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Nodes and layers PageRank centrality for multilayer networks*

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In this paper, we propose a new centrality algorithm that can simultaneously rank the nodes and layers of multilayer networks, referred to as the MRFNL centrality. The centrality of nodes and layers are obtained by developing a novel iterative algorithm for computing a set of tensor equations. Under some conditions, the existence and uniqueness of this centrality were proven by applying the Brouwer fixed point theorem. Furthermore, the convergence of the proposed iterative algorithm was established. Finally, numerical experiments on a simple multilayer network and two real-world multilayer networks (*i.e.*, Pierre Auger Collaboration and European Air Transportation Networks) are proposed to illustrate the effectiveness of the proposed algorithm and to compare it to other existing centrality measures.

Keywords: multilayer networks, PageRank centrality, random walks, transition probability tensors

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1. Introduction

Centrality measures can be used to rank nodes according to their importance, and are also very useful for understanding the behavior of systems.[1] A larger centrality value of a node means that it has a greater importance. In recent years, many centrality measures have been proposed to determine the relative importance of nodes, such as degree centrality,[2] betweenness centrality,[3,4] closeness centrality,[5] eigenvector centrality,[6] PageRank centrality,[7,8] and HITS centrality,[9] and so on. Meanwhile, identifying the central nodes is widely applied in many practical problems, which contain ranking for sports teams or individual athletes,[10–12] genetic and protein targets,[13] creating successful advertisements for e-commercial product,[14] *etc*.

In the real world, researchers have found that most complex networks are multilayer networks, which characterized by different types of layers. Therefore, it is important to generalize centrality measures of single layer networks for ranking the nodes of multilayer networks. Recently, this topic has attracted interest from researchers and numerous classical ranking algorithms have been extended to determine the key nodes in multilayer networks, such as betweenness, degree and eigenvector centrality measures.[15–17]

The other popular centrality measures that have been proposed for multiplex (*i.e.*, multilayer networks without interlayer coupling) or multilayer networks are PageRank and HITS centrality. By considering a random walk in multiplex networks, Ng *et al.*[18] proposed a generalization of PageRank for multiple networks. This centrality can be obtained by solving a set of tensor equations. Tudisco *et al.*[19] established a new eigenvector centrality for ranking nodes and layers in multiplex networks. The centrality values of nodes and layers were obtained by computing the Perron eigenvector of a multi-homogeneous map. Rahmede *et al.*[20] formulated and proposed a new multirank to simultaneously ranking the nodes and layers in multiplex networks. The novel multirank algorithm can be obtained by solving a coupled set of equations. Recently, Ding *et al.*[21] proposed topologically biased multiplex PageRank for multilayer networks by introducing topologically biased random walks in multilayer networks. However, in Refs. [18]–[20], they only consider the multilayer networks without interlayer coupling. Meanwhile, motivated by the idea of classical HITS method,[9] Wang *et al.*[22] proposed a Singular Vector of Tensor (SVT) centrality to quantify the hub and authority of nodes and layers in multilayer networks with interlayer coupling. The hub and authority values of nodes and layers can be get by computing the singular vector of 4th-order adjacency tensor of multilayer networks. Valdeolivas *et al.*[23] proposed a new centrality based on random walks with restart to rank the nodes in multilayer biological networks.

In this paper, motivated by the idea of Ng *et al.*,[18] we extend the PageRank centrality to study the centrality of nodes and layers in multilayer networks with interlayer coupling. The centrality values of nodes and layers can be obtained by developing an iterative algorithm to solve a set of tensor equations. Then, by using Brouwer fixed point theorem, we prove the existence and uniqueness of the proposed centrality under some conditions. The convergence of the proposed iterative algorithm is established. Finally, numerical experiments show that the effectiveness of the proposed centrality on a simple multilayer network and two real-world multilayer networks.

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The rest of this paper is organized as follows. Tensor representation of multilayer networks is presented in Section 2. In Section 3, we show some notations and preliminary knowledge on tensor. Section 4 establishes the proposed centrality measure and iterative algorithm. Numerical experiments are reported in Section 5. Finally, we draw a conclusion in the last section.

2. Tensor representation of multilayer networks

This paper focuses on multilayer networks that contain a fixed set of nodes connected by different types of layers.[24] A simple multilayer network can be found in Fig. 1(a) and the corresponding aggregated network is presented in Fig. 1(b). A multilayer network with L layers and N nodes can be represented by a pair $W=(\Phi,C)$, where $\Phi=\{G_1,G_2,\ldots,G_L\}$ is the set of all different layers and

$$C=\left\{E_{\alpha\beta}\subseteq V_\alpha\times V_\beta;\alpha,\beta\in\{1,2,\ldots,L\}\right\}$$

is the set of interconnections between nodes of different layers. Here, each layer is a single-network $G_k=(V,E_k)$, with $V=\{v_1,v_2,\ldots,v_N\}$ (*i.e.*, each layer has the same N nodes), and E_k is the set of intralayer connections in the layer G_k.

Similar to the adjacency matrix that represents the single layer networks, in this paper, a 4th-order adjacency tensor $\mathcal{T}=(\mathcal{T}_{i\alpha j\beta})\in R^{N\times L\times N\times L}$ is used to represent the multilayer networks W. Each entry of this tensor $\mathcal{T}$ is defined by

$$\mathcal{T}_{i\alpha j\beta}=\begin{cases}t_{i\alpha j\beta}, & \text{if } v_j^\beta\to v_i^\alpha,\\ 0, & \text{otherwise,}\end{cases}\tag{1}$$

where $1\le i,j\le N, 1\le\alpha,\beta\le L$, $t_{i\alpha j\beta}$ represents the weight of the edge that node j in layer β points to node i in layer α, v_i^α represents node i in layer α.

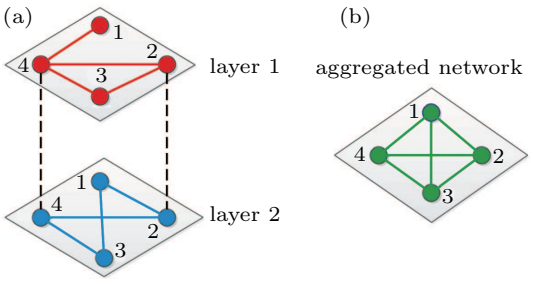


Fig. 1. (a) An example of a multilayer network that consists of four nodes connected by two different types of links and (b) the corresponding aggregated network.

3. Preliminary

In this section, some notations and some knowledge of tensor are given as follows. More generally, the high order tensor is the generalization of matrix; for example, a tensor of order two is a matrix. Moreover, a tensor is called non-negative if all of its entries are non-negative. Similar to the matrix-vector product, we show the tensor–vector product in the following.

Given two vectors $x\in R^N$ and $y\in R^L$ (R is the real field), the definition of $\mathcal{T}xyy$ and $\mathcal{T}xxy$ are given by

$$(\mathcal{T}xyy)_i=\sum_{\alpha=1}^{L}\sum_{j=1}^{N}\sum_{\beta=1}^{L}\mathcal{T}_{i,\alpha,j,\beta}x_jy_\alpha y_\beta,\quad i=1,2,\ldots,N,$$

$$(\mathcal{T}xxy)_\alpha=\sum_{i=1}^{N}\sum_{\beta=1}^{L}\sum_{j=1}^{N}\mathcal{T}_{i,\alpha,j,\beta}x_ix_jy_\alpha,\quad \alpha=1,2,\ldots,L,$$

and $\mathcal{T}xyy\in R^N,\mathcal{T}xxy\in R^L$.

Before establishing our centrality measure, we first construct two tensors $\mathcal{O}=(o_{i,\alpha,j,\beta})$ and $\mathcal{R}=(r_{i,\alpha,j,\beta})$ by normalizing the elements of tensor $\mathcal{T}$:

$$o_{i,\alpha,j,\beta}=\frac{t_{i,\alpha,j,\beta}}{\sum_{i=1}^{N}t_{i,\alpha,j,\beta}},\quad i=1,2,\ldots,N,$$

$$r_{i,\alpha,j,\beta}=\frac{t_{i,\alpha,j,\beta}}{\sum_{\alpha=1}^{L}t_{i,\alpha,j,\beta}},\quad \alpha=1,2,\ldots,L.$$

Each element $o_{i,\alpha,j,\beta}$ of $\mathcal{O}$ can be viewed as the probability of arriving at i-th node in layer α by given that the j-th node in layer β is currently arrived, and the each entry $r_{i,\alpha,j,\beta}$ of $\mathcal{R}$ can be view as the probability of using the layer α by given that the i-th node is arrived from the j-th node in layer β. Note that if $t_{i,\alpha,j,\beta}=0$ for all $1\le i\le N$, then let $o_{i,\alpha,j,\beta}=1/N$. Similarly, let $r_{i,\alpha,j,\beta}=1/L$ if $t_{i,\alpha,j,\beta}=0$ for all $1\le\alpha\le L$.

According to the construction of tensors $\mathcal{O}$ and $\mathcal{R}$, we can obtain

$$0\le o_{i,\alpha,j,\beta}\le 1,\quad\sum_{i=1}^{N}o_{i,\alpha,j,\beta}=1,$$

$$0\le r_{i,\alpha,j,\beta}\le 1,\quad\sum_{\alpha=1}^{L}r_{i,\alpha,j,\beta}=1.$$

It is obvious that tensors $\mathcal{O}$ and $\mathcal{R}$ are non-negative tensors. Furthermore, these two tensors are called transition probability tensors. Before ending this section, we introduce the concept of irreducible tensors[25] as follows:

Definition 1 An $(n_1\times\cdots\times n_m)$-dimensional tensor $\mathcal{T}$ is called irreducible if for any j and j' (the other indices are fixed), the $n_j\times n_{j'}$ matrices $(t_{\ldots,i_j,\ldots,i_{j'},\ldots})$ are irreducible. If $\mathcal{T}$ is not irreducible, then we call $\mathcal{T}$ reducible.

If tensor $\mathcal{T}$ is irreducible, it is easy to get that the corresponding tensors $\mathcal{H}$, $\mathcal{A}$, $\bar{\mathcal{H}}$, and $\bar{\mathcal{A}}$ are also irreducible.

4. The proposed centrality (MRFNL) and Algorithm

In this section, we establish a new multirank method for ranking the nodes and layers in multilayer networks, referred to as the MRFNL centrality. Suppose that vectors x and y are centrality values of nodes and layers in multilayer networks,

respectively. Then, we obtain centrality vectors x and y by solving following tensor equations

$$x_i = \sum_{\alpha=1}^{L}\sum_{j=1}^{N}\sum_{\beta=1}^{L} o_{i,\alpha,j,\beta}x_j y_\alpha y_\beta, \quad 1 \le i \le N, \tag{2}$$

$$y_\alpha = \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\beta=1}^{L} r_{i,\alpha,j,\beta}x_i x_j y_\beta, \quad 1 \le \alpha \le L. \tag{3}$$

Equivalently, equations (2) and (3) can be reformulated as the following tensor equations:

$$\mathcal{O}xy^2 = x \quad \text{and} \quad \mathcal{R}x^2y = y, \tag{4}$$

with

$$x \in \Omega_N, \quad y \in \Omega_L, \tag{5}$$

where $\Omega_N = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_N) \in R^N | x_i \ge 0, 1 \le i \le N, \sum_{i=1}^{N} x_i = 1\}$ and $\Omega_L = \{\boldsymbol{y} = (y_1, y_2, \ldots, y_L) \in R^L | y_i \ge 0, 1 \le i \le L, \sum_{i=1}^{L} y_i = 1\}$. If $(\bar{x}, \bar{y})$ is a solution to tensor equations (4), then $\bar{x}_i$ denotes the centrality value of node i and $\bar{y}_\alpha$ denotes the centrality value of layer α. The larger value of x_i and y_α indicates a relatively greater importance for each node and layer within the whole network, respectively. In particular, the proposed centrality will reduce to classical PageRank centrality[7] when the number of layers $L = 1$ (*i.e.*, the network is a single layer structure).

As we can see, from the tensor equations (4), there are two main differences between the MRFNL centrality and the existing PageRank centrality.[7,15] First, we obtain centrality values of nodes and layers without the need of aggregating or unfolding data by working directly on the tensor. Second, the proposed ranking method can simultaneously obtain the centrality values of nodes and layers in multilayer networks.

It is easy to obtain that if x and y are probability distributions, then $\mathcal{O}xy^2$ and $\mathcal{R}x^2y$ are also probability distributions. We show this by the following theorem.

Theorem 1 Assume $\mathcal{O}$ and $\mathcal{R}$ are constructed in Section 3, For any $x \in \Omega_N$ and $y \in \Omega_L$, then $\mathcal{O}xy^2 \in \Omega_N$ and $\mathcal{O}x^2y \in \Omega_L$.

Proof According to the definition of tensor–vector product, we have

$$(\mathcal{O}xy^2)_i = \sum_{\alpha=1}^{L}\sum_{j=1}^{N}\sum_{\beta=1}^{L} \mathcal{O}_{i,\alpha,j,\beta}x_j y_\alpha y_\beta, \quad i = 1, 2, \ldots, N,$$

$$(\mathcal{R}x^2y)_\alpha = \sum_{i=1}^{N}\sum_{\beta=1}^{L}\sum_{j=1}^{N} \mathcal{R}_{i,\alpha,j,\beta}x_i x_j y_\beta, \quad \alpha = 1, 2, \ldots, L.$$

Because $\mathcal{O}$ and $\mathcal{R}$ are transition probability tensors and $x \in \Omega_N, y \in \Omega_L$, we can obtain

$$\sum_{i=1}^{N}(\mathcal{O}xy^2)_i = \sum_{i=1}^{N}\sum_{\alpha=1}^{L}\sum_{j=1}^{N}\sum_{\beta=1}^{L} \mathcal{O}_{i,\alpha,j,\beta}x_j y_\alpha y_\beta = 1,$$

$$\sum_{\alpha=1}^{L}(\mathcal{R}x^2y)_\alpha = \sum_{\alpha=1}^{L}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\beta=1}^{L} \mathcal{R}_{i,\alpha,j,\beta}x_i x_j y_\beta = 1.$$

Hence, we get $\mathcal{O}xy^2 \in \Omega_N$ and $\mathcal{R}x^2y \in \Omega_L$. □

By applying Theorem 1, we propose the following Theorem 2 to show that the tensor equations (4) has a non-zero non-negative solution $(\bar{x}, \bar{y})$. The proof of Theorem 2 is similar to the Theorem 2 in Ref. [18].

Theorem 2 Suppose $\mathcal{O}$ and $\mathcal{R}$ are constructed in Section 3, then there exist non-zero non-negative vectors $\bar{x} \in \Omega_n$ and $\bar{y} \in \Omega_L$ such that $\mathcal{O}\bar{x}\bar{y}^2 = \bar{x}$ and $\mathcal{R}\bar{x}^2\bar{y} = \bar{y}$. In particular, if $\mathcal{T}$ is irreducible, then $\bar{x}$ and $\bar{y}$ are positive vectors.

Proof The proof of this Theorem can be converted into a fixed point problem as follows. We first define a mapping $T : \Omega \to \Omega$ by

$$T([x, y]) = [\mathcal{O}xy^2, \mathcal{R}x^2y].$$

where $\Omega = \{[x, y] \in R^{N+L} | x \in \Omega_N, y \in \Omega_L\}$. According to the Theorem 1, it is easy to obtain that T is well-defined (*i.e.*, when $[x, y] \in \Omega$, $T([x, y]) \in \Omega$) and continuous. Note that Ω_N, Ω_L, and Ω are closed convex set. Thus, by applying the Brouwer fixed point theorem, there exists $[\bar{x}, \bar{y}] \in \Omega$ such that $T([\bar{x}, \bar{y}]) = [\bar{x}, \bar{y}]$, *i.e.*, $\mathcal{O}\bar{x}\bar{y}^2 = \bar{x}$ and $\mathcal{R}\bar{x}^2\bar{y} = \bar{y}$.

Now, we assume that $\bar{x}$ and $\bar{y}$ are not positive, *i.e.*, some elements of $\bar{x}$ and $\bar{y}$ are zero. Let $I = \{i | \bar{x}_i = 0\}$ and $J = \{\alpha | \bar{y}_\alpha = 0\}$. It is apparent that I, J are proper subsets of $\{1, 2, \ldots, N\}$ and $\{1, 2, \ldots, L\}$, respectively. Suppose

$$\delta = \min\{\min\{\bar{x}_i | i \notin I\}, \min\{\bar{y}_\alpha | \alpha \notin J\}\}.$$

It is easy to get that $\delta > 0$. Since $\bar{x}$ and $\bar{y}$ satisfy $\mathcal{O}\bar{x}\bar{y}^2 = \bar{x}$, we have

$$\sum_{\alpha=1}^{L}\sum_{j=1}^{N}\sum_{\beta=1}^{L} \mathcal{O}_{i,\alpha,j,\beta}\bar{x}_j\bar{y}_\alpha\bar{y}_\beta = \bar{x}_i, \quad \forall i \in I.$$

Then, we consider the following inequality:

$$\begin{aligned} &\delta^3 \sum_{\alpha\notin J}\sum_{j\notin I}\sum_{\beta\notin J} \mathcal{O}_{i,\alpha,j,\beta} \\ &\le \sum_{\alpha\notin J}\sum_{j\notin I}\sum_{\beta\notin J} \mathcal{O}_{i,\alpha,j,\beta}\bar{x}_j\bar{y}_\alpha\bar{y}_\beta \\ &\le \sum_{\alpha=1}^{L}\sum_{j=1}^{N}\sum_{\beta=1}^{L} \mathcal{O}_{i,\alpha,j,\beta}\bar{x}_j\bar{y}_\alpha\bar{y}_\beta = 0, \forall i \in I. \end{aligned} \tag{6}$$

According to the inequality (6), we have

$$\mathcal{O}_{i,\alpha,j,\beta} = 0, \quad \forall i \in I, \forall j \notin I, \forall \alpha \notin J, \beta \notin J.$$

i.e., $\mathcal{O}$ is reducible. By applying similar argument and studying the another tensor equation $R\bar{x}^2\bar{y} = \bar{y}$, we can also obtain that $\mathcal{R}$ is reducible. From these results, we have a contradiction. Thus, $\bar{x}$ and $\bar{y}$ must be positive. The proof is now completed. □

Kellogg[26] has provided a general condition to guarantee the uniqueness of the fixed point in the Brouwer fixed point theorem, namely, (i) 1 is not an eigenvalue of the Jacobian

matrix of the mapping T, and (ii) for each point in the boundary of the domain of the mapping T, it is not a fixed point. Theorem 2 has shown that all the fixed points of T are positive when tensors $\mathcal{H}$, $\mathcal{A}$, $\bar{\mathcal{H}}$, and $\bar{\mathcal{A}}$ are irreducible, *i.e.*, they do not lie on the boundary $\partial\Omega$ of Ω. Based on these results, we obtain the following theorem that shows the tensor equations (4) has a unique solution.

Theorem 3 Suppose tensor $\mathcal{T}$ is irreducible. If 1 is not an eigenvalue of the Jacobian matrix of the mapping T, then the tensor equations (4) has a unique solution.

4.1. The proposed algorithm

To obtain $\bar{x}$ and $\bar{y}$ for the centrality scores of nodes and layers, in this subsection, we develop an efficient iterative algorithm for computing the tensor equations (4). We show the iterative algorithm as follows.

Algorithm 1: Computing MRFNL centrality in multilayer networks

1: Input: Given two transition probability tensors $\mathcal{O}$ and $\mathcal{R}$, two initial vectors $x_0 \in \Omega_N$ and $y_0 \in \Omega_L$, the tolerance ε.
2: Output: Two centrality vectors $\bar{x}$(centrality scores of nodes) and $\bar{y}$(centrality scores of layers).
3: Set $t = 1$;
4: Compute $x_t = (\mathcal{O}x_{t-1}y_{t-1}^2)$;
5: Compute $y_t = \mathcal{R}x_t^2 y_{t-1}$;
6: If $\|x_t - x_{t-1}\|_1 + \|y_t - y_{t-1}\|_1 < \varepsilon$, then stop, otherwise set $t = t+1$ and goto step 4.

In Algorithm 1, the main computational cost of the Algorithm 1 depends on the cost of performing tensor operations in steps 4 and 5. Suppose that there exist $O(N)$ nonzero entries in $\mathcal{O}$ and $\mathcal{R}$, then the cost of these tensor calculations are of $O(N)$ arithmetic operations.

Now, we establish the convergence of Algorithm 1 in following theorem.

Theorem 4 Suppose tensor $\mathcal{T}$ is irreducible and 1 is not an eigenvalue of the Jacobian matrix of the mapping T. Let $[\bar{x},\bar{y}]$ be the unique solution to tensor equations (4) and $[x_t,y_t]$ is generated by Algorithm 1. Given any initial point $[x_0,y_0]$, we have

$$\lim_{t\to\infty}[x_t,y_t] = [\bar{x},\bar{y}].$$

Proof Suppose $x_t = x_{t-1}$ and $y_t = y_{t-1}$ in the Algorithm 1, then we can get the unique solution vectors $\bar{x}$ and $\bar{y}$ for $\mathcal{O}\bar{x}\bar{y}^2 = \bar{x}$ and $\mathcal{R}\bar{x}^2\bar{y} = \bar{y}$ by using the Theorem 3. Meanwhile, if $x_t \neq x_{t-1}$ and $y_t \neq y_{t-1}$, there exists a subsequence $[x_{t_s},y_{t_s}]$ converges to the solution $[\bar{x},\bar{y}]$, the reason is that Ω is a compact space in R^{N+L}. Thus, $[x_t,y_t]$ will converges(up to a subsequence) to $[\bar{x},\bar{y}]$ for any given initial point $[x_0,y_0]$. □

5. Numerical experiments

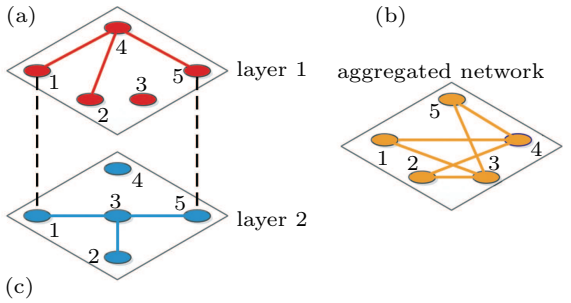
In this section, we will show three experiments to illustrate the effectiveness of the proposed method. In the first experiment, a simple multilayer network is constructed. In the second and third experiments, we apply MRFNL centrality for two real-world multilayer networks: the Pierre Auger Multilayer Collaboration Network[27] and the European Air Transportation Multilayer Network.[28] Note that these three networks are unweighted, and the $t_{i\alpha j\beta} = 1$ if node j in layer β points to node i in layer α, and otherwise $t_{i\alpha j\beta} = 0$. The initial points x_0, y_0 are set to $ones(N,1)/N$, $ones(L,1)/L$ respectively.

5.1. A simple example

In this subsection, we construct a simple example(shown in Fig. 2(a)) to show the difference behaviors of the proposed method and other methods based on an aggregated single layer structure.[29] Figure 2(a) includes five individuals connected by two different layers. Figure 2(b) shows the corresponding aggregated network. "In this paper, the multilayer network is aggregated into a single layer network by projecting each layer of multilayer network to one layer, and the adjacency matrix of aggregated network is obtained by summing the adjacency matrix of each layer (*i.e.*, $A = \sum_{\alpha=1}^{L} A^{\alpha}$, where A^{α} is the adjacency matrix of layer α in multilayer network)". The ranking results for four different cases are given in Fig. 2(c). MRFNL centrality is used for the multilayer network, and eigenvector centrality for the other three cases.

As we can see, from Fig. 2(c), nodes 3 and 4 are ranked number one when this multilayer network is aggregated into a single layer network. Meanwhile, we observe that nodes 1, 2, and 5 have the same centrality values in the aggregated network. In layer 1 and layer 2, nodes 4 and 3 are considered to be the most important points, respectively. The results of the MRFNL centrality show that node 1 and node 5 are ranked number one, *i.e.*, they play the most key roles in this simple multilayer network. In fact, from the Fig. 2(a), it is obvious that nodes 1 and 5 are the most important nodes in this simple multilayer network, the reason is that they bridge between these two layers. These results show that neglecting the relationships between the different layers will lead to incorrect results. However, our ranking algorithm is able to effectively

get the important nodes in multilayer networks.



(c)

Nodes	Ranking			
	Multiayer	Aggregate	Layer 1	Layer 2
1	1	2	2	2
2	3	2	2	2
3	2	1	3	1
4	2	1	1	3
5	1	2	2	2

Fig. 2. A simple multilayer network. (a) This multilayer network contains five individuals connected by two different layers. (b) The corresponding aggregated network. (c) The ranking results of nodes for four different cases, *i.e.*, MRFNL is applied for this multilayer network and eigenvector centrality is used for the other three cases.

5.2. Real world multilayer networks

To further compare the different behaviors of the proposed centrality measure and other centrality measures, we apply the proposed ranking algorithm and the other existing methods for two real world multilayer networks, which are studied separately in the following two subsections. Here, the other existing centrality measures contain Singular Vector of Tensor (SVT) centrality,[22] PageRank centrality (PC-S) based on aggregated single-layer structure and the corresponding centrality algorithm based on multilayer structure (called PC-M),[15] where the adjacency tensor $\mathcal{T}$ in SVT is same as the adjacency tensor in our centrality.

5.2.1. Pierre Auger Multilayer Collaboration Network

The Pierre Auger Multilayer Collaboration Network[27] (PieAug) describes the scientific collaboration network established in the framework of the Pierre Auger experiment, which contains 16 layers and 514 nodes. Nodes represent scientists and layers represent scientific topics. Each layer is a network characterizing the collaborations in a given scientific topic, and each layer in this multilayer network is undirected and unweighted. In this multilayer network, we connect the nodes in these three layers Enhancements, Detector and SD-reconstruction to their counterparts in other layers. In other words, the nodes in these three layers are able to spread information from one layer to any other layer.

Table 1 shows the experiment results when the MRFNL and SVT centrality measures are applied for Pierre Auger Multilayer Collaboration Network. However, Table 1 only displays the ranking results of the top 10 scientific topics, this is because nodes labels and identities are not disclosed. Note that the reason for only comparing MRFNL with the SVT centrality is that the other existing methods cannot simultaneously rank nodes and layers of multilayer networks with interlayer coupling.

Table 1. Numerical results of PieAug multilayer network for MRFNL and SVT centrality.

Ranking	MRFNL	SVT
1	Enhancements	Enhancements
2	Detector	Detector
3	SD-reconstruction	SD-reconstruction
4	Mass-composition	Point-source
5	Hybrid-reconstruction	Anisotropy
6	Point-source	Atmospheric
7	Spectrum	Hybrid-reconstruction
8	Anisotropy	Photons
9	Neutrinos	Neutrinos
10	Horizontal	Spectrum

As we can see, from Table 1, we observe that the top three scientific topics are the same for these two centrality measures. This is not surprising because these three scientific topics (*i.e.*, Enhancements, Detector, and SD-reconstruction) contain more scientists than the other scientific topics. In particular, Enhancements includes the largest number of scientists (*i.e.*, 5432) and rank number one. Another reason is that information can spread from one scientific topic to the other through these three layers. On the other hand, in the SVT results, Mass-composition does not appear in the list of the top 10 scientific topics, but appears in MRFNL (rank number four). Indeed, Mass-composition plays an important role in PieAug, which contains more scientists (*i.e.*, 190) than the other scientific topics, except Enhancements,Detector and SD-reconstruction. This result shows that the use of SVT for PieAug may not be effective.

To analyze the linear correlation between the MRFNL and SVT, we introduced the Pearson's correlation coefficient. The value of this coefficient between the MRFNL and SVT is equal to 0.1024. It means that there does not exist linear correlation between these two centrality measures. The reason for this is that the new centrality is proposed by introducing random walks in multilayer networks, and the SVT is obtained by solving the singular vector of adjacency tensor of multilayer networks. To better illustrate this result, we show the scatter plots between these two methods in Fig. 3.

We further compare ranking results of nodes between MRFNL and SVT centrality by introducing *intersection similarity*.[30] This index is applied to compare the top k elements of two ranking vectors. We show the definition of this

index as follows: given any two ranking vectors x and y, and x_k, y_k represent the top k entries of x and y, respectively. Then, the top k intersection similarity is defined as

$$isim_k(x,y) = \frac{1}{k}\sum_{i=1}^{k}\frac{|x_i \Delta y_i|}{2i}, \tag{7}$$

where $|x_i \Delta y_i|$ is the symmetric difference between x_i and y_i and $|S|$ denotes the cardinality of the set S. According to Eq. (7), it is easy to obtain that the intersection similarity between the two ranking vectors must be in the range $0 \le \tau \le 1$. In particular, this index is equal to 1 when the ordered sequences included in x and y are completely different. On the other hand, if $isim_k(x,y) = 0$ for all k, then x and y are called identical.

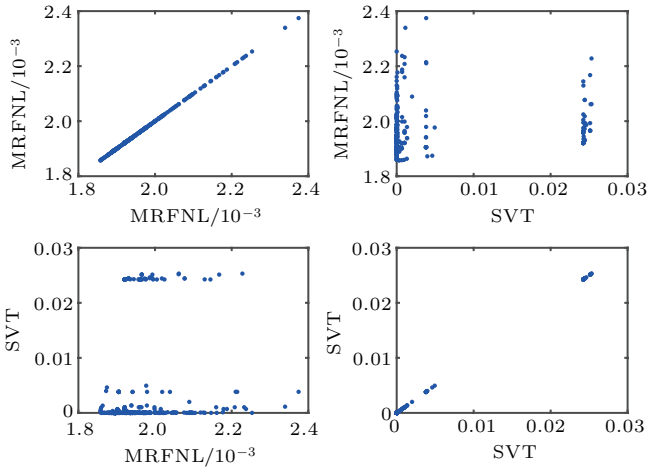


Fig. 3. Scatter plots between MRFNL and SVT.

Figure 4 presents the values of intersection similarity versus k for the ranking of nodes obtained by the proposed centrality and SVT. Figure 4 shows that the values of intersection similarity between these two centrality measures are large for small k. This means that the list of the top ranked nodes obtained by MRFNL differs significantly from the SVT centrality.

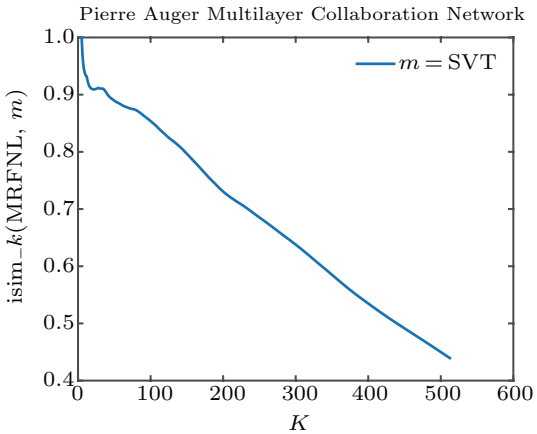


Fig. 4. Intersection similarity between MRFNL and SVT.

Before ending this subsection, we display the convergence rate of MRFNL and SVT centrality measures in Fig. 5. Figure 5 shows that MRFNL centrality method is faster than SVT, where MRFNL algorithm terminates after 14 iterations.

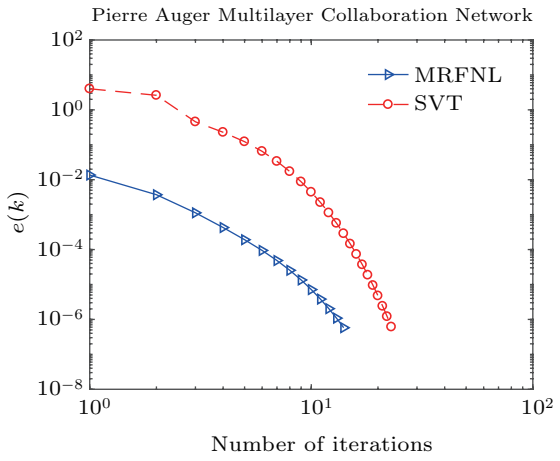


Fig. 5. The convergence history of the algorithms for PieAug network.

5.2.2. European Air Transportation Multilayer Network

In this subsection, we apply the proposed centrality and other existing centrality measures for ranking the nodes of European Air Transportation Multilayer Network (EUAir) and illustrate the effectiveness of the MRFNL centrality.

First, we describe the European Air Transportation Multilayer Network as follows. This network contains 37 layers and 450 nodes. Nodes represent European airports and layers represent European airline companies. Each layer is unweighted and undirected and is formed by the flight connections operated by a given European airline company. Furthermore, we connect the LFPG (Charles de Gaulle) and EBBR (Brussels) airports in one layer to their counterparts in other layers, *i.e.*, the interlayer links of this multilayer network only exist between these two nodes and their counterparts in other layers.

Table 2. Numerical results of EUAir multilayer network for all centrality measures.

Ranking	MRFNL	SVT	PC-S	PC-M
1	LFPG	EGSS	EHAM	LFPG
2	EBBR	EIDW	LGAV	EBBR
3	EHAM	LIME	LEMD	EHAM
4	LOWW	EBCI	LEBL	LOWW
5	LTBA	LEMD	EDDM	LEMD
6	LEMD	LEAL	EGSS	LGAV
7	LGAV	EDFH	LTBA	LTBA
8	EGKK	LIRA	LOWW	LEBL
9	LKPR	LEGE	EGKK	EDDM
10	EGLL	LEPA	EDDF	LIRF

Table 2 presents the top 10 European airports in EUAir network when MRFNL, SVT, PC-S, and PC-M centrality measures are used. According to the Table 2, we find the following results:

(i) By observing the ranking results of MRFNL and SVT centrality measures, the lists of top 10 airports obtained by these two measures are quite different. Only one airport (LEMD) appears in both lists. In particular, the LFPG (Charles de Gaulle) and EBBR (Brussels) airports do not appear in the top 10 European airports list of SVT, but appear in the list of MRFNL (rank number one and two, respectively). In fact, these two airports play a key role in EUAir network, because their flights are operated by almost all airlines. Meanwhile, they bridge between the all airlines. In other words, information can spread from one airline to the other through these two airports. These results illustrate that using SVT for EUAir may not be effective.

(ii) The same four European airports that appear in both lists are in the BORW and PC-S results. However, the LFPG and EBBR airports cannot be identified by PC-S. The reason for this is that PC-S centrality neglects the multiple relationships of EUAir network. However, our method can capture these two important airports.

(iii) Now, we analyze the results of MRFNL and PC-M centrality measures. The lists of the top 10 airports obtained by these two ranking methods are similar, and three different airports that appear in both lists. The top four airports are same for these two centrality measures and LFPG and EBBR airports can be identified by them. This is not surprising, because the basic idea of MRFNL and PC-M is same, *i.e.*, they consider a random walk in multilayer networks. The difference between them is that our centrality method considers the importance of layers.

Table 3. Pearson correlation coefficient between any two centrality measures for EUAir network.

	MRFNL	SVT	PC-S	PC-M
MRFNL	1	0.1205	0.3337	0.3468
SVT	0.1205	1	0.2050	0.1844
PC-S	0.3337	0.2050	1	0.4606
PC-M	0.3468	0.1844	0.4606	1

Now, the Pearson's correlation coefficients between the different centrality measures are provided in Table 3. Table 3 presents that no pair of centrality measures display linear correlation. In order to further illustrate these results, the scatter plots of nodes are shown in Fig. 6. Figure 6 also tells us that no pair of centrality measures shows linear correlation.

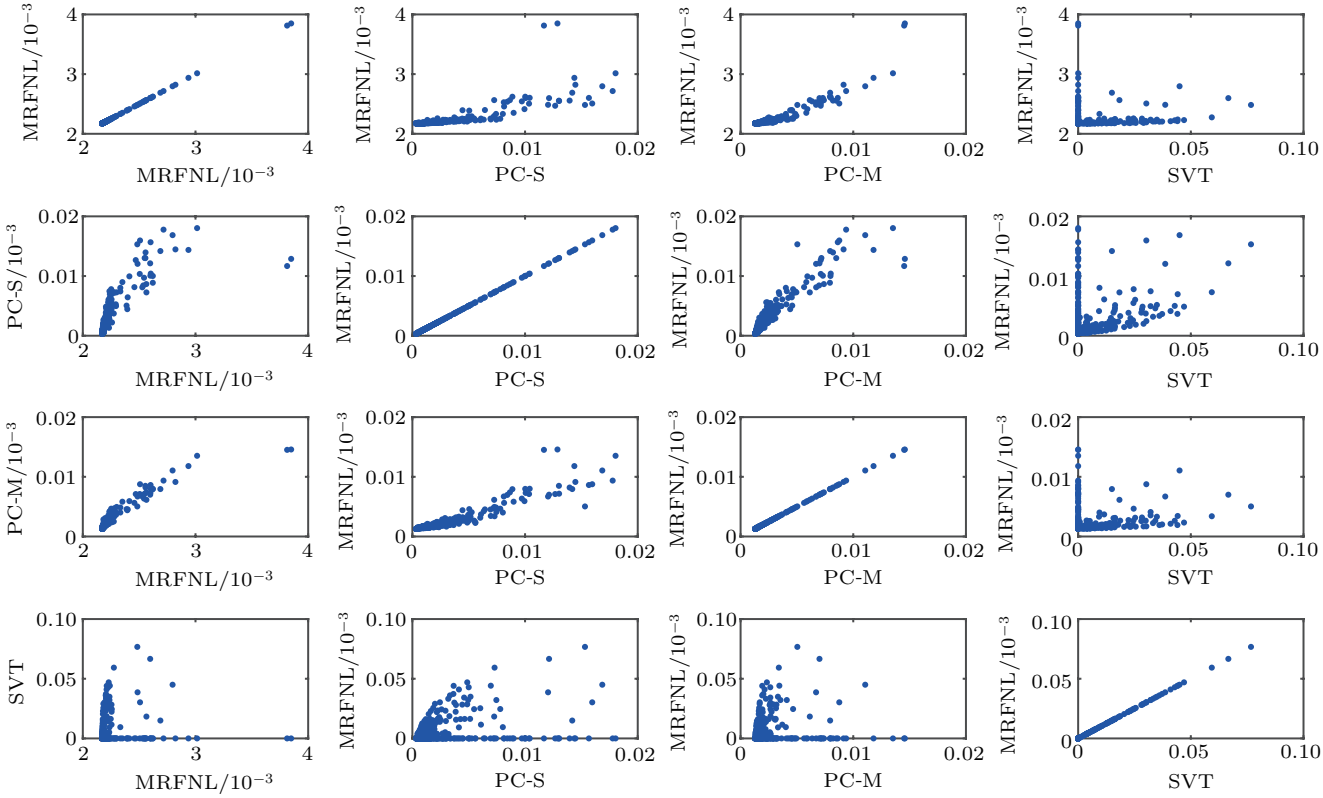


Fig. 6. Scatter plots between different centrality measures.

Furthermore, figure 7 shows the results of intersection similarity *versus* k for the ranking of nodes obtained by different methods. As we can see, from Fig. 7, the values of intersection similarity between these three centrality measures MRFNL, SVT, and PC-S are high for small k. This means that the lists of the top ranked airports obtained by these three centrality measures are significantly different. However, for MRFNL and PC-M methods, the values of intersection simi-

larity between them are small for small values of k. This indicates that the top airports are similar for these two centrality. These results can also be observed from Table 2.

Finally, the convergence rate of these algorithms are presented in Fig. 8. We find that the MRFNL centrality is the fastest among these four methods, where the proposed method terminates after 10 iterations.

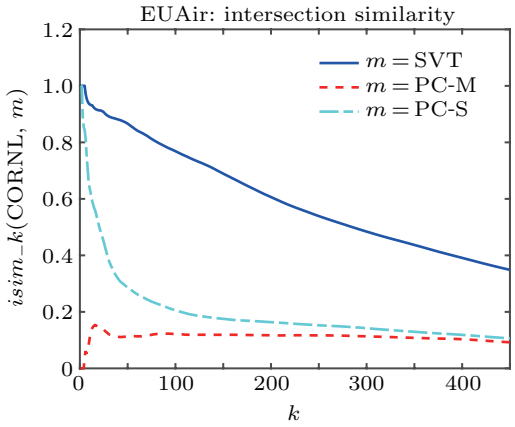


Fig. 7. Intersection similarity between MRFNL, SVT, PC-S, and PC-M.

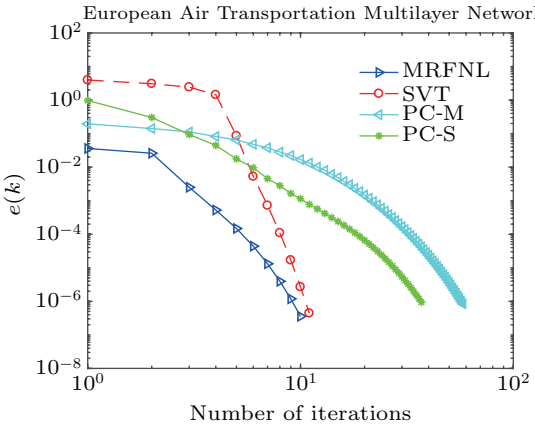


Fig. 8. The convergence history of the algorithms for EUAir network.

6. Conclusion

In this paper, we have extended the classical PageRank to rank the nodes and layers in multilayer networks with simultaneous interlayer coupling, referred to as the MRFNL centrality. We first constructed a set of tensor equations based on two transition probability tensors, and the new centrality can be obtained by developing a novel iterative algorithm that can be used to solve tensor equations. Then, the existence and uniqueness of the proposed centrality can be guaranteed by using Brouwer fixed point theorem. Furthermore, we have theoretically proved the convergence of the proposed iterative algorithm. The results of the numerical experiments have shown that the proposed method is efficient and outperforms some existing methods for the identification of essential nodes.

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