = \pi_{\rho_2(0)}$, then

$$\begin{aligned} D_{\min}(\rho_1(0)||\rho_2(0)) &= -\log \mathrm{Tr}(\pi_{\rho_1(0)}\rho_2(0)) \\ &= -\log \mathrm{Tr}(\rho_2(0)) = 0. \end{aligned}$$

Thus, the min-relative entropy cannot be a suitable witness of the non-Markovian process in some cases.

It was shown that the superposition states $\rho_1(0) = |+\rangle\langle+|$ and $\rho_2(0) = |-\rangle\langle-|$ are the optimal states for measuring non-Markovian processes in some cases, such as the trace norm.[2,37] Due to the hardness for optimizing the function Eq. (7), in future research we will consider studying whether the excited state $\rho_1(0) = |+\rangle\langle+|$ and ground state $\rho_2(0) = |-\rangle\langle-|$ are also the optimal states as a measure of non-Markovian processes based on the max-relative entropy of the examples.

Finally, we hope that non-Markovianity based on the max-relative entropy can be useful for future research in open quantum systems. Despite the quantity of new results in the quantification of non-Markovian processes in recent years, the relation between those non-Markovian quantifications is still a worthwhile subject for study in the future.

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