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Discrete symmetrical perturbation and variational algorithm of disturbed Lagrangian systems*

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We investigate the perturbation to discrete conformal invariance and the adiabatic invariants of Lagrangian systems. A variational algorithm is proposed for a system subjected to the perturbation quantities. The discrete determining equations of the perturbations to conformal invariance are established. For perturbed Lagrangian systems, the condition of the existence of adiabatic invariant is derived from the discrete perturbation to conformal invariance. The numerical simulations demonstrate that the variational algorithm has the higher precision and the longer time stability than the standard numerical method.

Keywords: variational integrator, discrete perturbation to conformal invariance, discrete adiabatic invariants, disturbed Lagrangian systems

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1. Introduction

The adiabatic perturbation theory is utilized to give an approximate description of the dynamics of a system containing fast and slow variables.[1] If systems have no exact invariants, scholars focus on adiabatic (approximate) invariants of the perturbation systems.[2,3] Kruskal[4] and many others in later years[5,6] dealt with adiabatic invariants of the Hamiltonian systems. Djukić[7] and Cveticanin[8] developed the theory leading to adiabatic invariants of non-conservative systems which cannot be described by Hamiltonian canonical equations. The theory of the symmetries and conserved quantities gives a simpler method to solve the equations of dynamical systems.[9] The perturbation to the symmetries and adiabatic invariants develops rapidly because of the close relations between the integrability and the variations. As a whole, there are three types of perturbations to symmetries: the perturbation to Noether symmetries,[10–12] the perturbation to Lie symmetries,[13,14] and the perturbation to Mei symmetries.[15,16] Conformal invariance is built on the scale invariance, the translation invariance, the rotational invariance, and a variety of interactions.[17–21] Perturbation to conformal invariance is a modern approach to find the adiabatic invariants for dynamical systems. The solutions of the determining equations of the perturbation to conformal invariance should satisfy both the conformal invariance and the additional restriction equations. Although it can provide a new integral method, there has been no report on the perturbation to conformal invariance and the adiabatic invariants of dynamical systems.

The discrete equations serve as fundamental mathematical models in mechanics. The discrete symmetries offer the exact solutions for the discrete equations of dynamical systems through the discrete Noether theorem.[22] The Lie symmetries[23,24] and the Mei symmetries[25] also provide the exact invariants of the systems. Theory of perturbations to these three types of symmetries yields methods of getting the solutions of the difference equations. However, theory of perturbation to conformal invariance has not been proposed to solve discrete dynamical systems. The present work focuses on the perturbation to conformal invariance and the adiabatic invariants for discrete Lagrangian systems. Section 2 presents the discrete Lagrangian equations according to the continuous variables discretization. The definition of conformal invariance of the discrete Lagrangian systems is reviewed. The discrete determining equations of conformal invariance are established. Section 3 defines conformal invariance for discrete Lagrangian systems subjected to the perturbation quantities. Section 4 derives exact invariants directly from the conformal invariance of the systems without perturbation. The discrete perturbed Lagrangian systems and the z-th order adiabatic invariants are obtained based on the symmetrical transformations of the initial systems. Section 5 presents a numerical example to demonstrate the application of the proposed approach.

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2. Conformal invariance of discrete Lagrangian systems

Consider the space $\tilde{Z}$ of sequence $(t,\boldsymbol{q})$, a mechanical system whose configurations are determined by one independent variable t and dependent variable $\boldsymbol{q}=(q_1,q_2,\ldots,q_n)$. Let us rewrite a finite-difference functional in the form

$$L=\sum_{\Omega}L_{s,d}\left(t,t^{+},q_s,q_s^{+}\right)h^{+}, \tag{1}$$

with spacing $h^{+}=t^{+}-t$, and $q_s^{+}=\underset{+h}{S}(q_s)$. Let us take a variation of the difference functional (1) along some curve $q_s=\psi_s(t)$ at some point (t,q_s). The variation will affect only two terms in the sum of Eq. (1),

$$L=\cdots+L_{s,d}\left(t,t^{-},q_s,q_s^{-}\right)h^{-}+L_{s,d}\left(t,t^{+},q_s,q_s^{+}\right)h^{+}+\cdots, \tag{2}$$

with spacing $h^{-}=t-t^{-}$, and $q_s^{-}=\underset{-h}{S}(q_s)$. Thus, we obtain the following expression for the total variation of the difference functional:

$$\Delta L=\frac{\partial L}{\partial t}\Delta t+\frac{\partial L}{\partial q_s}\Delta q_s. \tag{3}$$

For Eq. (2), we have

$$\frac{\partial L}{\partial t}=\left(t-t^{-}\right)\frac{\partial L_{s,d}^{-}}{\partial t}+L_{s,d}^{-}+\left(t^{+}-t\right)\frac{\partial L_{s,d}}{\partial t}-L_{s,d}, \tag{4}$$

$$\begin{aligned}\frac{\partial L}{\partial q_s}=&\left(t-t^{-}\right)\frac{\partial L_{s,d}^{-}}{\partial q_s}+\frac{\partial\left(t-t^{-}\right)}{\partial q_s}L_{s,d}^{-}\\&+\left(t^{+}-t\right)\frac{\partial L_{s,d}}{\partial q_s}+\frac{\partial\left(t^{+}-t\right)}{\partial q_s}L_{s,d},\end{aligned} \tag{5}$$

with $L_{s,d}=L_{s,d}\left(t,t^{+},q_s,q_s^{+}\right)$ and $L_{s,d}^{-}=\underset{-h}{S}\left(L_{s,d}\right)=L_{s,d}\left(t,t^{-},q_s,q_s^{-}\right)$. Substitute Eqs. (4) and (5) into Eq. (3) yields

$$\begin{aligned}\Delta L=&\left(h^{+}\frac{\partial L_{s,d}}{\partial q_s}+h^{-}\frac{\partial L_{s,d}^{-}}{\partial q_s}\right)\Delta q_s\\&+\left(h^{+}\frac{\partial L_{s,d}}{\partial t}+h^{-}\frac{\partial L_{s,d}^{-}}{\partial t}+L_{s,d}^{-}-L_{s,d}\right)\Delta t.\end{aligned} \tag{6}$$

The total variation of the difference functional (1) satisfies $\Delta L=0$. Considering the independence of Δq_s and Δt, we have

$$\frac{\partial L_{s,d}}{\partial q_s}+\frac{h^{-}}{h^{+}}\frac{\partial L_{s,d}^{-}}{\partial q_s}=0, \tag{7}$$

$$\frac{\partial L_{s,d}}{\partial t}+\frac{h^{-}}{h^{+}}\frac{\partial L_{s,d}^{-}}{\partial t}-\underset{+h}{D}\left(L_{s,d}^{-}\right)=0, \tag{8}$$

where $\underset{+h}{D}(L_{s,d}^{-})=\frac{L_{s,d}-L_{s,d}^{-}}{h^{+}}$, $\underset{+h}{D}$ is the right difference derivative in the lattices space $Z=(t,t^{+},t^{-},q_s,q_s^{+},q_s^{-})$. Expressions (7) and (8) are the equations for the difference Lagrangian systems as was reported in Ref. [26]. To consider the discrete Lagrangian equations on the uniform difference mesh $h^{+}=h^{-}=h$, equations (7) and (8) become

$$\frac{\partial L_{s,d}}{\partial q_s}+\frac{\partial L_{s,d}^{-}}{\partial q_s}=0, \tag{9}$$

$$\frac{\partial L_{s,d}}{\partial t}+\frac{\partial L_{s,d}^{-}}{\partial t}-\underset{+h}{D}\left(L_{s,d}^{-}\right)=0. \tag{10}$$

Equations (9) and (10) require the discrete equations of the Lagrangian systems on the uniform difference mesh. Note that equations (9) are the first order difference equations, which are the difference Lagrangian equations. Equation (10) defines the lattice on which the Lagrangian equations are discretized, which is the energy equation. In the continuous limit, the lattice equation disappears, and both of them become the differential Euler equation.

The variation of the functional (1) along the orbit of a group is generated by the operator

$$X=\xi_{00}\frac{\partial}{\partial t}+\eta_{s0}\frac{\partial}{\partial q_s}, \tag{11}$$

where $\xi_{00}=\xi_{00}(t,q_s)$ and $\eta_{s0}=\eta_{s0}(t,q_s)$. The prolongation of the Lie group operator to the neighboring points (t^{-},q_s^{-}) and (t^{+},q_s^{+}) is as follows:

$$\begin{aligned}\mathrm{pr}X=&\xi_{00}\frac{\partial}{\partial t}+\eta_{s0}\frac{\partial}{\partial q_s}+\xi_{00}^{-}\frac{\partial}{\partial t^{-}}+\eta_{s0}^{-}\frac{\partial}{\partial q_s^{-}}\\&+\xi_{00}^{+}\frac{\partial}{\partial t^{+}}+\eta_{s0}^{+}\frac{\partial}{\partial q_s^{+}},\end{aligned} \tag{12}$$

where $\xi_{00}^{-}=\xi_{00}^{-}(t^{-},q_s^{-})$, $\eta_{s,0}^{-}=\eta_{s,0}(t^{-},q_s^{-})$, $\xi_{00}^{+}=\xi(t^{+},q_s^{+})$, and $\eta_{s0}^{+}=\eta_{s0}(t^{+},q_s^{+})$.

The conformal invariance and conserved quantities for discrete Lagrangian systems under the infinitesimal transformation of Lie groups have been studied in Ref. [27]. Here we introduce this straightforward method of the conformal invariance of Mei symmetry when the perturbation exists. Mei symmetry (the form invariance) is a new type of symmetry theory proposed by Mei and his collaborators, relating to the form invariance of differential equations under dynamical transformations.[28,29] For the Lagrangian $L_{s,d}=L_{s,d}\left(t,t^{+},q_s,q_s^{+}\right)$, the determining equations of Mei symmetry of the discrete Lagrangian systems are

$$\frac{\partial \mathrm{pr}X(L_{s,d})}{\partial q_s}+\frac{\partial \mathrm{pr}X(L_{s,d}^{-})}{\partial q_s}=0, \tag{13}$$

$$\frac{\partial \mathrm{pr}X(L_{s,d})}{\partial t}+\frac{\partial \mathrm{pr}X(L_{s,d}^{-})}{\partial t}-\underset{+h}{D}\left(\mathrm{pr}X(L_{s,d}^{-})\right)=0. \tag{14}$$

The expanded forms of the determining equations of conformal invariance are given as follows.

Definition 1 For the difference functional (1), if there exists a nonsingular matrix $M_{s,d}^{k}$ satisfying

$$\mathrm{pr}X(L_d)=M_{s,d}^{k}\left(L_{k,d}\right), \tag{15}$$

then the discrete Lagrangian equations maintain the conformal invariance of Mei symmetry under the infinitesimal transformations (12). Equation (15) is called the determining equations of the conformal invariance of Mei symmetry for the discrete Lagrangian systems, and $M^k_{s,d}$ is called the conformal factor of the discrete Lagrangian systems.

Theorem 1 For the discrete Lagrangian $L_{s,d} = L_{s,d}(t,t^+,q_s,q_s^+)$, if the generators of the infinitesimal transformations (12) satisfy

$$\mathrm{pr}X(L_{s,d}) = ML_{s,d}, \tag{16}$$

then the discrete Lagrangian systems (9) and (10) satisfy the conformal invariance of Mei symmetry, where M are constants.

Proof From Eqs. (15) and (16), we can obtain

$$\begin{aligned}
&\frac{\partial \mathrm{pr}X(L_{s,d})}{\partial q_s} + \frac{\partial \mathrm{pr}X(L^-_{s,d})}{\partial q_s} = M\left(\frac{\partial L_{s,d}}{\partial q_s} + \frac{\partial L^-_{s,d}}{\partial q_s}\right)\\
&= M^k_{s,d}\left\{\frac{\partial L_{k,d}}{\partial q_k} + \frac{\partial L^-_{k,d}}{\partial q_k}\right\}, \quad s,k = 1,\ldots,n, \qquad (17)\\
&\frac{\partial \mathrm{pr}X(L_{s,d})}{\partial t} + \frac{\partial \mathrm{pr}X(L^-_{s,d})}{\partial t} - \underset{+h}{D}\left(\mathrm{pr}X(L^-_{s,d})\right)\\
&= M\left(\frac{\partial L_{s,d}}{\partial t} + \frac{\partial L^-_{s,d}}{\partial t} - \underset{+h}{D}\left(L^-_{s,d}\right)\right)\\
&= M^k_{s,d}\left\{\frac{\partial L_{k,d}}{\partial t} + \frac{\partial L^-_{k,d}}{\partial t} - \underset{+h}{D}\left(L^-_{k,d}\right)\right\}, \qquad (18)
\end{aligned}$$

where $M^k_{s,d} = \delta^k_s M$ $\left(\delta^k_s = \begin{cases} 1, & s = k \\ 0, & s \neq k \end{cases}\right)$. So the discrete Lagrangian systems (9) and (10) satisfy the conformal invariance of Mei symmetry.

Theorem 2 If a Lagrangian system maintains the conformal invariance of Mei symmetry, then it maintains the Mei symmetry too.

Proof Substituting Eqs. (9) and (10) into Eqs. (17) and (18), one can obtain Eqs. (13) and (14), which are the determining equations of Mei symmetry.

3. Perturbation to conformal invariance of the discrete Lagrangian systems

Suppose that the discrete Lagrange systems are perturbed by small quantities

$$F_d = \varepsilon W_{s,d} = \varepsilon W_{s,d}\left(t,t^+,q_s,q_s^+\right). \tag{19}$$

The total variation of the difference functional (1) satisfies

$$\Delta L = -\sum_{\Omega} \varepsilon W_{s,d}\left(t,t^+,q_s,q_s^+\right)h_+\delta q_s, \tag{20}$$

where δq_s is the discrete virtual placement satisfying

$$\delta q_s = \Delta q_s - \frac{q_s^+ - q_s}{h^+}\Delta t. \tag{21}$$

Substituting Eqs. (21) and (6) into Eq. (20), we have

$$\begin{aligned}
&\left(h^+\frac{\partial L_{s,d}}{\partial q_s} + h^-\frac{\partial L^-_{s,d}}{\partial q_s}\right)\Delta q_s\\
&+\left(h^+\frac{\partial L_{s,d}}{\partial t} + h^-\frac{\partial L^-_{s,d}}{\partial t} + L^-_{s,d} - L_{s,d}\right)\Delta t\\
&= -\varepsilon W_{s,d}h_+\left(\Delta q_s - \frac{q_s^+ - q_s}{h_+}\Delta t\right). \qquad (22)
\end{aligned}$$

We obtain the following expression for the variation of the difference functional:

$$\frac{\partial L_{s,d}}{\partial q_s} + \frac{h^-}{h^+}\frac{\partial L^-_{s,d}}{\partial q_s} + \varepsilon W_{s,d} = 0, \tag{23}$$

$$\frac{\partial L_{s,d}}{\partial t} + \frac{h^-}{h^+}\frac{\partial L^-_{s,d}}{\partial t} + \frac{L^-_{s,d} - L_{s,d}}{h^+} - \frac{(q_s^+ - q_s)}{h^+}\varepsilon W_{s,d} = 0. \tag{24}$$

The equations of the discrete perturbed Lagrangian systems on the uniform difference mesh become

$$\frac{\partial L_{s,d}}{\partial q_s} + \frac{\partial L^-_{s,d}}{\partial q_s} = -\varepsilon W_{s,d}, \tag{25}$$

$$\frac{\partial L_{s,d}}{\partial t} + \frac{\partial L^-_{s,d}}{\partial t} - \underset{+h}{D}\left(L^-_{s,d}\right) = \underset{+h}{D}(q_s)\,\varepsilon W_{s,d}. \tag{26}$$

For the discrete perturbed Lagrangian systems, the original symmetries and the invariants of the systems may vary because of the action of $\varepsilon W_{s,d}$. Assume the variation is a perturbation based on the symmetrical transformation of the initial system, then $\xi_0 = \xi_0(t,q_s)$ and $\eta_s = \eta_s(t,q_s)$, which denote the new generators after being perturbed and can be expressed as

$$\xi_0 = \xi_{00} + \varepsilon\xi_{01} + \varepsilon^2\xi_{02} + \cdots, \tag{27}$$

$$\eta_s = \eta_{s0} + \varepsilon\eta_{s1} + \varepsilon^2\eta_{s2} + \cdots. \tag{28}$$

The prolongation of the Lie group operators for the discrete perturbed system can be written as

$$\mathrm{pr}\tilde{X} = \varepsilon^m \mathrm{pr}X_m, \tag{29}$$

where

$$\begin{aligned}
\mathrm{pr}X_m =\ & \xi_{0m}\frac{\partial}{\partial t} + \eta_{sm}\frac{\partial}{\partial q_s} + \xi^-_{0m}\frac{\partial}{\partial t^-} + \eta^-_{sm}\frac{\partial}{\partial q_s^-}\\
&+\xi^+_{0m}\frac{\partial}{\partial t^+} + \eta^+_{sm}\frac{\partial}{\partial q_s^+}. \qquad (30)
\end{aligned}$$

According to the definition of the Mei symmetry, the functions $L_{s,d}$ and $\varepsilon W_{s,d}$ become $\tilde{L}_{s,d}$ and $\varepsilon\tilde{W}_{s,d}$. Expanding the functions $\tilde{L}_{s,d} = L_{s,d} + \varepsilon\mathrm{pr}\tilde{X}(L_{s,d}) + O(\varepsilon^2)$ and $\varepsilon\tilde{W}_{s,d} = \varepsilon W_{s,d} + \varepsilon^2\mathrm{pr}\tilde{X}(W_{s,d}) + O(\varepsilon^3)$, substituting them into Eqs. (25) and (26), and ignoring high order small quantities, we can obtain the determining equations of Mei symmetry of the discrete Lagrangian systems

$$\frac{\partial \mathrm{pr}\tilde{X}(L_{s,d})}{\partial q_s} + \frac{\partial \mathrm{pr}\tilde{X}(L^-_{s,d})}{\partial q_s} + \varepsilon\mathrm{pr}\tilde{X}\left(W_{s,d}\right) = 0, \tag{31}$$

$$\frac{\partial \mathrm{pr}\tilde{X}\left(L_{s,d}\right)}{\partial t}+\frac{\partial \mathrm{pr}\tilde{X}\left(L_{s,d}^{-}\right)}{\partial t}-\underset{+h}{D}\left(\mathrm{pr}\tilde{X}(L_{s,d}^{-})\right)$$
$$=\underset{+h}{D}(q_s)\,\varepsilon \mathrm{pr}\tilde{X}\left(W_{s,d}\right). \tag{32}$$

In the following definitions and theorems, the generators are the conformal invariance of Mei symmetry because the generators' vector (29) is the Mei symmetry firstly.

Definition 2 For the discrete perturbed Lagrangian systems (25) and (26), if there exists a nonsingular matrix $M_{s,d}^{k}$ satisfying

$$\mathrm{pr}\tilde{X}(L_{s,d}+L_{s,d}^{-})=M_{s,d}^{k}\left(L_{k,d}+L_{k,d}^{-}\right), \tag{33}$$
$$\mathrm{pr}\tilde{X}(\varepsilon W_{s,d})=\varepsilon M_{s,d}^{k}W_{k,d}, \tag{34}$$

then the Lagrangian systems maintain the perturbation to the conformal invariance of Mei symmetry under expressions (27) and (28). Equations (33) and (34) are called the determining equations of the perturbation to conformal invariance for the discrete Lagrangian systems, and $M_{s,d}^{k}$ is called the conformal factor of the discrete perturbed Lagrangian systems.

Theorem 3 For the discrete perturbed Lagrangian $L_{s,d}=L_{s,d}\left(t,t^{+},q_s,q_s^{+}\right)$, if the generators ξ_{0m} and η_{sm} of expressions (27) and (28) satisfy

$$\mathrm{pr}\tilde{X}(L_{s,d}+L_{s,d}^{-})=M\left(L_{s,d}+L_{s,d}^{-}\right), \tag{35}$$
$$\mathrm{pr}\tilde{X}(\varepsilon W_{s,d})=\varepsilon M W_{s,d}, \tag{36}$$

then the discrete perturbed Lagrangian systems (25) and (26) satisfy the perturbation to conformal invariance of Mei symmetry, where M are constants.

Proof According to Eqs. (33) and (34), equations (31) and (32) can be expressed as

$$\frac{\partial \mathrm{pr}\tilde{X}\left(L_{s,d}\right)}{\partial q_s}+\frac{\partial \mathrm{pr}\tilde{X}\left(L_{s,d}^{-}\right)}{\partial q_s}+\varepsilon \mathrm{pr}\tilde{X}\left(W_{s,d}\right)$$
$$=\frac{\partial\left(ML_{s,d}\right)}{\partial q_s}+\frac{\partial\left(ML_{s,d}^{-}\right)}{\partial q_s}+\varepsilon M W_{s,d}$$
$$=M_{s,d}^{k}\left(\frac{\partial L_{k,d}}{\partial q_k}+\frac{\partial L_{k,d}^{-}}{\partial q_k}+\varepsilon W_{k,d}\right), \tag{37}$$

$$\frac{\partial \mathrm{pr}\tilde{X}\left(L_{s,d}\right)}{\partial t}+\frac{\partial \mathrm{pr}\tilde{X}\left(L_{s,d}^{-}\right)}{\partial t}$$
$$-\underset{+h}{D}\left(\mathrm{pr}\tilde{X}(L_{s,d}^{-})\right)-\underset{+h}{D}(q_s)\,\varepsilon \mathrm{pr}\tilde{X}\left(W_{s,d}\right)$$
$$=\frac{\partial\left(ML_{s,d}\right)}{\partial t}+\frac{\partial\left(ML_{s,d}^{-}\right)}{\partial t}-\underset{+h}{D}\left(ML_{s,d}^{-}\right)-\varepsilon\underset{+h}{D}(q_s)MW_{s,d}$$
$$=M_{s,d}^{k}\left(\frac{\partial L_{k,d}}{\partial t}+\frac{\partial L_{k,d}^{-}}{\partial t}-\underset{+h}{D}\left(L_{k,d}^{-}\right)-\underset{+h}{D}(q_k)\,\varepsilon W_{k,d}\right), \tag{38}$$

with $M_{s,d}^{k}=\delta_s^k M$ $\left(\delta_s^k=\begin{cases}1, & s=k\\ 0, & s\neq k\end{cases}\right)$, $s,k=1,\ldots,n$. So the discrete perturbed Lagrangian systems (25) and (26) satisfy the perturbation to conformal invariance of Mei symmetry.

4. Adiabatic invariants of discrete Lagrangian systems

The Noether theorems state the relationship between the conservation laws and the symmetries of the dynamical systems.[9–11] For the discrete Lagrangian systems, the discrete analogues of Noether theorem for difference equations were reported in Refs. [26] and [30]. For this perturbed system on uniform lattice, the new generators (27) and (28) satisfy

$$\mathrm{pr}\tilde{X}(L_d)+L_d\underset{+h}{D}(\xi_0)+\varepsilon W_{s,d}\left(\eta_s-\underset{+h}{D}(q_s)\,\xi_0\right)+\underset{+h}{D}(G_{s,d})$$
$$\equiv\xi_0\left(\frac{\partial L_d}{\partial t}+\frac{\partial L_d^{-}}{\partial t}-\underset{+h}{D}(L_d^{-})-\varepsilon\underset{+h}{D}(q_s)W_{s,d}\right)$$
$$+\eta_s\left(\frac{\partial L_d}{\partial q_s}+\frac{\partial L_d^{-}}{\partial q_s}+\varepsilon W_{s,d}\right)$$
$$+\underset{+h}{D}\left(h\eta_s\frac{\partial L_d^{-}}{\partial q_s}+h\xi_0\frac{\partial L_d^{-}}{\partial t}+\xi_0 L_d^{-}+G_{s,d}\right). \tag{39}$$

After being perturbed, the gauge function comes into

$$G_{s,d}=G_{s0,d}+\varepsilon G_{s1,d}+\varepsilon^2 G_{s2,d}+\cdots. \tag{40}$$

Substituting Eqs. (27), (28), and (40) into Eq. (39), we have

$$\varepsilon^m\Big[\mathrm{pr}X_m(L_d)+L_d\underset{+h}{D}(\xi_{0m})$$
$$+W_{s,d}\left(\eta_{s(m-1)}-\underset{+h}{D}(q_s)\,\xi_{0(m-1)}\right)+\underset{+h}{D}(G_{sm,d})\Big]$$
$$\equiv\varepsilon^m\xi_{0m}\left(\frac{\partial L_d}{\partial t}+\frac{\partial L_d^{-}}{\partial t}-\underset{+h}{D}(L_d^{-})-\varepsilon\underset{+h}{D}(q_s)W_{s,d}\right)$$
$$+\varepsilon^m\eta_{sm}\left(\frac{\partial L_d}{\partial q_s}+\frac{\partial L_d^{-}}{\partial q_s}+\varepsilon W_{s,d}\right)$$
$$+\underset{+h}{D}\left(\varepsilon^m h\eta_{sm}\frac{\partial L_d^{-}}{\partial q_s}+\varepsilon^m h\xi_{0m}\frac{\partial L_d^{-}}{\partial t}\right.$$
$$\left.+\varepsilon^m\xi_{0m}L_d^{-}+\varepsilon^m G_{sm,d}\right),\quad (m=0,1,2,\ldots), \tag{41}$$

when $m=0$, the condition $W_{s,d}=0$. If the left-hand side of Eq. (41) equals zero, i.e.,

$$\mathrm{pr}X_m(L_d)+L_d\underset{+h}{D}(\xi_{0m})+W_{s,d}\left(\eta_{s(m-1)}-\underset{+h}{D}(q_s)\,\xi_{0(m-1)}\right)$$
$$+\underset{+h}{D}(G_{sm,d})=0, \tag{42}$$

which is called the discrete Noether identity. Here the right-hand side of Eq. (41) equals zero, based on Eqs. (25) and (26), the discrete versions of the conservation law are

$$I_d^z=\sum_{m=0}^{z}\varepsilon^m\left(h\eta_{sm}\frac{\partial L_d^{-}}{\partial q_s}+h\xi_{0m}\frac{\partial L_d^{-}}{\partial t}+\xi_{0m}L_d^{-}+G_{sm,d}\right). \tag{43}$$

The discrete equations (43) are called the difference version of Noether conservation laws associated with the perturbed Lagrangian systems.

Theorem 4 If a discrete gauge function $G_{s0,d} = G_{s0,d}(t,t^+,q_s,q_s^+)$ exists such that the symmetry operators $\mathrm{pr}X_m$ satisfy the discrete Noether identity (42), then the discrete perturbed Lagrangian systems (25) and (26) possess the discrete adiabatic invariants (43).

The discrete Noether identity (42) is the sufficient condition for obtaining adiabatic invariants (43). We need to have the appropriate Lie group operators to satisfy them, as well as make sure that the adiabatic invariants are non-trivial.

The definition of discrete adiabatic invariants for the Lagrangian systems is obtained as follows.

Definition 3 For systems (25) and (26), if a physical quantity $I_d^z = I_d^z(t,t^+,q_s,q_s^+,\varepsilon)$ satisfies

$$\underset{+h}{D}\left(I_d^z\right) = O\left(\varepsilon^{z+1}\right), \tag{44}$$

where $I_d^z = I_{d0}^0 + \varepsilon I_{d1}^1 + \cdots + \varepsilon^z I_{dz}^z$, I_d^z is called a z-th order adiabatic invariant of the systems.

Based on Theorem 4 and Definition 3, Noether theorems for the discrete Lagrangian systems subjected to the perturbation quantities are obtained as follows.

Theorem 5 For the discrete Lagrangian systems (25) and (26), if a discrete gauge function $G_{sm,d} = G_{sm,d}(t,t^+,q_s,q_s^+)$ exists such that the infinitesimal transformation generators ξ_{0m} and η_{sm} satisfy the discrete Noether identity (42), then the discrete perturbed Lagrangian systems have the z-th order adiabatic invariants (43).

Proof Considering Eqs. (25), (26), and (42), the discrete derivative of I_d^z can be expressed as

$$\begin{aligned}
&\underset{+h}{D}\left(I_d^z\right)\\
&= \underset{+h}{D}\sum_{m=0}^{z}\varepsilon^m\left\{h\eta_{sm}\frac{\partial L_d^-}{\partial q_s} + h\xi_{0m}\frac{\partial L_d^-}{\partial t} + \xi_{0m}L_d^- + G_{sm,d}\right\}\\
&= \sum_{m=0}^{z}\varepsilon^m\left\{h\underset{+h}{D}\left(\eta_{sm}\frac{\partial L_d^-}{\partial q_s}\right) + h\underset{+h}{D}\left(\xi_{0m}\frac{\partial L_d^-}{\partial t}\right)\right.\\
&\quad + \underset{+h}{D}\left(\xi_{0m}L_d^-\right) - \mathrm{pr}X_m\left(L_d\right) - L_d\underset{+h}{D}\left(\xi_{0m}\right)\\
&\quad \left. - W_{s,d}\left(\eta_{s(m-1)} - \underset{+h}{D}\left(q_s\right)\xi_{0(m-1)}\right)\right\}\\
&= \sum_{m=0}^{z}\varepsilon^m\left\{\xi_{0m}\left(-\frac{\partial L_{s,d}}{\partial t} - \frac{\partial L_{s,d}^-}{\partial t} + \underset{+h}{D}\left(L_d^-\right)\right)\right.\\
&\quad + \eta_{sm}\left(-\frac{\partial L_{s,d}}{\partial q_s} - \frac{\partial L_{s,d}^-}{\partial q_s}\right)\\
&\quad \left. - W_{s,d}\left(\eta_{s(m-1)} - \underset{+h}{D}\left(q_s\right)\xi_{0(m-1)}\right)\right\}.
\end{aligned}$$

Using Eqs. (25) and (26), we have

$$\begin{aligned}
\underset{+h}{D}\left(I_d^z\right) &= \sum_{m=0}^{z}\varepsilon^m\left\{\varepsilon W_{s,d}\left(\eta_{sm} - \underset{+h}{D}\left(q_s\right)\xi_{0m}\right)\right.\\
&\quad \left. - W_{s,d}\left(\eta_{s(m-1)} - \underset{+h}{D}\left(q_s\right)\xi_{0(m-1)}\right)\right\}\\
&= \sum_{m=0}^{z}\varepsilon^{m+1}W_{s,d}\left(\eta_{sm} - \underset{+h}{D}\left(q_s\right)\xi_{0m}\right)\\
&\quad - \sum_{m=0}^{z}\varepsilon^m W_{s,d}\left(\eta_{s(m-1)} - \underset{+h}{D}\left(q_s\right)\xi_{0(m-1)}\right).
\end{aligned} \tag{45}$$

The first term of Eq. (45) can be expressed as

$$\begin{aligned}
&\sum_{m=0}^{z}\varepsilon^{m+1}W_{s,d}\left(\eta_{sm} - \underset{+h}{D}\left(q_s\right)\xi_{0m}\right)\\
&= \varepsilon W_{s,d}\left(\eta_{s0} - \underset{+h}{D}\left(q_s\right)\xi_{00}\right) + \varepsilon^2 W_{s,d}\left(\eta_{s1} - \underset{+h}{D}\left(q_s\right)\xi_{01}\right)\\
&\quad + \cdots + \varepsilon^z W_{s,d}\left(\eta_{s(z-1)} - \underset{+h}{D}\left(q_s\right)\xi_{0(z-1)}\right)\\
&\quad + \varepsilon^{z+1}W_{s,d}\left(\eta_{sz} - \underset{+h}{D}\left(q_s\right)\xi_{0z}\right).
\end{aligned} \tag{46}$$

Expand the second term of Eq. (45) as

$$\begin{aligned}
&\sum_{m=0}^{z}\varepsilon^m W_{s,d}\left(\eta_{s(m-1)} - \underset{+h}{D}\left(q_s\right)\xi_{0(m-1)}\right)\\
&= \varepsilon^0 W_{s,d}\left(\eta_{s(-1)} - \underset{+h}{D}\left(q_s\right)\xi_{0(-1)}\right)\\
&\quad + \varepsilon W_{s,d}\left(\eta_{s0} - \underset{+h}{D}\left(q_s\right)\xi_{00}\right)\\
&\quad + \varepsilon^2 W_{s,d}\left(\eta_{s1} - \underset{+h}{D}\left(q_s\right)\xi_{01}\right)\\
&\quad + \cdots + \varepsilon^z W_{s,d}\left(\eta_{s(z-1)} - \underset{+h}{D}\left(q_s\right)\xi_{0(z-1)}\right).
\end{aligned} \tag{47}$$

Substitute Eqs. (46) and (47) into Eq. (45) yields

$$\begin{aligned}
\underset{+h}{D}\left(I_d^z\right) &= \sum_{m=0}^{z}\varepsilon^{m+1}W_{s,d}\left(\eta_{sm} - \underset{+h}{D}\left(q_s\right)\xi_{0m}\right)\\
&\quad - \sum_{m=0}^{z}\varepsilon^m W_{s,d}\left(\eta_{s(m-1)} - \underset{+h}{D}\left(q_s\right)\xi_{0(m-1)}\right)\\
&= \varepsilon^{z+1}W_{s,d}\left(\eta_{sz} - \underset{+h}{D}\left(q_s\right)\xi_{0z}\right)\\
&\quad - \varepsilon^0 W_{s,d}\left(\eta_{s(-1)} - \underset{+h}{D}\left(q_s\right)\xi_{0(-1)}\right),
\end{aligned} \tag{48}$$

the perturbations $W_{s,d} = 0$ hold when $z = 0$, so

$$\underset{+h}{D}\left(I_d^z\right) = \varepsilon^{z+1}W_{s,d}\left(\eta_{sz} - \underset{+h}{D}\left(q_s\right)\xi_{0z}\right). \tag{49}$$

Expression (49) is deduced by the results of Definition 3, the Leibniz rule, and Eqs. (25), (26), and (42). It shows that I_d^z is in direct proportional to $z+1$, so I_d^z is the discrete analogue of the z-th order adiabatic invariants for the discrete disturbed Lagrangian systems.

5. An example

As motivation for the present work, consider a classical mechanical system with one degree of freedom, whose motion is given by the Newton equation

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{\mathrm{d}}{\mathrm{d}x}V\left(x\right) = 0. \tag{50}$$

The integration of Eq. (50) provides the time dependence of the coordinate $x(t)$ of a mass $m = 1$ particle in the potential $V(x)$. Suppose the systems are subjected to the perturbation forces $F_d = \varepsilon W = -\varepsilon\left(\mathrm{d}x/\mathrm{d}t\right)^2$. The Lagrangian

function is $L=\frac{1}{2}\left(\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+x^2\right)$. In the case of the independent variable $t=(t,t^+,t^{++},\ldots)$ and depended variables $q=(q,q^+,q^{++},\ldots)$, the discrete force is $\varepsilon W_d=-\varepsilon\left(\frac{q^+-q}{h}\right)^2$.

When the systems do not have the perturbation forces, the difference Euler–Lagrangian equation can be written as Eqs. (9) and (10). The prolongation of the Lie group operators for the discrete system is $\mathrm{pr}X=q\frac{\partial}{\partial q}+q^-\frac{\partial}{\partial q^-}+q^+\frac{\partial}{\partial q^+}$, the conformal factor is $M^k_{s,d}=2\delta^k_s M=\begin{pmatrix}2&0\\0&2\end{pmatrix}$ according Theorem 4, the conserved quantity is $I^z_d=\frac{1}{h}q\left(\frac{q-q^-}{h}\right)-q\left(\frac{q^+-q}{h}\right)=\text{const.}$, which is the conservation of angular momentum. Equations (9) and (10) describe the motion of the system away from the impact point by implicitly defining a map $(q^-,q)\mapsto(q,q^+)$ and are also the variational numerical algorithm. In order to implement the variational integrator for the harmonic oscillator, equation (9) can be written as $\partial L_d/\partial q=-\partial L_d^-/\partial q$. By taking $b=\partial L_d/\partial q$ for each k, this equation is simply $b=-\partial L_d^-/\partial q$, together with the next update $b^+=-\partial L_d/\partial q^+$. Solving the implicit equation for q^+ and then evaluating the explicit equation that gives b^+, we update (q,b) to (q^+,b^+). More numerical algorithms show that the discrete variational methods have good performances both on the accuracy and long-term tracking.[31,32] In Fig. 1, we can obtain more precise numerical results than those from the standard RK4 for this free harmonic oscillator under the initial conditions $(q_1,b_1)=(0.01,0.01)$. The variational algorithms demonstrate the high precision and the long time stability with comparisons to the standard method.

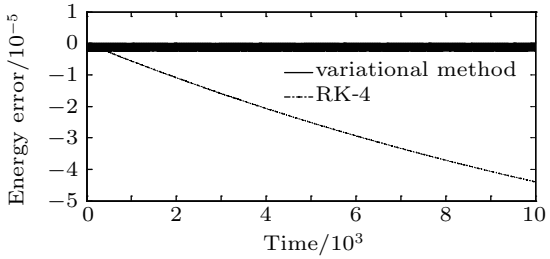


Fig. 1. Error in the energy for free harmonic oscillator versus time.

When adding the perturbation quantities, the difference Euler–Lagrangian equation of the systems can be written as

$$\frac{\partial L_d}{\partial q}+\frac{\partial L_d^-}{\partial q}-\varepsilon\left(\frac{q^+-q}{h^+}\right)^2=0, \tag{51}$$

$$\underset{+h}{D}\left(L_d^-\right)=\varepsilon\underset{+h}{D}(q)\left(\frac{q^+-q}{h^+}\right)^2. \tag{52}$$

The difference equations (51) and (52) are also the discrete variational algorithms for this system. Then we can check the variational invariance of Eqs. (51) and (52) with regard to the symmetry of Lagrangian equation

$$\mathrm{pr}X_1=(q+\varepsilon q)\frac{\partial}{\partial q}+\left(q^-+\varepsilon q^-\right)\frac{\partial}{\partial q^-}+\left(q^++\varepsilon q^+\right)\frac{\partial}{\partial q^+}. \tag{53}$$

One can readily verify the invariance of the functions $L_{s,d}$ and $\varepsilon W_{s,d}$, that is

$$\begin{aligned}\mathrm{pr}X_1(L_d)&=\left(\frac{q^+\xi^+-q\xi^+-q^+\xi+q\xi}{h^2}\right)+q\xi\\&=2(1+\varepsilon)L_d,\end{aligned} \tag{54}$$

$$\begin{aligned}\mathrm{pr}X_1(\varepsilon W_d)&=2\left(\frac{q^+-q}{h}\right)\cdot\left(\frac{\xi^+-\xi}{h}\right)\\&=2(1+\varepsilon)\varepsilon W_d.\end{aligned} \tag{55}$$

The conformal factor is $M^k_{s,d}=2\delta^k_s M=2\begin{pmatrix}1+\varepsilon&0\\0&1+\varepsilon\end{pmatrix}$. We have the solution of Eq. (42) with the prolongation of the Lie group operators $\mathrm{pr}\tilde{X}$. From the adiabatic invariants (43), we have

$$I^z_d=-q(1+\varepsilon)\left(\frac{q^+-2q+q^-}{h}\right), \tag{56}$$

with the gauge function $G_{1,d}=(1+\varepsilon)q\left(\frac{q^+-q}{h}\right)$.

We implement the discrete equations (51) and (52) as numerical integrators of the perturbed Lagrangian systems. Figure 2 shows the adiabatic invariants (56) with different coefficients of ε under the initial conditions $(q_1,b_1)=(0.01,8)$. As we go forward in time, the perturbed parameter ε keeps on increasing, leading to a decrease in accuracy.

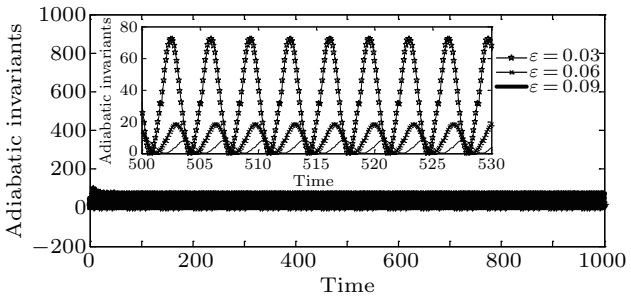


Fig. 2. Adiabatic invariants in Eq. (56) versus time.

6. Conclusions

We construct the conformal invariance perturbation and the adiabatic invariants of the discrete Lagrangian systems. The discrete equations of the perturbed Lagrangian systems are obtained utilizing the discrete variational approach. The discrete conformal invariances of both the free Lagrangian systems and the perturbed ones are proposed respectively. The adiabatic invariants of the discrete perturbed Lagrangian systems are obtained based the discrete Noether theorem of the systems. The discrete Noether theorem of the perturbed Lagrangian version gives the conditions of finding first integrals when their symmetries are known. The discrete conformal invariance and the numerical computation of the systems

demonstrate the following points. (i) The conformal invariance proposes one way of searching the solutions of discrete perturbed Lagrangian mechanical systems. (ii) The variational algorithm is a reasonable structure-preserving scheme for analyzing the dynamical behaviors of the perturbed mechanical systems.

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