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# Energy-optimal problem of multiple nonholonomic wheeled mobile robots via distributed event-triggered optimization algorithm*

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The distributed event-triggered optimization problem for multiple nonholonomic robots has been studied to minimize the global battery energy consumption. Each robot possesses its own cost function which depends on the state of the hand position and represents battery energy consumption. By coordinate transformation, the dynamics of the hand positions can be formulated into two groups of first-order integrators. Then the distributed event-triggered optimization algorithm is designed such that the states of robots' hand positions exponentially converge to the optimizer of the global cost function. Meanwhile, the velocity and orientation of each robot are ensured to reach zero and a certain constant, respectively. Moreover, the inter-execution time is lower bounded and the Zeno behavior is therefore naturally avoided. Numerical simulations show the effectiveness of the proposed algorithm.

**Keywords:** distributed optimization, nonholonomic robots, event-triggered, energy-optimal, consensus

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## 1. Introduction

In robot systems, mobile robots are expected to operate in environments where it may not be easy to replace batteries or refill the fuel, such as hostile environments.[1] Therefore, to ensure that the robot team can complete their missions, energy consumption problems must be taken into account at the design phase. Energy consumption caused by both motion and communication belong to the field of energy-optimal problems.[2–7] In the existing research, the energy-optimal problems are mostly studied by optimal control theory. For example, in Ref. [4], the authors proposed a power control and channel allocation optimization algorithm with low energy consumption for channel allocation game model. Tokekar *et al.*[6] studied the problem of finding velocity profiles for car-like robots to minimize the energy consumption while traveling along a given path. In Ref. [7], Egerstedt *et al.* introduced an open-loop control protocol for multi-robots to achieve rendezvous while minimizing the overall energy consumption. However, this type of open-loop controller cannot achieve adjustment and compensation for deviations of the controlled variables. Therefore, the closed-loop controller is proven to be more favorable to actual robot systems.

Over the past few decades, distributed optimization problems have increasingly attracted attention in science and engineering. In distributed optimization problems, each agent has a local cost function, which is only known by itself. The goal of distributed optimization is to cooperatively optimize a global cost function in the form of the sum of local cost functions of all agents. In the process of optimization, each agent can only use the local information of its neighbors. There are a large number of studies for distributed optimization algorithms in the context of multi-agent systems. Most of these studies focus on the protocols with first-order or second-order linear systems. The adopted methods include sub-gradient method,[8–10] gradient method,[11–13] primal-dual method[14,15], zero-gradient-sum (ZGS) method,[16–18] and so forth. For example, Nedic *et al.*[8] first put forward the distributed discrete-time sub-gradient optimization algorithm with diminishing step size. In Ref. [12], a distributed gradient optimization algorithm was presented for continuous-time multi-agent systems with single-integrator dynamics, which took into account the gradient gain and finite-time convergence. In Ref. [13], the authors proposed a gradient-based distributed optimization algorithm for first-order integrators. A continuous-time zero-gradient-sum algorithm was presented in Ref. [16] to solve the time-varying distributed convex optimization problem. However, the ZGS algorithm needs to restrict the initial states of agents.

In practice, each agent is usually equipped with simple embedded micro-processor which has limited energy and communication bandwidth. Most existing algorithms are updated continuously, which might be inefficient or impractical since they can lead to large communication load and waste excessive resources. To reduce the unnecessary waste of communication and computation resources, event-triggered control laws have been applied to single and double-integrators.[19–22]

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Similarly, event-triggered algorithms have also been applied to distributed optimization problems.[23–29] Lü *et al.*[23] focused on the distributed event-triggered sub-gradient algorithm for solving a class of convex optimization problem based on first-order discrete-time multi-agent systems. In Ref. [25], an event triggered zero-gradient-sum algorithm was proposed to solve the optimization problem. To avoid calculating the threshold of the triggering times continuously, Liu *et al.*[26] presented a novel distributed event-triggered optimization algorithm based on sampled-data state. In Ref. [29], the distributed event-triggered optimization algorithm was extended from first-order dynamics to second-order dynamics, thereby accomplishing the optimal coordination problems.

Motivated by these considerations, in this paper we focus on the distributed event-triggered optimization problem for multiple nonholonomic wheeled mobile robots. The local cost function of each robot represents its own battery energy consumption, which is related to the state of the hand position. By coordinate transformation, the dynamics of the hand positions of robots are formulated as two groups of first-order integrators. We will solve the energy-optimal problem by using the distributed event-triggered optimization algorithm, and at the same time the states of the hand positions exponentially converge to the optimal point.

The main contributions of this paper follow. (i) Most existing results related to distributed optimization problems have concentrated on first-order or second-order linear systems. However, there are abundant nonlinear and even nonholonomic systems in practice, such as mobile robots. Therefore, we propose the distributed event-triggered optimization algorithm to solve the energy-optimal problem for multiple nonholonomic mobile robots. (ii) A distributed event-triggered control scheme is designed to minimize the energy consumption. For each robot, the controller is updated only at its own triggering times. Moreover, the event-triggered algorithm can avoid the Zeno behavior. (iii) It can be proven that the optimal solution is the system equilibrium point and also the consensus point. Furthermore, the exponential convergence rates of position and velocity are presented with the help of Lyapunov stability theory, which is different from the asymptotical convergence in Ref. [13].

The rest of this paper is organized as follows. In Section 2, some basic concepts on algebraic graph theory are introduced and the problem is formulated. Section 3 shows the main results. The distributed event-triggered optimization algorithm for the energy-optimal problem of multiple robots and the detailed stability analysis are also presented. Simulations are given to validate the theoretical results in Section 4. Finally, Section 5 gives a conclusion of this paper.

**Notations** Let $\mathbb{R}^{N\times N}$, $\mathbb{R}^N$, and $\mathbb{R}$ be the set of $N\times N$ real matrices, $N$-dimensional real column vectors, and real numbers, respectively. $\mathbf{1}_N$ denotes the $N$-dimensional vector with all entries being one, and $\boldsymbol{I}_N$ is the $N$-dimensional identity matrix. $\otimes$ represents the Kronecker product. $\|\cdot\|$ denotes the Euclidean norm. $W^{\mathrm{T}}$ and $W^{-1}$ denote the transpose and inverse of a matrix, respectively. $\mathrm{col}(x_1,\ldots,x_N)=[x_1^{\mathrm{T}},\ldots,x_N^{\mathrm{T}}]^{\mathrm{T}}$ is the column vector stacked with vectors $x_1,\ldots,x_N$.

## 2. Preliminaries and problem formulation

### 2.1. Algebraic graph theory

In this paper, we use a graph to model a communication network. $\mathscr{G}=(\mathscr{V},\mathscr{E})$ represents an undirected graph, where $\mathscr{V}=\{1,2,\ldots,N\}$ and $\mathscr{E}\subseteq\mathscr{V}\times\mathscr{V}$ are the node set and the edge set, respectively. If $(i,j)\in\mathscr{E}$ exists, then the pair of nodes $i$ and $j$ are neighbors and they can transmit their information to each other. The neighbor set of node $i$ is defined by $\mathscr{N}_i=\{j\subseteq\mathscr{V}:(i,j)\subseteq\mathscr{E},j\neq i\}$. $A=[a_{ij}]\in\mathbb{R}^{N\times N}$ is a weighted adjacency matrix of graph $\mathscr{G}$, where $a_{ij}=a_{ji}>0$ if $(i,j)\in\mathscr{E}$, otherwise $a_{ij}=0$. The degree matrix of $\mathscr{G}$ is given by $D=\mathrm{diag}\{b_1,\ldots,b_N\}\in\mathbb{R}^{N\times N}$ with $b_i=\sum_{j=1}^{N}a_{ij}$. The Laplacian matrix $L$ is defined as $L=D-A$. For undirected graphs, $L$ is symmetric and positive semi-definite. Graph $\mathscr{G}$ is called connected if there is one path between any two nodes in $\mathscr{G}$. Then, the associated Laplacian matrix has a simple zero eigenvalue and $\mathbf{1}_N$ is the corresponding eigenvector; i.e., $L\mathbf{1}_N=0$. Assume that the eigenvalues of $L$ are $\lambda_1,\ldots,\lambda_N$ with an increasing order, $0=\lambda_1<\lambda_2\leq\lambda_3\leq\cdots\leq\lambda_N$. More details can be found in Ref. [30].

**Assumption 1** Graph $\mathscr{G}$ is undirected and connected.

**Remark 1** Under the Assumption 1, there exists an orthogonal matrix $Q=[r,R]$ with $R\in\mathbb{R}^{N\times(N-1)}$, $r=\frac{\mathbf{1}_N}{\sqrt{N}}$, and $RR^{\mathrm{T}}=\boldsymbol{I}_N-\frac{1}{N}\mathbf{1}_N\mathbf{1}_N^{\mathrm{T}}$ such that $Q^{\mathrm{T}}LQ=\mathrm{diag}\{0,\lambda_2,\ldots,\lambda_N\}$, then $\lambda_2\boldsymbol{I}_{N-1}\leq R^{\mathrm{T}}LR\leq\lambda_N\boldsymbol{I}_{N-1}$.

### 2.2. Problem formulation

In this subsection, we put forward an energy-optimal strategy for robots to move some object to one location and ensure the battery energy consumption to reach the minimum.

First, consider a group of $N$ nonholonomic wheeled mobile robots moving on a horizontal plane. The kinematics of robot $i$ is described by

$$\begin{aligned}\dot{q}_{i1}(t)&=v_i(t)\cos(q_{i3}(t)),\\ \dot{q}_{i2}(t)&=v_i(t)\sin(q_{i3}(t)),\\ \dot{q}_{i3}(t)&=\omega_i(t),\end{aligned}\tag{1}$$

where $q_i(t)=[q_{i1}(t),q_{i2}(t)]^{\mathrm{T}}\in\mathbb{R}^2$ and $q_{i3}(t)\in\mathbb{R}$ are the position and orientation of robot $i$, respectively. $v_i(t),\omega_i(t)\in\mathbb{R}$ are the linear and angular velocity control inputs of robot $i$, respectively.

Assume that the local cost function relies on the state of the hand position and shows the battery energy consumption.

Combined with the practical significance of distributed optimization, the energy-optimal problem can be formulated as the following distributed optimization problem:

$$\min_{x\in\mathbb{R}^2}\sum_{i=1}^{N} f_i(x), \tag{2}$$

where $f_i:\mathbb{R}^2\to\mathbb{R}$ is the local cost function of robot $i$.

**Remark 2** In this paper, we consider the case that the robots move some object from one place to another by the splints fixed at the hand positions. Consequently, the local cost functions representing the battery energy consumption of robots rely on the states of the hand positions rather than the centroid of the robots. Furthermore, the consensus of the hand positions has more practical interest, see Ref. [31].

**Assumption 2** The battery level of each robot is sufficient to guarantee that the robot team will rendezvous at the optimal point where the overall energy consumption reaches the minimum.

Next, we show the dynamics of the hand positions. According to the conversion of Ref. [32], assume the hand position of each robot is a point located at a fixed distance $\rho>0$ from the centroid of the robot and on the axis of the orientation (see Fig. 1).

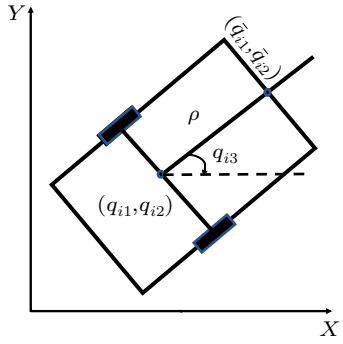



**Fig. 1.** Illustration of the hand position.

For robot $i$, the coordinates of the hand position $x_i(t)=[\bar{q}_{i1}(t),\bar{q}_{i2}(t)]^{\mathrm{T}}$ are

$$\begin{aligned}\bar{q}_{i1}(t)&=q_{i1}(t)+\rho\cos(q_{i3}(t)),\\ \bar{q}_{i2}(t)&=q_{i2}(t)+\rho\sin(q_{i3}(t)),\end{aligned} \tag{3}$$

where $\rho>0$ is a positive constant.

Differentiating Eq. (3) yields

$$\begin{aligned}\dot{\bar{q}}_{i1}(t)&=v_i(t)\cos(q_{i3}(t))-\rho\omega_i(t)\sin(q_{i3}(t)):=u_{i1}(t),\\ \dot{\bar{q}}_{i2}(t)&=v_i(t)\sin(q_{i3}(t))+\rho\omega_i(t)\cos(q_{i3}(t)):=u_{i2}(t),\end{aligned} \tag{4}$$

where $u_{i1}(t)$ and $u_{i2}(t)$ are the virtual control inputs. As $\rho>0$, the above transformation is hence globally invertible, and the true control inputs $v_i(t)$ and $\omega_i(t)$ can be deduced as

$$\begin{aligned}v_i(t)&=u_{i1}(t)\cos(q_{i3}(t))+u_{i2}(t)\sin(q_{i3}(t)),\\ \omega_i(t)&=\rho^{-1}(-u_{i1}(t)\sin(q_{i3}(t))+u_{i2}(t)\cos(q_{i3}(t))).\end{aligned} \tag{5}$$

Hence, the dynamics of the hand positions can be expressed as

$$\begin{aligned}\dot{\bar{q}}_{i1}(t)&=u_{i1}(t),\\ \dot{\bar{q}}_{i2}(t)&=u_{i2}(t),\\ \dot{q}_{i3}(t)&=\rho^{-1}(-u_{i1}(t)\sin(q_{i3}(t))+u_{i2}(t)\cos(q_{i3}(t))).\end{aligned} \tag{6}$$

Obviously, the dynamics of the hand position of each robot has been modelled in the form of single integrator

$$\dot{x}_i(t)=u_i(t), \tag{7}$$

where $u_i(t)=[u_{i1}(t),u_{i2}(t)]^{\mathrm{T}}$.

The distributed energy-optimal problem for multiple nonholonomic wheeled mobile robots is stated as: design an event-triggered control law, as well as an event-triggering condition for each robot such that the hand positions of all robots tend to the optimal point of the cost function while the global energy consumption reaches the minimum. In addition, the velocities and orientations converge to zero and some fixed values, respectively; that is,

$$\begin{aligned}&x_i(t)\to x^*,\ v_i(t)\to 0,\ q_{i3}(t)\to q_{is},\\ &\omega_i(t)\to 0,\ i=1,2,\ldots,N,\end{aligned} \tag{8}$$

where $x^*=\arg\min_{x\in\mathbb{R}^2}\sum_{i=1}^{N} f_i(x)$, $q_{is}$ is a certain constant.

Assume that the local cost function $f_i$ is only available to robot $i$, and may be different. We make the following assumption on these cost functions.

**Assumption 3** The local cost functions $f_i$, $i=1,\ldots,N$, are differentiable and strongly convex with constant $\chi_i>0$; that is,

$$(x-y)^{\mathrm{T}}(\nabla f_i(x)-\nabla f_i(y))\geq\chi_i\|x-y\|^2,\ \ \forall x,y\in\mathbb{R}^2.$$

Furthermore, their gradients are Lipschitz with constant $\sigma_i>0$; that is,

$$\|\nabla f_i(x)-\nabla f_i(y)\|\leq\sigma_i\|x-y\|,\ \ \forall x,y\in\mathbb{R}^2.$$

**Lemma 1**[33] (Young's inequality) For any $x,y\in\mathbb{R}^n,\zeta>0$, we have

$$x^{\mathrm{T}}y\leq\frac{x^{\mathrm{T}}x}{2\zeta}+\frac{\zeta}{2}y^{\mathrm{T}}y.$$

## 3. Distributed event-triggered optimization algorithm for minimum energy consumption

In this section, a distributed event-triggered optimization algorithm is proposed to achieve the control objective in Eq. (8) and thus solve the optimization problem in Eq. (2).

To avoid the unnecessary resource waste, we construct the control protocol based on the distributed event-triggered strategy. For robot $i$, the control law is updated only at some dis-

crete times $\{t_0^i, t_1^i, \ldots\}$, called the event-triggering sequence, satisfying a certain triggering condition.

To construct the triggering condition, we define the state measurement error for robot $i$ as

$$e_i(t) = x_i(t_k^i) - x_i(t), \quad t \in [t_k^i, t_{k+1}^i), \tag{9}$$

where $t_k^i$ is the $k$-th event-triggering time of robot $i$.

The event-triggering times $\{t_0^i, t_1^i, \ldots\}$ for robot $i$ are specified when the following triggering condition

$$\|e_i(t)\| \leq \beta_i \left\| \sum_{j \in \mathcal{N}_i} a_{ij} \left( x_i(t_k^i) - x_j(t_{k'(t)}^j) \right) \right\| \tag{10}$$

is violated; that is,

$$t_0^i = 0,$$
$$t_{k+1}^i = \inf \left\{ t > t_k^i : \|e_i(t)\| > \beta_i \left\| \sum_{j \in \mathcal{N}_i} a_{ij} \left( x_i(t_k^i) - x_j(t_{k'(t)}^j) \right) \right\| \right\}, \quad k \in \mathbb{N},$$

where $\beta_i > 0$ will be determined later and $k'(t) = \arg \min_{l \in \mathbb{N}: t \geq t_l^j} \{t - t_l^j\}$. For $t \in [t_k^i, t_{k+1}^i)$, $t_{k'(t)}^j$ is the latest triggering instant of robot $j$.

Then, the distributed event-triggered optimization control law for robot $i$ is constructed as

$$u_i(t) = \alpha \sum_{j \in \mathcal{N}_i} a_{ij} \left( x_j(t_{k'(t)}^j) - x_i\left(t_k^i\right) \right) - \gamma \nabla f_i \left( x_i(t_k^i) \right), \quad t \in [t_k^i, t_{k+1}^i), \tag{11}$$

where $\alpha$ and $\gamma$ are positive constants. The control law for each robot depends on itself and its neighbours' latest event-triggering instants.

**Remark 3** In Ref. [34], the authors considered the distributed optimization problem with delayed sampled-data, where the communication among agents is periodic. Compared to the periodic sampled communication, the event-triggered control protocol proposed in this paper can greatly reduce the communication times among robots.

Let $\boldsymbol{x} = \mathrm{col}(x_1, \ldots, x_N)$, $e = \mathrm{col}(e_1, \ldots, e_N)$, and $\nabla F(\boldsymbol{x}) = \mathrm{col}(\nabla f_1(x_1), \ldots, \nabla f_N(x_N))$. The dynamics of the hand positions can be rewritten in the compact form

$$\dot{\boldsymbol{x}} = -\alpha(L \otimes \boldsymbol{I}_2)(\boldsymbol{x} + e) - \gamma \nabla F(\boldsymbol{x} + e). \tag{12}$$

The following lemma illustrates a basic necessary condition for the optimality.

**Lemma 2**[35] Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function, and $x^*$ be the minimum of $f$ over $\mathbb{R}^n$. Then

$$\nabla f(x^*) = 0.$$

The following lemma indicates the relationship between the equilibrium point of system in Eq. (12) and the optimal solution of problem in Eq. (2).

**Lemma 3** Suppose that Assumption 1, Assumption 2, and Assumption 3 hold, then the equilibrium point of system in Eq. (12) is equivalent to the optimal solution of problem in Eq. (2).

**Proof** Let $x^* \in \mathbb{R}^2$ be the optimal solution of problem in Eq. (2) and denote $\boldsymbol{x}^* = \mathrm{col}(x^*, x^*, \ldots, x^*)$. Obviously, $\alpha \sum_{j \in \mathcal{N}_i} a_{ij}(x^* - x^*) - \gamma \nabla f_i(x^*) = 0$, which shows that the optimal solution $\boldsymbol{x}^*$ of problem in Eq. (2) is the equilibrium point of system in Eq. (12).

Conversely, for any equilibrium point $\hat{\boldsymbol{x}} = \mathrm{col}(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_N)$ of system in Eq. (12), we have $e = 0$ once the equilibrium of system in Eq. (12) is achieved. Hence, we obtain

$$-\alpha(L \otimes \boldsymbol{I}_2)\hat{\boldsymbol{x}} - \gamma \nabla F(\hat{\boldsymbol{x}}) = 0. \tag{13}$$

From Assumption 1, we have $\mathbf{1}_N^{\mathrm{T}} L = 0$. Multiplying the left of Eq. (13) by $(\mathbf{1}_N^{\mathrm{T}} \otimes \boldsymbol{I}_2)$ gives

$$-\alpha(\mathbf{1}_N^{\mathrm{T}} \otimes \boldsymbol{I}_2)(L \otimes \boldsymbol{I}_2)\hat{\boldsymbol{x}} - \gamma(\mathbf{1}_N^{\mathrm{T}} \otimes \boldsymbol{I}_2)\nabla F(\hat{\boldsymbol{x}}) = 0. \tag{14}$$

Therefore,

$$(\mathbf{1}_N^{\mathrm{T}} \otimes \boldsymbol{I}_2)\nabla F(\hat{\boldsymbol{x}}) = 0,$$

which means

$$\sum_{i=1}^{N} \nabla f_i(\hat{x}_i) = 0.$$

According to Lemma 2, we obtain that the equilibrium point $\hat{\boldsymbol{x}}$ of system in Eq. (12) is the optimal solution of problem in Eq. (2).

Employ the transformation

$$\bar{\boldsymbol{x}}(t) = \boldsymbol{x}(t) - \boldsymbol{x}^*, \tag{15}$$

and system equation (12) is transformed into

$$\dot{\bar{\boldsymbol{x}}} = -\alpha(L \otimes \boldsymbol{I}_2)(\bar{\boldsymbol{x}} + e) - \gamma h, \tag{16}$$

where $h = h_1 + h_2$ with $h_1 = \nabla F(\boldsymbol{x} + e) - \nabla F(\boldsymbol{x})$ and $h_2 = \nabla F(\boldsymbol{x}) - \nabla F(\boldsymbol{x}^*)$.

Let

$$\begin{aligned} X(t) &= [X_1^{\mathrm{T}}, X_2^{\mathrm{T}}]^{\mathrm{T}} \\ &= (Q^{\mathrm{T}} \otimes \boldsymbol{I}_2)\bar{\boldsymbol{x}}(t), \\ E(t) &= [E_1^{\mathrm{T}}, E_2^{\mathrm{T}}]^{\mathrm{T}} \\ &= (Q^{\mathrm{T}} \otimes \boldsymbol{I}_2)e(t), \end{aligned} \tag{17}$$

where $X_1, E_1 \in \mathbb{R}^2, X_2, E_2 \in \mathbb{R}^{2(N-1)}$, and $Q$ is determined by Remark 1. From Eq. (16) and Remark 1, we obtain

$$\begin{aligned} \dot{X}_1 &= -\gamma(r^{\mathrm{T}} \otimes \boldsymbol{I}_2)h, \\ \dot{X}_2 &= -\gamma(R^{\mathrm{T}} \otimes \boldsymbol{I}_2)h - \alpha(R^{\mathrm{T}} L R \otimes \boldsymbol{I}_2)(X_2 + E_2). \end{aligned} \tag{18}$$

**Theorem 1** Suppose Assumption 1, Assumption 2, and Assumption 3 hold, and choose appropriate parameters

$\alpha,\gamma,\beta_i>0$ satisfying

$$0<\beta_i\le\frac{\chi}{\lambda_N\sqrt{2(2\sigma^2+\chi^2)}},$$
$$\mu=\frac{1}{4}\chi\gamma-\frac{\chi^2\alpha\lambda_N}{4\sigma^2}>0, \tag{19}$$

where $\chi=\min_{i\in\mathcal{V}}\{\chi_i\}$, $\sigma=\max_{i\in\mathcal{V}}\{\sigma_i\}$. Then, the distributed optimization control law in Eq. (11) with the triggering condition in Eq. (10) can solve the optimization problem in Eq. (2), and meanwhile the control objective in Eq. (8) can be achieved for any initial state $q_i(0)$.

**Proof** Choose the following Lyapunov function

$$V(t)=\frac{1}{2}X^{\mathrm{T}}(t)X(t)=\frac{1}{2}(X_1^{\mathrm{T}}X_1+X_2^{\mathrm{T}}X_2), \tag{20}$$

where $X(t)$ is defined in Eq. (17).

The derivative of $V$ along the system Eq. (18) yields

$$\begin{aligned}\dot V &= -\gamma X_1^{\mathrm{T}}(r^{\mathrm{T}}\otimes\boldsymbol{I}_2)h-\gamma X_2^{\mathrm{T}}(R^{\mathrm{T}}\otimes\boldsymbol{I}_2)h\\&\quad-\alpha X_2^{\mathrm{T}}(R^{\mathrm{T}}LR\otimes\boldsymbol{I}_2)(X_2+E_2)\\&= -\gamma\overline{\boldsymbol{x}}^{\mathrm{T}}h-\alpha X_2^{\mathrm{T}}(R^{\mathrm{T}}LR\otimes\boldsymbol{I}_2)X_2\\&\quad-\alpha X_2^{\mathrm{T}}(R^{\mathrm{T}}LR\otimes\boldsymbol{I}_2)E_2.\end{aligned}$$

Recalling that $h=h_1+h_2=(\nabla F(\boldsymbol{x}+e)-\nabla F(\boldsymbol{x}))+(\nabla F(\boldsymbol{x})-\nabla F(\boldsymbol{x}^*))$, using Lemma 1 and Assumption 3 along with $\|Q\otimes\boldsymbol{I}_2\|=1$, we obtain the following estimation:

$$\begin{aligned}-\gamma\overline{\boldsymbol{x}}^{\mathrm{T}}h &= -\gamma\overline{\boldsymbol{x}}^{\mathrm{T}}(\nabla F(\boldsymbol{x}+e)-\nabla F(\boldsymbol{x}))\\&\quad-\gamma\overline{\boldsymbol{x}}^{\mathrm{T}}(\nabla F(\boldsymbol{x})-\nabla F(\boldsymbol{x}^*))\\&\le\sigma\gamma\|\overline{\boldsymbol{x}}\|\|e\|-\chi\gamma\|\overline{\boldsymbol{x}}\|^2\\&\le-\frac{1}{2}\chi\gamma\|\overline{\boldsymbol{x}}\|^2+\frac{\sigma^2\gamma}{2\chi}\|e\|^2.\end{aligned}$$

In addition,

$$-\alpha X_2^{\mathrm{T}}(R^{\mathrm{T}}LR\otimes\boldsymbol{I}_2)E_2\le\frac{\alpha}{2}\left(\frac{1}{\lambda_N}X_2^{\mathrm{T}}\left((R^{\mathrm{T}}LR)^2\otimes\boldsymbol{I}_2\right)X_2+\lambda_NE_2^{\mathrm{T}}E_2\right).$$

Note that $R^{\mathrm{T}}LR$ is symmetric and positive definite matrix with eigenvalues $\lambda_2,\dots,\lambda_N$ and thus the eigenvalues of $R^{\mathrm{T}}LR-\frac{1}{2\lambda_N}(R^{\mathrm{T}}LR)^2$ are $\lambda_i-\frac{1}{2\lambda_N}\lambda_i^2, i=2,\dots,N$. Obviously, $\lambda_i-\frac{1}{2\lambda_N}\lambda_i^2\ge\lambda_2-\frac{1}{2\lambda_N}\lambda_2^2\ge\frac{1}{2}\lambda_2$. Then we have

$$\begin{aligned}&-\alpha X_2^{\mathrm{T}}(R^{\mathrm{T}}LR\otimes\boldsymbol{I}_2)X_2-\alpha X_2^{\mathrm{T}}(R^{\mathrm{T}}LR\otimes\boldsymbol{I}_2)E_2\\&\le-\alpha\left(X_2^{\mathrm{T}}\left(\left(R^{\mathrm{T}}LR-\frac{1}{2\lambda_N}(R^{\mathrm{T}}LR)^2\right)\otimes\boldsymbol{I}_2\right)X_2\right)\\&\quad+\frac{\alpha}{2}\lambda_NE_2^{\mathrm{T}}E_2\\&\le-\frac{1}{2}\alpha\lambda_2X_2^{\mathrm{T}}X_2+\frac{1}{2}\alpha\lambda_N\|e\|^2.\end{aligned}$$

Therefore, we obtain

$$\dot V(t)\le-\frac{1}{2}\chi\gamma X_1^{\mathrm{T}}X_1-\left(\frac{1}{2}\chi\gamma+\frac{1}{2}\alpha\lambda_2\right)X_2^{\mathrm{T}}X_2+\left(\frac{1}{2}\alpha\lambda_N+\frac{\sigma^2\gamma}{2\chi}\right)\|e\|^2. \tag{21}$$

It follows from the triggering condition in Eq. (10) that

$$\begin{aligned}\|e_i\| &\le\beta_i\left\|\sum_{j\in\mathcal{N}_i}a_{ij}\left(x_i(t_k^i)-x^*+x^*-x_j(t_{k'(t)}^j)\right)\right\|\\&\le\beta_i\left\|\sum_{j\in\mathcal{N}_i}a_{ij}\left(\bar x_i(t_k^i)-\bar x_j(t_{k'(t)}^j)\right)\right\|.\end{aligned}$$

Hence,

$$\begin{aligned}\|e\|^2&\le\beta^2(\bar{\boldsymbol{x}}+e)^{\mathrm{T}}(L^{\mathrm{T}}L\otimes\boldsymbol{I}_2)(\bar{\boldsymbol{x}}+e)\\&\le\beta^2(2\bar{\boldsymbol{x}}^{\mathrm{T}}(L^{\mathrm{T}}L\otimes\boldsymbol{I}_2)\bar{\boldsymbol{x}}+2e^{\mathrm{T}}(L^{\mathrm{T}}L\otimes\boldsymbol{I}_2)e)\\&\le2\beta^2\lambda_N^2\|\bar{\boldsymbol{x}}\|^2+2\beta^2\lambda_N^2\|e\|^2,\end{aligned}$$

where $\beta=\max_{i\in\mathcal{V}}\{\beta_i\}$. Recall that $\|Q\otimes\boldsymbol{I}_2\|=1$, one has

$$\|e\|^2\le\frac{2\beta^2\lambda_N^2}{1-2\beta^2\lambda_N^2}\|X\|^2. \tag{22}$$

By substituting Eq. (22) into Eq. (21), we obtain

$$\begin{aligned}\dot V&\le-\left[\frac{1}{2}\chi\gamma-\left(\frac{1}{2}\alpha\lambda_N+\frac{\sigma^2\gamma}{2\chi}\right)\frac{2\beta^2\lambda_N^2}{1-2\beta^2\lambda_N^2}\right]X_1^{\mathrm{T}}X_1\\&\quad-\left[\frac{1}{2}\chi\gamma+\frac{1}{2}\alpha\lambda_2-\left(\frac{1}{2}\alpha\lambda_N+\frac{\sigma^2\gamma}{2\chi}\right)\frac{2\beta^2\lambda_N^2}{1-2\beta^2\lambda_N^2}\right]X_2^{\mathrm{T}}X_2.\end{aligned}$$

Since $0<\beta_i\le\frac{\chi}{\lambda_N\sqrt{2(2\sigma^2+\chi^2)}}$, then $\frac{\beta^2\lambda_N^2}{1-2\beta^2\lambda_N^2}\le\frac{\chi^2}{4\sigma^2}$. Therefore, one has

$$\begin{aligned}\dot V&\le-\left(\frac{1}{4}\chi\gamma-\frac{\chi^2\alpha\lambda_N}{4\sigma^2}\right)X_1^{\mathrm{T}}X_1\\&\quad-\left(\frac{1}{4}\chi\gamma+\frac{1}{2}\alpha\lambda_2-\frac{\chi^2\alpha\lambda_N}{4\sigma^2}\right)X_2^{\mathrm{T}}X_2\\&\le-\left(\frac{1}{4}\chi\gamma-\frac{\chi^2\alpha\lambda_N}{4\sigma^2}\right)\|X\|^2.\end{aligned}$$

Denote $\mu=\frac{1}{4}\chi\gamma-\frac{\chi^2\alpha\lambda_N}{4\sigma^2}$, and we can select appropriate parameters $\alpha,\gamma$, and $\beta_i$ such that $\mu>0$. One knows from Eq. (20) that

$$V(t)\le c\exp(-2\mu t), \tag{23}$$

where $c=V(0)$. This implies that $\bar{\boldsymbol{x}}$ exponentially converges to 0, that is, $x_i$ exponentially converges to $x^*$.

From Eq. (11) and $\nabla F(\boldsymbol{x}^*)=0$, $u(t)=\mathrm{col}(u_1(t),\dots,u_N(t))$ can be written as

$$u(t)=-\alpha(L\otimes\boldsymbol{I}_2)(\overline{\boldsymbol{x}}(t)+e(t))-\gamma(\nabla F(\boldsymbol{x}+e)-\nabla F(\boldsymbol{x}^*)).$$

According to Assumption 3, we obtain

$$\begin{aligned}\|u(t)\| &\le\alpha\|L\|(\|\overline{\boldsymbol{x}}(t)\|+\|e(t)\|)+\gamma\|\nabla F(\boldsymbol{x}+e)-\nabla F(\boldsymbol{x}^*)\|\\&\le(\alpha\|L\|+\gamma\sigma)\|\overline{\boldsymbol{x}}(t)\|+(\alpha\|L\|+\gamma\sigma)\|e(t)\|.\end{aligned}$$

$\bar{\boldsymbol{x}}(t)$ exponentially converges to zero and meanwhile $e(t)$ exponentially tends to zero; therefore, $u(t)$ exponentially reaches

zero. From Eq. (5), we know that $v_i(t)$ and $\omega_i(t)$ exponentially converge to zero. Hence, the orientation $q_{i3}(t)$ tends to a certain stationary value.

Next, we will show that any inter-event intervals are lower bounded by a positive number, which guarantees that there is no Zeno behavior.

**Theorem 2** Consider the nonholonomic mobile robots with the hand position dynamics in Eq. (12), the controller in Eq. (11), and the triggering condition in Eq. (10). Then, the Zeno behavior cannot occur in system Eq. (12).

**Proof** From Eq. (23), we have $\|\boldsymbol{x}-\boldsymbol{x}^*\| \leq \sqrt{2c}\exp(-\mu t)$. For robot $i$, $i\in 1,2,\ldots,N$, at time $t\in[t_k^i,t_{k+1}^i)$,

$$\begin{aligned}\|\dot{e}_i(t)\| &= \|\dot{x}_i(t)\| \\ &= \left\|-\alpha\sum_{j\in\mathcal{N}_i}a_{ij}(x_i(t_k^i)-x_j(t_{k'(t)}^j))-\gamma\nabla f_i(x_i(t_k^i))\right\| \\ &\leq \alpha\left\|\sum_{j\in\mathcal{N}_i}a_{ij}\left[(x_i(t_k^i)-x^*)-(x_j(t_{k'(t)}^j)-x^*)\right]\right\| \\ &\quad+\gamma\sigma\left\|x_i(t_k^i)-x^*\right\| \\ &\leq 2\alpha b\sqrt{2c}+\gamma\sigma\sqrt{2c}\exp(-\mu t),\end{aligned}$$

where $b=\max_{i\in\mathcal{V}}\{b_i\}$ and recall that $b_i=\sum_{j=1}^{N}a_{ij}$.

By applying Newton–Leibnitz formula and noting that $e_i(t_k^i)=0$, we deduce that

$$\begin{aligned}\|e_i(t)\| &= \left\|\int_{t_k^i}^{t}\dot{e}_i(s)\mathrm{d}s\right\| \\ &\leq \int_{t_k^i}^{t}\|\dot{e}_i(s)\|\mathrm{d}s \\ &\leq 2\alpha b\sqrt{2c}(t-t_k^i)-\frac{\gamma\sigma\sqrt{2c}}{\mu}\left(\exp(-\mu t)-\exp(-\mu t_k^i)\right).\end{aligned}$$

According to the above inequality, we have

$$\begin{aligned}\lim_{t\to t_{k+1}^{i-}}\|e_i(t)\| &\leq 2\alpha b\sqrt{c}(t_{k+1}^i-t_k^i) \\ &\quad+\frac{\gamma\sigma\sqrt{c}}{\mu}\left(\exp(-\mu t_k^i)-\exp(-\mu t_{k+1}^i)\right). \quad (24)\end{aligned}$$

Note that $\lim_{t\to t_{k+1}^{i-}}\|e_i(t)\|=\beta_i\|\sum_{j\in\mathcal{N}_i}a_{ij}(x_i(t_k^i)-x_j(t_{k'(t)}^j)\|$ on the basis of the triggering condition in Eq. (10), we hence obtain

$$\begin{aligned}&\beta_i\left\|\sum_{j\in\mathcal{N}_i}a_{ij}(x_i(t_k^i)-x_j(t_{k'(t)}^j))\right\| \\ &\leq 2\alpha b\sqrt{2c}(t_{k+1}^i-t_k^i) \\ &\quad+\frac{\gamma\sigma\sqrt{2c}}{\mu}\left(\exp(-\mu t_k^i)-\exp(-\mu t_{k+1}^i)\right). \quad (25)\end{aligned}$$

Multiplying Eq. (25) by $\exp(\mu t_k^i)$ and denote $\varepsilon_k^i=t_{k+1}^i-t_k^i$, one knows

$$\alpha_k^i\leq s_k^i\varepsilon_k^i+m(1-\exp(-\mu\varepsilon_k^i)),$$

where $\alpha_k^i=\beta_i\|\sum_{j\in\mathcal{N}_i}a_{ij}(x_i(t_k^i)-x_j(t_{k'(t)}^j)\|\exp(\mu t_k^i)\geq 0$, $s_k^i=2\alpha b\sqrt{2c}\exp(\mu t_k^i)>0$, and $m=\frac{\gamma\sigma\sqrt{2c}}{\mu}>0$.

Since $\exp(q)\geq 1+q$ for any $q\in\mathbb{R}$, we have $\exp(-\mu\varepsilon_k^i)\geq 1-\mu\varepsilon_k^i$. For any robot $i$,

$$0\leq\frac{\alpha_k^i}{s_k^i+\mu m}\leq\varepsilon_k^i=t_{k+1}^i-t_k^i.$$

In the conclusion, the Zeno behavior can be excluded.

## 4. Numerical simulations

In this section, some numerical results are given to verify the theoretical results.

Consider the unicycle system with five wheeled mobile robots satisfying Eq. (1), whose interaction topology is depicted in Fig. 2 with weights of all edges as 1. Thus, the largest eigenvalue of the Laplacian matrix is $\lambda_5=5$. The initial states

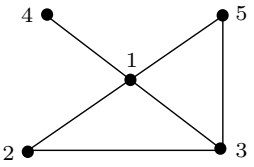



**Fig. 2.** Communication topology for robots.

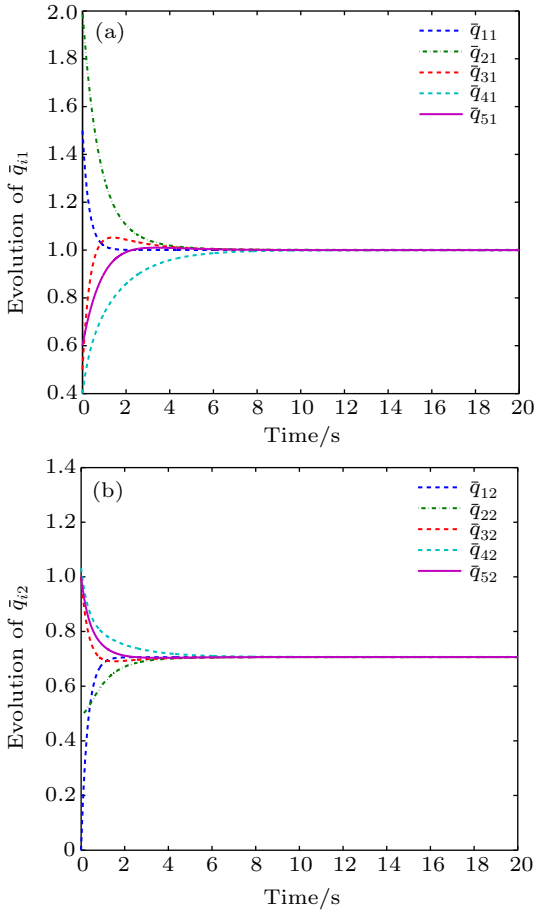



**Fig. 3.** State trajectories of $x_i$.

are chosen as $(q_{11},q_{12},q_{13})(0)=(1,0,0)$, $(q_{21},q_{22},q_{23})(0)=(1.5,0.5,0)$, $(q_{31},q_{32},q_{33})(0)=(1,1,\pi)$, $(q_{41},q_{42},q_{43})(0)=(0.15,0.6,\frac{\pi}{3})$, and $(q_{51},q_{52},q_{53})(0)=(0.1,1,0)$. Define the hand position by $\rho=0.5$. The local battery consumption functions are defined as $f_i(x_i)=50\|x_i(t)-x_i(0)\|^2$. It can be easily calculated that the optimal value $x^*=[1,0.71]^{\mathrm{T}}$. It is observed that the above local cost functions are strongly convex and their gradients are Lipschitz. After the calculation of the above local cost function $f_i(x_i), i\in\mathscr{V}$, the minimum strongly-convex constant is $\chi=100$ and the maximum Lipschitz constant is $\sigma=200$. Choose $\alpha=0.6$, $\gamma=0.01$, $\beta_1=0.04$, $\beta_2=0.04$, $\beta_3=0.03$, $\beta_4=0.02$, and $\beta_5=0.045$ satisfying Eq. (19). Then, by Theorem 1, the optimization problem in Eq. (2) can be solved by the distributed optimization control law in Eq. (11) with the event-triggering condition in Eq. (10). The simulation results are shown in Figs. 3–6, indicating the effectiveness of the proposed energy-optimal strategy. Figure 3 shows the states of five robots' hand positions converge to the optimal point $x^*=[1,0.71]^{\mathrm{T}}$. Figure 4 shows the event-triggering times of five robots. The measurement error and threshold evolution of the robots are shown in Fig. 5. From Fig. 6, we can see that the orientations converge to fixed values, and the linear velocities reach zero; that is, they will keep still when the robots reach the optimal point.

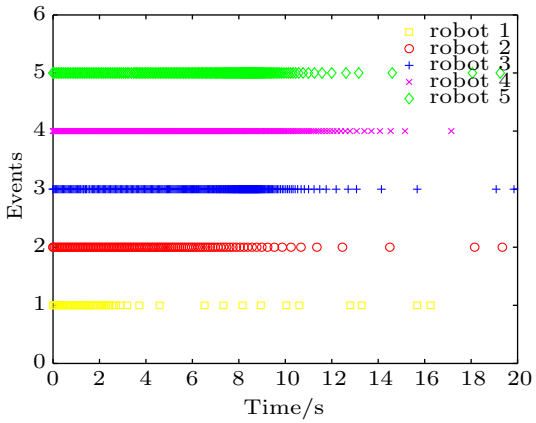



**Fig. 4.** Event-triggering times of five robots.

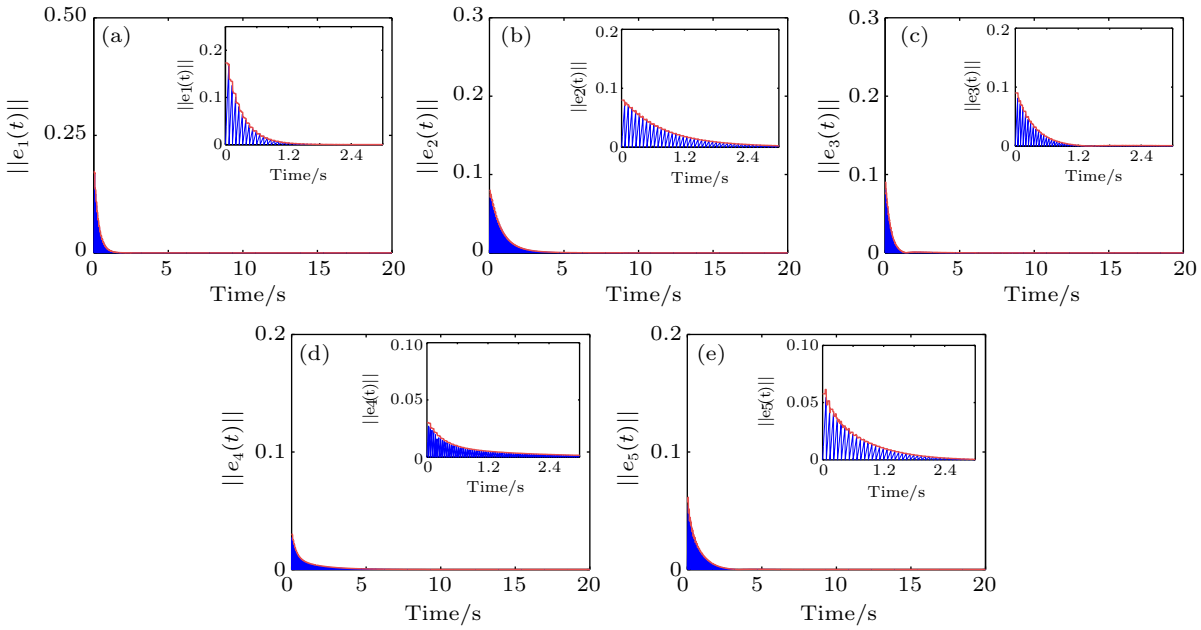



**Fig. 5.** Measurement error and threshold evolution of robots.

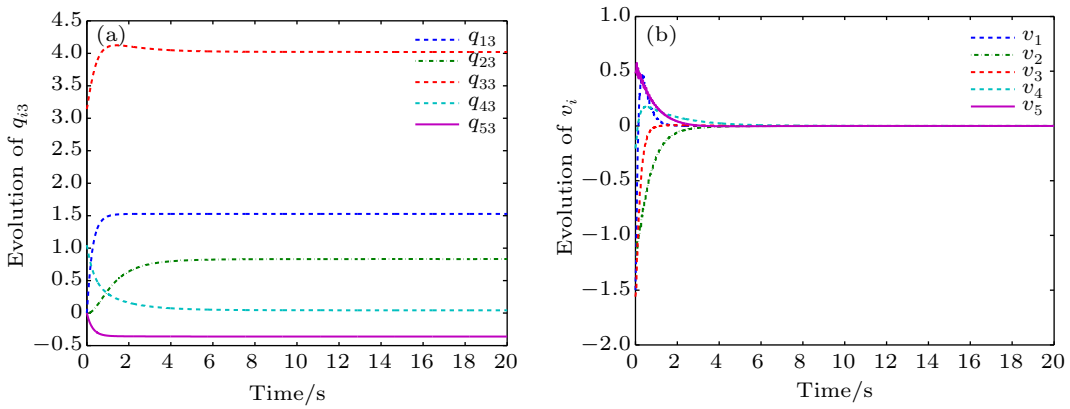



**Fig. 6.** The evolution of orientation states and linear velocities of robot $i$.

## 5. Conclusion

In this paper, we have presented an energy-optimal strategy such that the states of the robots' hand positions achieve consensus while minimizing the overall energy consumption of robots. We have formulated this problem as the distributed optimization problem. By coordinate transformation, the dynamics of the hand positions are formulated as two groups of first-order integrators. A distributed event-triggered optimization algorithm is hence presented and guaranteed that the equilibrium point is the optimal solution. Based on the Lyapunov theory, sufficient conditions are obtained to make the states of the robots' hand positions exponentially converge to the optimal point of the global cost function. At the same time, the velocities and orientations converge to zero and certain fixed values, respectively. Moreover, the Zeno behavior is avoided.

Our further research considers that the robots rendezvous at some point while the overall battery energy consumption reaches the minimum in a finite time. The constraint condition that each robot moves only in its own area will also be considered.

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