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Dynamics of three nonisospectral nonlinear Schrödinger equations*

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Dynamics of three nonisospectral nonlinear Schrödinger equations (NNLSEs), following different time dependencies of the spectral parameter, are investigated. First, we discuss the gauge transformations between the standard nonlinear Schrödinger equation (NLSE) and its first two nonisospectral counterparts, for which we derive solutions and infinitely many conserved quantities. Then, exact solutions of the three NNLSEs are derived in double Wronskian terms. Moreover, we analyze the dynamics of the solitons in the presence of the nonisospectral effects by demonstrating how the shapes, velocities, and wave energies change in time. In particular, we obtain a rogue wave type of soliton solutions to the third NNLSE.

Keywords: nonisospectral nonlinear Schrödinger equations, gauge transformations, bilinear forms, solitons, rogue waves

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1. Introduction

As an integrable partial differential equation, the celebrated nonlinear Schrödinger equation (NLSE) can be solved by means of the inverse scattering transform (IST),[1,2] and possesses soliton solutions. A handful of direct techniques are available to derive its soliton solutions, such as the Hirota's bilinear method,[3] Darboux–Bäcklund transformations, *cf.* Ref. [4], and the Wronskian technique, *cf.* Refs. [5]–[7]. The equation has a broad range of applications. For instance, the evolution of waves in nonlinear media is governed by the NLSE;[8] it can be used to model deep water waves[9] and vortex filaments.[10] The equation is also relevant to describe transmitting short optical pulses in a dielectric fiber[11,12] and the collapse of Langmuir waves in plasma physics.[13] The vector version of the NLSE, known as the Manakov model,[14] is further connected to areas in fiber optics and low-temperature physics. The NLSE exhibits rogue waves that are solutions of the rational type: Peregrine gave the first-order rational solutions,[15] and its higher order generalizations were given in Refs. [16] and [17]. The term "rogue waves" was introduced by oceanographers to depict large-amplitude waves that occur more frequently than expected for Gaussian-distributed statistical events. Rogue waves has been a hot topic of discussion and debate.[18] More details about the generating mechanisms of rogue waves in various physical contexts can be found in Ref. [19] and references therein.

It is known that some physical models possess time-varying wave packets in a nonuniform medium. In this context, some extensions of the NLSE[20–22] appeared to characterize these models, and have time-varying solitary wave solutions. Furthermore, like many other nonlinear evolution equations,[23–28] they have Lax pairs with the spectral parameter depending on time, which are referred to as nonisospectral nonlinear Schrödinger equations (NNLSEs). It is a natural question to study exact solutions of these equations as well as the nonisospectral dynamics.

In this paper, we investigate the solutions of the following three NNLSEs:

$$\mathrm{i}q_{1,t}+q_{1,xx}+2|q_1|^2q_1-\alpha xq_1=0, \tag{1}$$

$$\mathrm{i}q_{2,t}+q_{2,xx}+2|q_2|^2q_2-\mathrm{i}\beta(xq_2)_x=0, \tag{2}$$

$$\mathrm{i}q_{3,t}+x(q_{3,xx}+2|q_3|^2q_3)+2q_{3,x}+2q_3\partial^{-1}|q_3|^2=0, \tag{3}$$

where α and β are real constants, and $\partial^{-1}=\int_{-\infty}^{x}\mathrm{d}x$. We call the equations NNLSE-I, NNLSE-II, and NNLSE-III, respectively. In the ZS-AKNS Lax representations,[1,2] they correspond respectively to three different time dependencies of the spectral parameter η, *i.e.*, $\partial_t\eta=\eta^j$, $j=0,1,2$. The NNLSE-I was first introduced in the context of nonuniform media, and was solved using the IST in 1976 by Chen and Liu.[20] The IST was used to solve the NNLSE-II in 1979 by Gupta.[2] Calogero and Degasperis introduced the NNLSE-III in 1978.[22] This equation is considered to be gauge-equivalent to the cylindrical Heisenberg equation.[29] It is also one of the nonisospectral flows that are used to construct τ-symmetries for the isospectral hierarchy.[28,30]

In this paper, we first discuss the gauge transformations between the NNLSE-I, NNLSE-II, and NLSE, *i.e.*, the transformations between the evolution equations and their Lax pairs. Early applications of gauge transformations in soliton theory can be found in Ref. [31]. By employing the gauge transformations, we obtain the solutions and infinitely many conserved quantities for the two NNLSEs directly from the known results of the NLSE. However, we do not find such connections between the NLSE and NNLSE-III. Soliton solutions

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in double Wronskian forms to the NNLSE-I and NNLSE-II are provided based on those of the NLSE. For the NNLSE-III, its solutions in double Wronskian forms can be derived in a similar generalized procedure as that used for the NLSE.[7] Then we investigate the dynamics of these three NNLSEs and discuss the nonisospectral effects resulting from the different time dependencies of the spectral parameter. The nonisospectral effects are represented by changes in the shape, velocity, and wave energy of the solitons. In particular, the soliton solutions of the NNLSE-III have a rogue wave property due to the nonisospectral dependency ($\partial_t\eta=\eta^2$) of the parameter.

The paper is organized as follows. In Section 2, we discuss Lax representations of these three NNLSEs and their gauge transformations. In Section 3, we derive solutions in double Wronskian forms for the three NNLSEs. Nonisospectral behaviors of these solutions are investigated in Section 4. Finally, Section 5 concludes this paper.

2. Lax pairs, gauge transformations, and conserved quantities

In this section, we recall the Lax pairs of the NLSE and its three nonisospectral counterparts. We also discuss the gauge transformations between the NLSE and the first two NNLSEs, from which we can obtain solutions and infinitely many conserved quantities for those two NNLSEs.

2.1. Lax pairs

Consider the ZS-AKNS linear problems[1,2]

$$\Phi_x=\begin{pmatrix}-\eta & q\\ r & \eta\end{pmatrix}\Phi=M(\eta,q,r)\,\Phi, \tag{4}$$

$$\Phi_t=\begin{pmatrix}A & B\\ C & -A\end{pmatrix}\Phi=N(\eta,q,r)\,\Phi, \tag{5}$$

where $\Phi=(\phi_1,\phi_2)^{\mathrm{T}}$, η is the spectral parameter depending on t, and q,r are functions of x,t. The compatibility condition $M_t-N_x+[M,N]=0$ implies (after integration)

$$A=\partial^{-1}(r,q)\begin{pmatrix}-B\\ C\end{pmatrix}-\eta_t x+A_0, \tag{6}$$

and

$$\begin{pmatrix}q\\ r\end{pmatrix}_t=L\begin{pmatrix}-B\\ C\end{pmatrix}-2\eta\begin{pmatrix}-B\\ C\end{pmatrix}-2A_0\sigma_3\begin{pmatrix}q\\ r\end{pmatrix}-2\eta_t\sigma_3\begin{pmatrix}xq\\ xr\end{pmatrix}. \tag{7}$$

Here A_0 is a function of η and t, but is independent of x, and

$$L=-\sigma_3\partial+2\begin{pmatrix}q\\ -r\end{pmatrix}\partial^{-1}(r,q),$$
$$\sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \partial=\frac{\partial}{\partial x}.$$

To obtain evolution equations, we use the ansatz

$$\begin{pmatrix}B\\ C\end{pmatrix}=\sum_{j=1}^{2}\begin{pmatrix}b_j\\ c_j\end{pmatrix}\eta^{2-j}, \tag{8}$$

and substitute it into Eq. (7). In this isospectral case ($\eta_t=0$), letting $A_0=2\mathrm{i}\eta^2$, $(b_1,c_1)=-2\mathrm{i}(q,r)$, and $r=-q^*$, where the asterisk denotes the complex conjugation, we obtain the NLSE

$$\mathrm{i}q_t+q_{xx}+2|q|^2q=0. \tag{9}$$

The NNLSEs (1)–(3) correspond respectively to the choices

$$\begin{array}{llll}\eta_t=\mathrm{i}\alpha/2, & A_0=2\mathrm{i}\eta^2, & (b_1,c_1)=-2\mathrm{i}(q_1,r_1), & r_1=-q_1^*,\\ \eta_t=-\beta\eta, & A_0=2\mathrm{i}\eta^2, & (b_1,c_1)=-2\mathrm{i}(q_2,r_2), & r_2=-q_2^*,\\ \eta_t=-2\mathrm{i}\eta^2, & A_0=0, & (b_1,c_1)=-2\mathrm{i}x(q_3,r_3), & r_3=-q_3^*,\end{array}$$

where we substitute $\{q_j,r_j\}_{j=1,2,3}$ for q,r respectively. The above choices of η_t, A_0, and (b_1,c_1) allow us to get explicit expressions of $N(\eta,q,r)$ in the Lax pair. We have

$$N(\eta,q,r)=\mathrm{i}\begin{pmatrix}2\eta^2-qr & -2q\eta+q_x\\ -2r\eta-r_x & -2\eta^2+qr\end{pmatrix},\quad r=-q^*, \tag{10}$$

in the case of NLSE, and

$$N_1(\eta,q_1,r_1)=\mathrm{i}\begin{pmatrix}2\eta^2-q_1r_1-\frac{\alpha}{2}x & -2q_1\eta+q_{1,x}\\ -2r_1\eta-r_{1,x} & -2\eta^2+q_1r_1+\frac{\alpha}{2}x\end{pmatrix}, \tag{11}$$

$$N_2(\eta,q_2,r_2)=\mathrm{i}\begin{pmatrix}2\eta^2-\mathrm{i}\beta x\eta-q_2r_2 & -2q_2\eta+q_{2,x}+\mathrm{i}\beta xq_2\\ -2r_2\eta-r_{2,x}+\mathrm{i}\beta xq_2 & -2\eta^2+\mathrm{i}\beta x\eta+q_2r_2\end{pmatrix}, \tag{12}$$

$$N_3(\eta,q_3,r_3)=\mathrm{i}\begin{pmatrix}2x\eta^2-xq_3r_3-\partial^{-1}q_3r_3 & -2xq_3\eta+(xq_3)_x\\ -2xr_3\eta-(xr_3)_x & -2x\eta^2+xq_3r_3+\partial^{-1}q_3r_3\end{pmatrix}, \tag{13}$$

for NNLSE-I, NNLSE-II, and NNLSE-III, respectively, where the reduction $r_j=-q_j^*$, and $j=1,2,3$ would be imposed.

2.2. Gauge equivalence

Gauge transformations can be seen as a tool to connect different nonlinear evolution equations, *cf.* Ref. [31]. The connections between NLSE, NNLSE-I, and NNLSE-II are described in the following theorem.

Theorem 1 Under the transformation

$$\phi_1=\exp\left(\frac{\mathrm{i}}{2}\alpha tx+\frac{\mathrm{i}}{6}\alpha^2t^3\right)\psi_1,$$

$$\phi_2 = \exp\left(-\left(\frac{\mathrm{i}}{2}\alpha t x + \frac{\mathrm{i}}{6}\alpha^2 t^3\right)\right)\psi_2,$$
$$X = x + \alpha t^2, \quad T = t,$$
$$q = \exp\left(\mathrm{i}\left(\alpha t x + \frac{1}{3}\alpha^2 t^3\right)\right)q_1,$$
$$r = \exp\left(-\mathrm{i}\left(\alpha t x + \frac{1}{3}\alpha^2 t^3\right)\right)r_1, \tag{14}$$

where $\Phi = (\phi_1, \phi_2)^{\mathrm{T}}$, $\Psi = (\psi_1, \psi_2)^{\mathrm{T}}$ satisfies the Lax pairs of the NLSE and NNLSE-I, respectively, *i.e.*,

$$\Phi_x = M(k, q(T,X), r(T,X))\Phi,$$
$$\Phi_t = N(k, q(T,X), r(T,X))\Phi, \tag{15}$$
$$\Psi_x = M(\eta, q_1(t,x), r_1(t,x))\Psi,$$
$$\Psi_t = N_1(\eta, q_1(t,x), r_1(t,x))\Psi, \tag{16}$$

with $\eta = k + (\mathrm{i}/2)\alpha t$. Then the NNLSE-I is transformed to the NLSE

$$\mathrm{i}q_T + q_{XX} + 2|q|^2 q = 0. \tag{17}$$

Similarly, the transformation

$$\phi_1 = \psi_1, \quad \phi_2 = \psi_2, \quad X = x\mathrm{e}^{-\beta t}, \quad T = \frac{1 - \mathrm{e}^{-2\beta t}}{2\beta},$$
$$q = \mathrm{e}^{\beta t} q_2, \quad r = \mathrm{e}^{\beta t} r_2 \tag{18}$$

transforms the NNLSE-II to the NLSE equation (17), where $\Psi = (\psi_1, \psi_2)^{\mathrm{T}}$ satisfies the Lax pair

$$\Psi_x = M(\eta, q_2(t,x), r_2(t,x))\Psi,$$
$$\Psi_t = N_2(\eta, q_2(t,x), r_2(t,x))\Psi, \tag{19}$$

with $\eta = k\mathrm{e}^{-\beta t}$.

The proof can be done by direct computations. Note that the transformations (14) were given by Fred Tappert in the note added in Ref. [20]. These two gauge transformations are reversible, which reveals that the NLSE also bridges the gap between the NNLSE-I and NNLSE-II. We also stress that, from the viewpoint of Ref. [29], although the two nonisospectral equations NNLSE-I and NNLSE-II are gauge-connected to the standard NLSE, their soliton dynamics are still physically interesting.

2.3. Solutions and conserved quantities for the NNLSE-I and NNLSE-II

The previous gauge transformations allow us to obtain solutions of the NNLSE-I and NNLSE-II based on those of the NLSE. Precisely, for any $q(T,X)$ being a solution to the NLSE equation (14), then

$$q_1(t,x) := \exp\left[-\mathrm{i}\left(\alpha t x + \frac{1}{3}\alpha^2 t^2\right)\right] q(t, x + \alpha t^2), \tag{20}$$

$$q_2(t,x) = \mathrm{e}^{-\beta t} q\left(\frac{1 - \mathrm{e}^{-2\beta t}}{2\beta}, x\mathrm{e}^{-\beta t}\right), \tag{21}$$

are solutions to the NNLSE-I and NNLSE-II, respectively. The gauge transformations also allow us to derive conserved quantities for the NNLSE-I and NNLSE-II. Consider the AKNS isospectral evolution hierarchy related to Eq. (4) and Eq. (5). Here we use X and T as the space and time variables, and the conservation law can be written as, *cf.* Ref. [32]

$$(-\eta + q\omega)_T = (A + B\omega)_X, \tag{22}$$

where $\omega = \phi_2/\phi_1$, and $q\omega$ is determined by the Riccati equation[32,33]

$$q\omega_X = qr + 2\eta q\omega - (q\omega)^2. \tag{23}$$

Through the expansion

$$q\omega = \sum_{j=1}^{\infty} \frac{\omega_j}{(2\eta)^j}, \tag{24}$$

one has

$$\omega_{j+1} \equiv \omega_{j+1}(q,r) = -qr\delta_{j,0} + q\left(\frac{\omega_j}{q}\right)_X + \sum_{k=1}^{j-1} \omega_{j-k}\omega_k, \quad j = 0, 1, \ldots. \tag{25}$$

Then $q\omega(q,r)$ are the communal infinitely many conserved densities for all the AKNS isospectral evolution equations. For the NLSE equation (17), each conserved density can be represented as

$$\rho_j = \omega_j(q,r), \tag{26}$$

and the related conserved quantity is

$$I_j = \int_{-\infty}^{\infty} \omega_j(q,r)\,\mathrm{d}X, \quad r = -q^*. \tag{27}$$

By using the transformations in Eqs. (14) and (18), the conserved quantities for Eqs. (1) and (2) are given respectively by

$$I_j = \int_{-\infty}^{\infty} \omega_j\left(\exp\left[\mathrm{i}\left(\alpha t x + \frac{1}{3}\alpha^2 t^3\right)\right] q_1, \exp\left[-\mathrm{i}\left(\alpha t x + \frac{1}{3}\alpha^2 t^3\right)\right] r_1\right)\mathrm{d}x, \quad r_1 = -q_1^*, \tag{28}$$

$$I_j = \int_{-\infty}^{\infty} \mathrm{e}^{-\beta t}\omega_j(\mathrm{e}^{\beta t} q_2, \mathrm{e}^{\beta t} r_2)dx, \quad r_2 = -q_2^*. \tag{29}$$

Therefore the conserved densities for the NNLSE-I and NNLSE-II are

$$\rho_{1,j} = \omega_j\left(\exp\left(\mathrm{i}\left(\alpha t x + \frac{1}{3}\alpha^2 t^3\right)\right) q_1, \exp\left(-\mathrm{i}\left(\alpha t x + \frac{1}{3}\alpha^2 t^3\right)\right) r_1\right), \tag{30}$$

$$\rho_{2,j} = \mathrm{e}^{-\beta t}\omega_j(\mathrm{e}^{\beta t} q_2, \mathrm{e}^{\beta t} r_2), \tag{31}$$

where $r_j = -q_j^*$, and $j = 1,2$. The first two conserved densities for the NLSE, NNLSE-I, and NNLSE-II are given by

$$\rho_1 = -qr, \quad \rho_2 = -qr_X, \quad r = -q^*, \tag{32}$$

$$\rho_{1,1} = -q_1 r_1, \quad \rho_{1,2} = \mathrm{i}\alpha t q_1 r_1 - q_1 r_{1,x}, \quad r_1 = -q_1^*, \tag{33}$$

$$\rho_{2,1} = -\mathrm{e}^{\beta t} q_2 r_2, \quad \rho_{2,2} = -\mathrm{e}^{2\beta t} q_2 r_{2,x}, \quad r_2 = -q_2^*, \tag{34}$$

respectively. It is important to observe that when we compute the conserved densities in Eqs. (30) and (31) from Eqs. (25) and (26), the relations $\partial x/\partial X = 1$ and $\partial x/\partial X = \mathrm{e}^{\beta t}$ are employed respectively. Furthermore, the infinitely many conservation laws for the NNLSE-I and NNLSE-II can also be derived by introducing new spectral parameters, as shown in Ref. [34].

3. Solutions in double Wronskian form

The Wronskian technique is available for deriving solutions of nonisospectral equations as well, *cf.* Refs. [34]–[36]. In this section, we apply the Wronskian technique to derive solutions of the NNLSE-I, NNLSE-II, and NNLSE-III. All these results are obtained through a similar generalized procedure as given in Ref. [7].

By employing the transformation

$$q_j = \frac{g}{f}, \quad (j = 1,2,3), \tag{35}$$

we transform equations (1)–(3) respectively into the following bilinear forms:

$$(\mathrm{i}D_t + D_x^2 - \alpha x) g \cdot f = 0, \quad D_x^2 f \cdot f = g g^*, \tag{36}$$

$$(\mathrm{i}D_t + D_x^2 - \mathrm{i}\beta(D_x + 1)) g \cdot f = 0, \quad D_x^2 f \cdot f = g g^*, \tag{37}$$

$$(\mathrm{i}D_t + x D_x^2) g \cdot f = -2 g_x f, \quad D_x^2 f \cdot f = g g^*, \tag{38}$$

where f is a real function, g is complex, and D is the Hirota bilinear operator[37] defined by

$$D_x^j D_t^j \, f \cdot g \equiv (\partial_x - \partial_{x'})^j (\partial_t - \partial_{t'})^j f(x,t) g(x',t')|_{x'=x,t'=t}. \tag{39}$$

Note that to get the first equation in Eq. (38), one should use $\partial^{-1}|q|^2 = f_x/f$ due to the second one in Eq. (38).

As shown in Refs. [6] and [38], we employ the notation $|\widehat{M-1};\widehat{N-1}|$ to denote an $(M+N)\times(M+N)$ double Wronskian form established as

$$|\widehat{M-1};\widehat{N-1}| = |\phi, \partial_x\phi, \ldots, \partial_x^{M-1}\phi; \psi, \partial_x\psi, \ldots, \partial_x^{N-1}\psi|,$$

where ϕ and ψ are $(M+N)$ column vectors given as

$$\phi = (\phi_1, \phi_2, \ldots, \phi_{M+N})^{\mathrm{T}}, \ \psi = (\psi_1, \psi_2, \ldots, \psi_{M+N})^{\mathrm{T}}.$$

For the solutions in double Wronskian forms to the above three bilinear NNLSEs (36)–(38), we have the following theorem.

Theorem 2 Suppose that the $2N \times 2N$ double Wronskians

$$f = |\widehat{N-1};\widehat{N-1}|, \quad g = 2|\widehat{N-2};\widehat{N}| \tag{40}$$

satisfy

$$f = f^*, \quad g^* = 2|\widehat{N};\widehat{N-2}|, \tag{41}$$

and their entries meet the condition

$$\phi_x = A\phi, \quad \psi_x = -A\psi, \tag{42}$$

where $\alpha_{n,m}$ as the (n,m) entry of the matrix A is some function independent of x. Then equation (40) can respectively provide solutions for Eqs. (36)–(38) if ϕ and ψ further satisfy

$$\phi_t = -2\mathrm{i}\phi_{xx} + \frac{\mathrm{i}}{2}\alpha x\phi, \quad \psi_t = 2\mathrm{i}\psi_{xx} - \frac{\mathrm{i}}{2}\alpha x\psi, \tag{43}$$

$$\phi_t = -2\mathrm{i}\phi_{xx} - \beta x\phi, \quad \psi_t = 2\mathrm{i}\psi_{xx} + \beta x\psi, \tag{44}$$

$$\phi_t = -2\mathrm{i}x\phi_{xx}, \quad \psi_t = 2\mathrm{i}x\psi_{xx}, \tag{45}$$

respectively.

The proof of the above theorem is similar to the one for the NLSE in Ref. [7]. Note that the case where A is diagonal for the NNLSE-III was also obtained as a reduction in Ref. [36].

The discussion for the reality and conjugate condition in Eq. (41) is explained in Ref. [7]. The coefficient matrix A can also be considered as a lower triangular matrix, *i.e.*, $\alpha_{n,m} = 0$ for $n < m$. If we further impose

$$\phi = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \psi^*, \tag{46}$$

where I is an $N \times N$ unit matrix, then the reality and conjugate condition equation (41) can be satisfied. In this case, A has the form

$$A = \begin{pmatrix} -B & 0 \\ 0 & B^* \end{pmatrix}, \tag{47}$$

where B is also a lower triangular matrix. Thus, we can discuss solutions according to the eigenvalues of B, *i.e.*, the diagonal elements of B.

Now we give the explicit forms for ϕ_j and ψ_j. When B has N distinct eigenvalues depending on time, we consider B in Eq. (47) to be diagonal, *i.e.*,

$$B = D_N[b_1(t), b_2(t), \ldots, b_N(t)] = \mathrm{Diag}(b_1(t), b_2(t), \ldots, b_N(t)), \tag{48}$$

where $\{b_j(t)\}$ are N distinct complex functions of t. In this case, the double Wronskian entry vectors can be denoted as

$$\begin{aligned} \phi &= \phi_{2N}^D[b_1(t), b_2(t), \ldots, b_N(t)] \\ &= (\phi_1^D, \phi_2^D \ldots, \phi_N^D; -(\psi_1^D)^*, -(\psi_2^D)^*, \ldots, -(\psi_N^D)^*)^{\mathrm{T}}, \end{aligned} \tag{49}$$

$$\begin{aligned} \psi &= \psi_{2N}^D[b_1(t), b_2(t), \ldots, b_N(t)] \\ &= (\psi_1^D, \psi_2^D \ldots, \psi_N^D; (\phi_1^D)^*, (\phi_2^D)^*, \ldots, (\phi_N^D)^*)^{\mathrm{T}}, \end{aligned} \tag{50}$$

where for the bilinear NNLSE-I equation (36), we have, for $j=1,2,\ldots,N$,

$$b_j(t)=k_j-\frac{\mathrm{i}}{2}\alpha t \tag{51}$$

with arbitrary constant k_j, and we take

$$\phi_j^D=a_j^-\,\mathrm{e}^{-\theta_j},\quad \psi_j^D=a_j^+\,\mathrm{e}^{\theta_j}, \tag{52}$$

where

$$\theta_j=\left(k_j-\frac{\mathrm{i}}{2}\alpha t\right)x+2\mathrm{i}k_j^2t-\frac{\mathrm{i}}{6}\alpha^2t^3+\theta_j^{(0)}; \tag{53}$$

for the bilinear NNLSE-II equation (37), we have, for $j=1,2,\ldots,N$,

$$b_j(t)=k_j\,\mathrm{e}^{-\beta t}, \tag{54}$$

$$\phi_j^D=a_j^-\,\mathrm{e}^{-\zeta_j},\quad \psi_j^D=a_j^+\,\mathrm{e}^{\zeta_j} \tag{55}$$

with

$$\zeta_j=k_j\,\mathrm{e}^{-\beta t}x-\frac{\mathrm{i}k_j^2}{\beta}(\mathrm{e}^{-2\beta t}-1)+\zeta_j^{(0)}; \tag{56}$$

and for the bilinear NNLSE-III equation (38), we have, for $j=1,2,\ldots,N$,

$$b_j(t)=\frac{\mathrm{i}}{2(t+k_j)}, \tag{57}$$

$$\phi_j^D=a_j^-\,\mathrm{e}^{-\rho_j},\quad \psi_j^D=a_j^+\,\mathrm{e}^{\rho_j} \tag{58}$$

with

$$\rho_j=\frac{\mathrm{i}x}{2(t+k_j)}+\rho_j^{(0)}; \tag{59}$$

where k_j, $a_j^{\pm}$, $\theta_j^{(0)}$, $\zeta_j^{(0)}$, and $\rho_j^{(0)}$ are all complex constants. These explicit Wronskian entries contribute normal solitons or solitary waves that display nonisospectral dynamics.

When B has N identical eigenvalues depending on t, for the NNLSE-I and NNLSE-II, we consider B in Eq. (47) to be the following Jordan form, *i.e.*,

$$B=J_N[b_1(t)]=\begin{pmatrix} b_1(t) & 0 & 0 & \ldots & 0 & 0\\ \kappa_1 & b_1(t) & 0 & \ldots & 0 & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & 0 & \ldots & \kappa_1 & b_1(t)\end{pmatrix}_{N\times N}, \tag{60}$$

where κ_1 is some nonzero function of t. To obtain more choices for the double Wronskian entries ϕ_1,ψ_1, we first investigate properties of matrices commuting with a Jordan block.

Proposition 1[39,40] Let J_N be an $N\times N$ Jordan block, and $\mathcal{G}$ is an $N\times N$ complex matrix independent of t and x. Then $\mathcal{G}$ commutes with Jordan block J_N if and only if $\mathcal{G}$ is in the following form:

$$\mathcal{G}=\begin{pmatrix} g_0 & 0 & 0 & \ldots & 0 & 0\\ g_1 & g_0 & 0 & \ldots & 0 & 0\\ g_2 & g_1 & g_0 & \ldots & 0 & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ g_{N-1} & g_{N-2} & g_{N-3} & \ldots & g_1 & g_0\end{pmatrix}_{N\times N}, \tag{61}$$

where g_j, $j=0,1,\ldots,N-1$, are arbitrary complex numbers, *i.e.*, $\mathcal{G}$ is a complex lower-triangular Toeplitz matrix. Furthermore, all such $N\times N$ matrices $\mathcal{G}$ form an Abelian semigroup $\tilde{G}_N$ with an identity.

For the NNLSE-I and NNLSE-II, the double Wronskian entries ϕ and ψ can be taken as

$$\phi=\phi_{2N}^J=\begin{pmatrix}\Phi_N[b_1(t)]\\ -\Psi_N^*[b_1(t)]\end{pmatrix},\qquad \psi=\psi_{2N}^J=\begin{pmatrix}\Psi_N[b_1(t)]\\ \Phi_N^*[b_1(t)]\end{pmatrix} \tag{62}$$

with

$$\Phi_N[b_1(t)]=\mathcal{G}\mathcal{Q}_1^-,\quad \Psi_N[b_1(t)]=\mathcal{H}\mathcal{Q}_1^+,\quad \mathcal{G},\mathcal{H}\in\tilde{G}_N, \tag{63}$$

$$\mathcal{Q}_1^{\pm}=(\mathcal{Q}_{1,0}^{\pm},\mathcal{Q}_{1,1}^{\pm},\ldots,\mathcal{Q}_{N-1}^{\pm})^{\mathrm{T}},$$

$$\mathcal{Q}_{1,j}^-=\frac{1}{j!}\partial_{k1}^j\phi_1^D,\quad \mathcal{Q}_{1,j}^+=\frac{1}{j!}\partial_{k1}^j\psi_1^D, \tag{64}$$

where for the NNLSE-I, ϕ_1^D and ψ_1^D are defined as Eq. (52), $b_1(t)$ is given as Eq. (51) and $\kappa_1\equiv 1$ as Eq. (60). For the NNLSE-II, ϕ_1^D and ψ_1^D are defined as Eq. (55), $b_1(t)$ is given as Eq. (54) and $\kappa_1\equiv \mathrm{e}^{-\beta t}$. For the NNLSE-III, we consider B in Eq. (47) to be given as

$$B=J_N[b_1(t)]=(B_{sj})_{N\times N},$$

$$B_{sj}=\begin{cases}\dfrac{1}{(s-j)!}\partial_{k1}^{s-j}b_1(t)=\dfrac{\mathrm{i}(-1)^{s-j}}{2(t+k_1)^{s-j+1}}, & s\geq j,\\ 0, & s<j,\end{cases} \tag{65}$$

where $b_1(t)$ is given by Eq. (57). In this case, we determine that B commutes with any element $\mathcal{G}$ of $\tilde{G}_N$, the double Wronskian entries ϕ and ψ can still take the forms given in Eqs. (62)–(64), and ϕ_1^D and ψ_1^D are provided by Eq. (58). We note that solutions corresponding to B having N same eigenvalues can be called Jordan block solutions, which can also be derived from soliton solutions through certain limit procedures.[39,40]

The so-called mixed solution corresponds to the choice of B in the form

$$B=\mathrm{Diag}(D_s[b_1(t),b_2(t),\ldots,b_s(t)],J_{h_1}[b_{s+1}(t)],J_{h_2}[b_{s+2}(t)]), \tag{66}$$

where $s+\sum_{j=1}^2 h_j=N$, and to avoid a trivial case we always let $b_j(t)$ be distinct. Taking the Wronskian entry vectors as

$$\phi = \Big(\phi_1^D, \phi_2^D, \ldots, \phi_s^D, (\Phi_{h_1}[b_{s+1}(t)])^{\mathrm{T}}, (\Phi_{h_2}[b_{s+2}(t)])^{\mathrm{T}};$$
$$-(\psi_1^D)^*, -(\psi_2^D)^*, \ldots, -(\psi_s^D)^*, -(\Psi_{h_1}^*[b_{s+1}(t)])^{\mathrm{T}}, -(\Psi_{h_2}^*[b_{s+2}(t)])^{\mathrm{T}}\Big)^{\mathrm{T}},$$
$$\psi = \Big(\psi_1^D, \ldots, \psi_s^D, (\Psi_{h_1}[k_{s+1}])^{\mathrm{T}}, (\Psi_{h_2}[k_{s+2}])^{\mathrm{T}}; (\phi_1^D)^*, \ldots, (\phi_s^D)^*, (\Phi_{h_1}^*[k_{s+1}])^{\mathrm{T}}, (\Phi_{h_2}^*[k_{s+2}])^{\mathrm{T}}\Big)^{\mathrm{T}}.$$

The corresponding solution can be described as intersections between solutions and Jordan block solutions. Any mixed solution can be obtained from a soliton solution through a certain local limit procedure.

Now we have listed out soliton, Jordan block, and mixed solutions for the NNLSEs. Compared with the results given in Ref. [7] for the NLSE, and taking into account the gauge transformations in Eq. (14) and Eq. (18), the Wronskian solutions to the NNLSE-I and NNLSE-II are well consistent with the one to the NLSE.

4. Dynamics of the NNLSEs

In this section, we present an analysis of the dynamics of $|q_j|^2$, $j = 1,2,3$. We present nonisospectral effects resulting from the different choices of the time-dependent parameters η_j, through considering changes in the top trace and shape of the 1-soliton solution, and changes in wave energy. For comparison, first let us recall some solution results about the NLSE. Their typical characteristics are that for the 1-soliton solution, the amplitude and velocity are governed by the real and imaginary parts of the parameter, and for two parallel solitons there exists periodic interactions. The NLSE equation (17) has the bilinear form[3]

$$(\mathrm{i}D_T + D_X^2)g.f = 0,$$
$$D_X^2\, f.f = gg^*,$$

using the transformation $q = g/f$. Here, f and g have double Wronskian forms (40) obeying

$$\phi_X = A\phi,\ \psi_X = -A\psi,$$
$$\phi_T = -2\mathrm{i}\phi_{XX},\ \psi_T = 2\mathrm{i}\psi_{XX},$$

where A is defined in Eq. (47). Soliton solutions can be obtained by taking

$$\phi_j = a_j^-\, \mathrm{e}^{-\xi_j},\ \ \psi_j = a_j^+\, \mathrm{e}^{\xi_j},\ \ \phi_{N+j} = -\psi_j^*,$$
$$\psi_{N+j} = \phi_j^*,\ \ j = 1,2,\ldots,N, \tag{67}$$

with

$$\xi_j = k_j X + 2\mathrm{i}k_j^2 T + \xi_j^{(0)}, \tag{68}$$

where both k_j and $\xi_j^{(0)}$ are complex constants. The 1-soliton solution of the NLSE equation (17) is

$$|q|^2 = 4a_1^2\, \mathrm{sech}^2\big[2a_1\big(X - 4b_1T + h_1\big)\big], \tag{69}$$

where and in the following discussion, we take $a_j^{\pm} = 1$, $k_j = a_j + \mathrm{i}b_j$, $\xi_j^{(0)} = c_j + \mathrm{i}d_j$ and $h_j = c_j/a_j$ for convenience. This is a soliton traveling with a fixed amplitude $4a_1^2$, constant velocity $4b_1$, and top trace

$$X = 4b_1T - h_1.$$

We observe that a stationary soliton appears when $b_1 = 0$. When two solitons travel with the same velocity, *i.e.*, $b_1 = b_2 = b$, periodic interactions exist. In fact, the solution can be written as

$$|q(T, 4bT - h_1)|^2 = \frac{4(a_1^2 - a_2^2)^2 \cosh[2a_2(h_1 - h_2)]}{(a_1^2 + a_2^2)\cosh[2a_2(h_1 - h_2)] - 2a_1a_2\cos[2(d_1 - d_2) + 4(a_1^2 - a_2^2)T]}$$
$$+ \frac{4(a_1^2 - a_2^2)^2 a_2^2\left\{1 - \cosh^2[2a_2(h_1 - h_2)]\right\}}{\left\{(a_1^2 + a_2^2)\cosh[2a_2(h_1 - h_2)] - 2a_1a_2\cos[2(d_1 - d_2) + 4(a_1^2 - a_2^2)T]\right\}^2}. \tag{70}$$

Following the line $X = 4bT - h_1$, the value $|q|^2$ is periodic with the periodicity

$$P = \frac{\pi}{2|a_1^2 - a_2^2|}. \tag{71}$$

4.1. Dynamics for the NNLSE-I

Using the transformation equation (14) and the 1-soliton solution equation (69) of the NLSE, we get the 1-soliton solution for the NNLSE-I as follows:[20]

$$|q_1|^2 = 4a_1^2\, \mathrm{sech}^2\left[2a_1\left(x - 4b_1t + \alpha t^2 + \bar{h}_1\right)\right]. \tag{72}$$

Apparently, this can also be obtained from $|q_1|^2 = gg^*/f^2$ where

$$f = \begin{vmatrix} \mathrm{e}^{-\theta_1} & \mathrm{e}^{\theta_1} \\ -\mathrm{e}^{\theta_1^*} & \mathrm{e}^{-\theta_1^*} \end{vmatrix},\quad g = 2\begin{vmatrix} \mathrm{e}^{\theta_1} & \partial_x\mathrm{e}^{\theta_1} \\ \mathrm{e}^{-\theta_1^*} & \partial_x\mathrm{e}^{-\theta_1^*} \end{vmatrix}$$

and θ_1 is defined as in Eq. (53). Here for convenience we assume $a_j^{\pm} = 1$, $k_j = a_j + \mathrm{i}b_j$, but $\theta_j^{(0)} = \bar{c}_j + \mathrm{i}\bar{d}_j$ and $\bar{h}_j = \bar{c}_j/a_j$.

Obviously, the transformation equation (14) or the time-varying spectral parameter η for the NNLSE-I does not change the shape of the soliton, compared with Eq. (69), but changes its velocity. The top trace for Eq. (72) is no longer a straight line but follows a parabola, *i.e.*,

$$x = 4b_1 t - \alpha t^2 - \bar{h}_1, \tag{73}$$

with vertex $(t,x) = (2b_1/\alpha, 4b_1^2/\alpha - \bar{h}_1)$, where $\bar{h}_1$ is the initial position. At the vertex, the velocity of the 1-soliton solution changes sign. We cannot get stationary solitons in this case. Besides the shape, energy for the soliton is not changed. This can be understood from the transformation (14) and $\rho_{1,1}$ in Eq. (33) which show that $\int_{-\infty}^{+\infty} |q_1|^2 \mathrm{d}x = \int_{-\infty}^{+\infty} |q|^2 \mathrm{d}X$. Figure 1(a) shows a soliton moving along a parabola, while figure 1(b) shows the scattering of two solitons given by

$$\begin{aligned} &|q_1|^2 = \frac{gg^*}{f^2}, \quad f = |\phi,\ \partial_x\phi;\ \psi,\ \partial_x\psi|, \\ &g = 2|\phi;\ \psi,\ \partial_x\psi,\ \partial_x^2\psi| \end{aligned} \tag{74}$$

with

$$\begin{aligned} \phi &= (\mathrm{e}^{-\theta_1}, \mathrm{e}^{-\theta_2}; -\mathrm{e}^{\theta_1^*}, -\mathrm{e}^{\theta_2^*})^{\mathrm{T}}, \\ \psi &= (\mathrm{e}^{\theta_1}, \mathrm{e}^{\theta_2}; \mathrm{e}^{-\theta_1^*}, \mathrm{e}^{-\theta_2^*})^{\mathrm{T}}, \end{aligned} \tag{75}$$

where θ_j is defined as in Eq. (53).

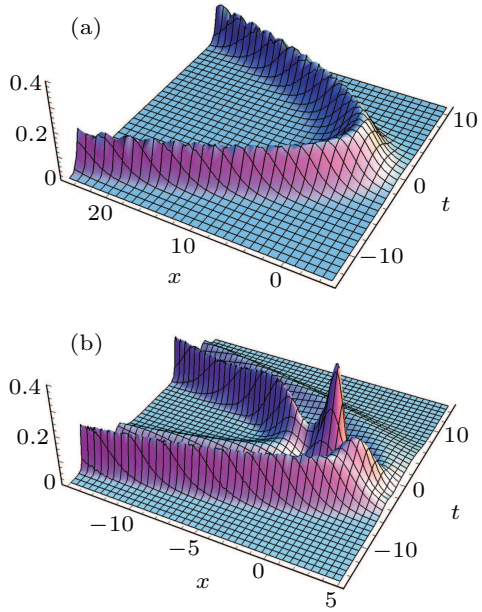


Fig. 1. Shape and motion of 1-soliton and 2-soliton solutions of the NNLSE-I: (a) a soliton given by Eq. (72) for $k_1 = 0.2$, $\theta_1^{(0)} = 0$, and $\alpha = 0.2$. (b) Scattering of two solitons given by Eqs. (74) and (75) for $k_1 = 0.3 - 0.3\mathrm{i}$, $k_2 = 0.15 + 0.3\mathrm{i}$, $\theta_1^{(0)} = \theta_2^{(0)} = 0$, and $\alpha = 0.2$.

In addition, noticing the coordinate transformation $X = x + \alpha t^2$, $T = t$ in Eq. (14), for the NNLSE-I, when two solitons have the same velocity, *i.e.*, $b_1 = b_2 = b$, the value of $|q_1|^2$ on the parabola $x = 4bt - \alpha t^2 - \bar{h}_1$ has the same expression as in Eq. (70); therefore, in this case two solitons interact periodically with the period given in Eq. (71). Figure 2 shows the periodic interactions of two solitons with same top trace $x(t) = -0.04t^2$.

For Jordan block solution of the NNLSE-I, we consider $N = 2$ and $\mathcal{G}, \mathcal{H}$ as unit matrices in Eqs. (62)–(64), *i.e.*,

$$\begin{aligned} \phi &= (\mathcal{Q}_{1,0}^{-}, \mathcal{Q}_{1,1}^{-}; -\mathcal{Q}_{1,0}^{+*}, -\mathcal{Q}_{1,1}^{+*})^{\mathrm{T}}, \\ \psi &= (\mathcal{Q}_{1,0}^{+}, \mathcal{Q}_{1,1}^{+}; \mathcal{Q}_{1,0}^{-*}, \mathcal{Q}_{1,1}^{-*})^{\mathrm{T}}, \end{aligned} \tag{76}$$

where

$$\begin{aligned} &\mathcal{Q}_{1,0}^{-} = \mathrm{e}^{-\theta_1}, \quad \mathcal{Q}_{1,1}^{-} = -(x + 4\mathrm{i}k_1 t + \alpha t^2)\mathrm{e}^{-\theta_1}; \\ &\mathcal{Q}_{1,0}^{+} = \mathrm{e}^{\theta_1}, \quad \mathcal{Q}_{1,1}^{+} = (x + 4\mathrm{i}k_1 t + \alpha t^2)\mathrm{e}^{\theta_1}, \end{aligned} \tag{77}$$

where θ_1 is defined as in Eq. (53).

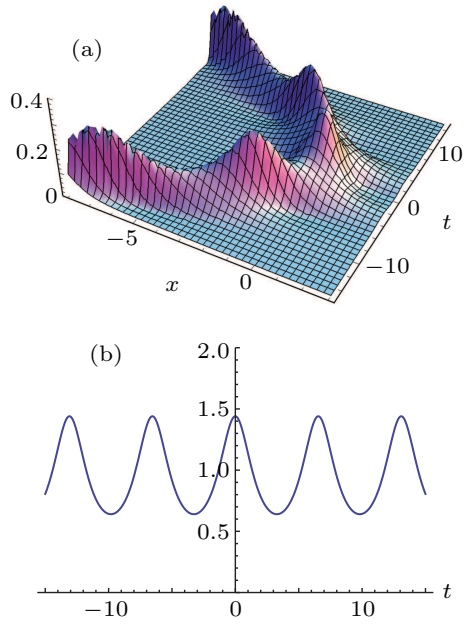


Fig. 2. Periodic interactions of two solitons of the NNLSE-I: (a) Periodic interactions of two solitons given by Eqs. (74) and (75) for $k_1 = 0.1$, $k_2 = 0.5$, $\theta_1^{(0)} = \theta_2^{(0)} = 0$, and $\alpha = 0.04$. (b) The plot of $|q_1|^2$ on the communal top trace $x(t) = -0.04t^2$.

The above-described solution can also be seen as a limit result of two soliton solutions under the procedure $\theta_2 \to \theta_1$ due to $k_2 \to k_1$. This indicates that in the Jordan block solution the periodic interactions disappear and the only interaction takes place in the area near the vertex of the parabola, as shown in Fig. 3.

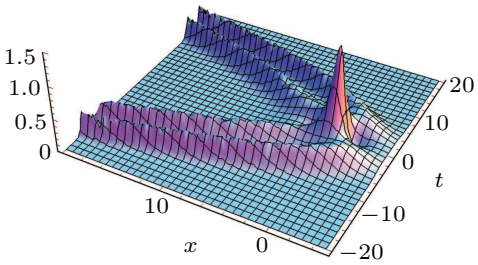


Fig. 3. Shape and motion for a Jordan block solution to the NNLSE-I given by Eqs. (74), (76), and (77) for $k_1 = 0.3$, $\theta_1^{(0)} = 0$, and $\alpha = 0.1$.

With regard to typical dynamics of the NLSE, in addition to amplitude–velocity independence and periodic interactions, the NLSE admits rogue waves (see Refs. [16], [41], and [42]

for general formulas). For the rogue waves of the NNLSE-I, we have the following.

Corollary 1 The gauge transformation equation (14) allows us to construct rogue wave solutions of NNLSE-I equation (1) from those of the NLSE (*e.g.*, Refs. [16], [41], and [42]).

Of course, the same rogue wave solutions for the NNLSE-I can also be derived using the Darboux transformation.[43]

4.2. Dynamics for the NNLSE-II

One soliton solution for the NNLSE-II is as follows:[21]

$$|q_2|^2 = 4a_1^2\,\mathrm{e}^{-2\beta t}\,\mathrm{sech}^2\left[2a_1\,\mathrm{e}^{-\beta t}\left(x-\frac{4b_1\sinh\beta t}{\beta}+\frac{\hat{c}_1\,\mathrm{e}^{\beta t}}{a_1}\right)\right], \tag{78}$$

which can be obtained by using Eqs. (18) and (69), or through the double Wronskian method similar to the NNLSE-I, *i.e.*, through the double Wronskians

$$f=\begin{vmatrix}\mathrm{e}^{-\zeta_1} & \mathrm{e}^{\zeta_1}\\ -\mathrm{e}^{\zeta_1^*} & \mathrm{e}^{-\zeta_1^*}\end{vmatrix} \quad\text{and}\quad g=2\begin{vmatrix}\mathrm{e}^{\zeta_1} & \partial_x\mathrm{e}^{\zeta_1}\\ \mathrm{e}^{-\zeta_1^*} & \partial_x\mathrm{e}^{-\zeta_1^*}\end{vmatrix},$$

where ζ_1 is defined by Eq. (56). In this subsection, we still take $a_j^{\pm}=1$, $k_j=a_j+ib_j$, but $\zeta_j^{(0)}=\hat{c}_j+\mathrm{i}\hat{d}_j$ and $\hat{h}_j=\hat{c}_j/a_j$ for convenience.

In what follows, we investigate nonisospectral effects on the shape, velocity, and energy of Eq. (78). Equation (78) provides a solitary wave traveling with amplitude $4a_1^2\,\mathrm{e}^{-2\beta t}$, top trace

$$x(t) = \frac{4b_1\sinh\beta t}{\beta}-\frac{\hat{c}_1\,\mathrm{e}^{\beta t}}{a_1} = \left(\frac{2b_1}{\beta}-\frac{\hat{c}_1}{a_1}\right)\mathrm{e}^{\beta t}-\frac{2b_1}{\beta}\,\mathrm{e}^{-\beta t}, \tag{79}$$

and velocity

$$x'(t)=\beta\left(\frac{2b_1}{\beta}-\frac{\hat{c}_1}{a_1}\right)\mathrm{e}^{\beta t}+2b_1\,\mathrm{e}^{-\beta t}. \tag{80}$$

The above top trace can further be described as follows: when $(2b_1/\beta-\hat{c}_1/a_1)\cdot 2b_1/\beta>0$, $x(t)$ has a shape similar to $\mathrm{sgn}[b_1/\beta]\sinh\beta t$; when $(2b_1/\beta-\hat{c}_1/a_1)\cdot 2b_1/\beta<0$, $x(t)$ has a shape similar to $-\mathrm{sgn}[b_1/\beta]\cosh\beta t$; when $2b_1/\beta=\hat{c}_1/a_1$, $x(t)=-2b_1/\beta\,\mathrm{e}^{-\beta t}$, and particularly, when $b_1=\hat{c}_1=0$, $x(t)=0$ corresponds to a stationary soliton, as shown in Fig. 4(a). Besides the amplitude and velocity, the energy of solitons is also time-varying. From Eq. (29), $I_{2,1}=\mathrm{e}^{\beta t}\int_{-\infty}^{+\infty}|q_2|^2\,\mathrm{d}x$ is a conserved quantity for the NNLSE-II, which indicates that when $\beta>0$ the energy of the wave $|q_2|^2$ decays exponentially, and in this case, the soliton equation (78) is a damped soliton. To obtain an energy-invariant wave, one has to consider $\mathrm{e}^{\beta t}|q_2|^2$ instead of $|q_2|^2$. If $|q_2|^2$ is defined as in Eq. (78) and $\beta>0$, $\mathrm{e}^{\beta t}|q_2|^2$ is a wave with invariant energy, but has a decreasing amplitude with speed $4\beta a_1^2\,\mathrm{e}^{-\beta t}$. Consequently, the wave length of $\mathrm{e}^{\beta t}|q_2|^2$ will get larger as t increases, which implies that $\mathrm{e}^{\beta t}|q_2|^2$ and $|q_2|^2$ have the same wave length characteristics. Therefore, in this case, we conclude that equation (78) provides a solitary wave traveling along the top trace equation (79) (or with velocity equation (80), with exponentially decaying energy, exponentially decaying amplitude and infinitely increasing wave width.

Figure 4(b) shows a head-on collision of two solitons given by

$$|q_2|^2=\frac{gg^*}{f^2},\quad f=|\phi,\ \partial_x\phi;\ \psi,\ \partial_x\psi|,\quad g=2|\phi;\ \psi,\ \partial_x\psi,\ \partial_x^2\psi| \tag{81}$$

with

$$\phi=(\mathrm{e}^{-\zeta_1},\mathrm{e}^{-\zeta_2};-\mathrm{e}^{\zeta_1^*},-\mathrm{e}^{\zeta_2^*})^{\mathrm{T}},\quad \psi=(\mathrm{e}^{\zeta_1},\mathrm{e}^{\zeta_2};\mathrm{e}^{-\zeta_1^*},\mathrm{e}^{-\zeta_2^*})^{\mathrm{T}}, \tag{82}$$

where ζ_j is defined as in Eq. (56).

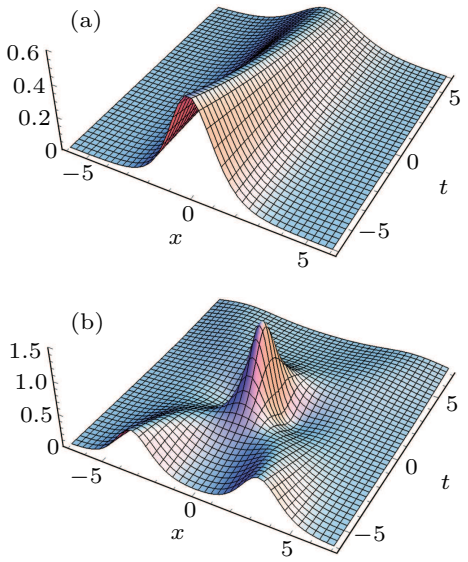


Fig. 4. Shape and motion of 1-soliton and 2-soliton solutions to the NNLSE-II: (a) A stationary soliton given by Eq. (78) for $k_1=0.3$, $\zeta_1^{(0)}=0$, and $\beta=0.05$. (b) A head-on collision of two solitons given by Eqs. (81) and (82) for $k_1=0.3-0.15\mathrm{i}$, $k_2=0.3+0.15\mathrm{i}$, $\zeta_1^{(0)}=\zeta_2^{(0)}=0$, and $\beta=0.05$.

When $b_1=b_2=b$ as well as $\hat{h}_1=\hat{h}_2=\hat{h}$ in Eq. (82), the two solitons travel with the same velocity. In this case, they also have the same top trace

$$x=\frac{4b\sinh\beta t}{\beta}-\hat{h}\,\mathrm{e}^{-\beta t}. \tag{83}$$

Using Eqs. (18) and (70), it is easy to obtain

$$\left|q_2\left(t,\frac{4b\sinh\beta t}{\beta}-\hat{h}\,\mathrm{e}^{-\beta t}\right)\right|^2=\frac{4\,\mathrm{e}^{-2\beta t}(a_1^2-a_2^2)^2}{a_1^2+a_2^2-2a_1a_2\cos[2(\hat{d}_1-\hat{d}_2)+2(a_1^2-a_2^2)(1-\mathrm{e}^{-2\beta t})/\beta]}.$$

Obviously, we cannot get an exact period. We call this quasiperiodic interactions, and give the points on Eq. (83) where $|q_2|^2$ reaches its extremums:

$$t_s=-\frac{1}{2\beta}\ln\left[1-\frac{\beta(s\pi-\hat{d}_1+\hat{d}_2)}{a_1^2-a_2^2}\right],$$

where s is some integer number. Such time-varying quasiperiodic interactions are depicted in Fig. 5.

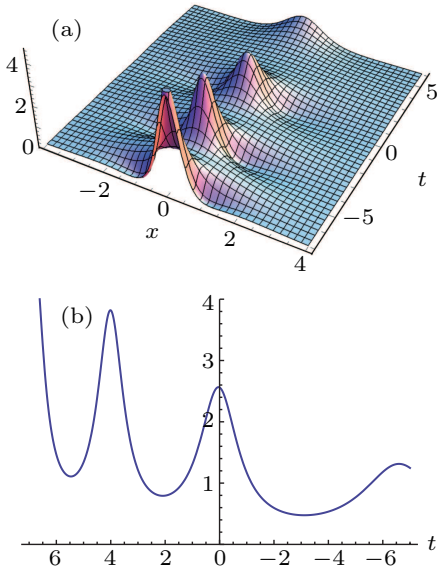


Fig. 5. Time-varying quasiperiodic interactions of two solitons of the NNLSE-II: (a) a stationary soliton given by Eq. (78) for $k_1=0.3$, $\zeta_1^{(0)}=0$, and $\beta=0.05$. (b) Plot for the value of $|q_2|^2$ on the top trace $x=0$.

For Jordan block solutions of the NNLSE-II, we still consider that $N=2$ and $\mathcal{G},\mathcal{H}$ are unit matrices in Eqs. (62)–(64), *i.e.*,

$$\begin{aligned}\phi&=(\mathcal{Q}_{1,0}^{-},\mathcal{Q}_{1,1}^{-};-\mathcal{Q}_{1,0}^{+*},-\mathcal{Q}_{1,1}^{+*})^{\mathrm{T}},\\ \psi&=(\mathcal{Q}_{1,0}^{+},\mathcal{Q}_{1,1}^{+};\mathcal{Q}_{1,0}^{-*},\mathcal{Q}_{1,1}^{-*})^{\mathrm{T}},\end{aligned}\tag{84}$$

where

$$\begin{aligned}&\mathcal{Q}_{1,0}^{-}=\mathrm{e}^{-\zeta_1},\quad \mathcal{Q}_{1,1}^{-}=-\left[x\mathrm{e}^{-\beta t}+\frac{2\mathrm{i}k_1(1-\mathrm{e}^{-2\beta t})}{\beta}\right]\mathrm{e}^{-\zeta_1},\\ &\mathcal{Q}_{1,0}^{+}=\mathrm{e}^{\zeta_1},\quad \mathcal{Q}_{1,1}^{+}=\left[x\mathrm{e}^{-\beta t}+\frac{2\mathrm{i}k_1(1-\mathrm{e}^{-2\beta t})}{\beta}\right]\mathrm{e}^{\zeta_1},\end{aligned}\tag{85}$$

and ζ_1 is defined by Eq. (56).

The solution can describe a limit result of two solitons under the procedure $\zeta_2\to\zeta_1$ due to $k_2\to k_1$. In this case, the quasiperiodic interactions disappear. However, as shown in Fig. 6, due to time-varying amplitude and wave width, the dynamics for this limit solution are completely different from those of the NLSE and NNLSE-I. Correspondingly, figure 6(b) is a contour plot of Fig. 6(a), where we assumed larger ranges for x and t. After interacting at $(t,x)=(0,0)$, the two waves disappear step by step.

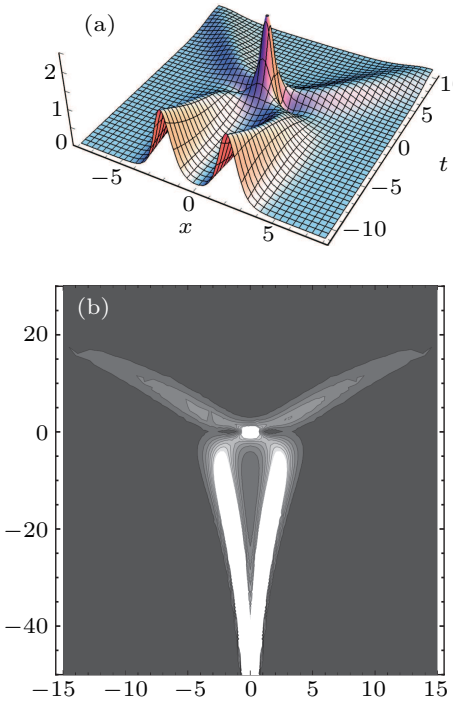


Fig. 6. Shape and motion of a Jordan block solution to the NNLSE-II: (a) the Jordan block solution given by Eq. (81) with Eqs. (84) and (85) for $k_1=0.4$, $\zeta_1^{(0)}=0$, and $\beta=0.05$. (b) A contour plot of (a) with $x\in[-15,15]$, $t\in[-50,30]$ and plot range $\{0,0.6\}$.

4.3. Rogue waves from the NNLSE-III

Rational solutions of the NLSE got great attention after optical rogue waves were found (*cf.* Ref. [44]). In this context, Peregrine[15] introduced the first-order rogue wave of NLSE, and based on this, some approaches were given to prove the nonsingularity of the solutions (*cf.* Refs. [16] and [42]). Note that mathematically rouge waves have been extended to higher dimensional systems, *e.g.*, Refs. [45] and [46].

In this part, we show how that the soliton solutions of the NNLSE-III allow us to generate solitary waves with rogue wave behavior. Let us calculate the 1-soliton solution for the NNLSE-III. We have

$$|q_3|^2=\frac{gg^*}{f^2},\quad f=\begin{vmatrix}\mathrm{e}^{-\rho_1} & \mathrm{e}^{\rho_1}\\ -\mathrm{e}^{\rho_1^*} & \mathrm{e}^{-\rho_1^*}\end{vmatrix},$$
$$g=2\begin{vmatrix}\mathrm{e}^{\rho_1} & \partial_x\mathrm{e}^{\rho_1}\\ \mathrm{e}^{-\rho_1^*} & \partial_x\mathrm{e}^{-\rho_1^*}\end{vmatrix},$$

where ρ_1 is defined as in Eq. (59); therefore, the 1-soliton solution for the NNLSE-III is

$$|q_3|^2=\left[\frac{b_1}{(a_1+t)^2+b_1^2}\right]^2$$

$$\times \operatorname{sech}^2\left[\frac{b_1 x}{(a_1+t)^2+b_1^2}+2\tilde{c}_1\right], \tag{86}$$

where we still take $a_j^{\pm}=1$, $k_j=a_j+\mathrm{i}b_j$, but $\rho_j^{(0)}=\tilde{c}_j+\mathrm{i}\tilde{d}_j$. In contrast to the NNLSE-I and NNLSE-II, here a_1 and b_1 seem to switch their roles because a_1 can be zero while b_1 cannot.

From Eq. (86) of $|q_3|^2$, and looking at the soliton part

$$\operatorname{sech}^2\left[\frac{b_1 x}{(a_1+t)^2+b_1^2}+2\tilde{c}_1\right],$$

it is obviously Gaussian-distributed with respect to x for a given t. Moreover, the part $\left[\frac{b_1}{(a_1+t)^2+b_1^2}\right]^2$ defines a time-varying amplitude with a quadratic rational function of t due to the nonisospectral effect, which tends to zero when $t\to\pm\infty$. Combining these two effects leads us to get a rogue wave from Eq. (86). This is a wave that "appears from nowhere and disappears without a trace" (*cf.* Ref. [47]) and admits an arbitrarily high amplitude. Precisely, the top trace of Eq. (86) is a parabola

$$x=-\frac{2\tilde{c}_1[(a_1+t)^2+b_1^2]}{b_1}, \tag{87}$$

if $\tilde{c}_1\neq 0$. This means that if $\tilde{c}_1\neq 0$, the soliton travels along the parabola and changes direction at its vertex $(t,x)=(-a_1,-2b_1\tilde{c}_1)$; hence, the amplitude is $1/b_1^2$ regardless of the value of $\tilde{c}_1$. Equation (87) also suggests that the soliton travels with speed

$$x'(t)=-\frac{4\tilde{c}_1(a_1+t)}{b_1},$$

and a stationary soliton only appears when $\tilde{c}_1=0$.

Figures 7(a) and 7(b) depict a stationary rogue wave and a moving rogue wave, respectively. Obviously, their wave energies are concentrated on the area near $(t,x)=(-a_1,-2b_1\tilde{c}_1)$. Figure 7(c) shows the scattering of two solitons given by

$$|q_3|^2=\frac{gg^*}{f^2},\quad f=|\phi,\ \partial_x\phi;\ \psi,\ \partial_x\psi|,$$
$$g=2|\phi;\ \psi,\ \partial_x\psi,\ \partial_x^2\psi| \tag{88}$$

with

$$\phi=\left(\mathrm{e}^{-\rho_1},\mathrm{e}^{-\rho_2};-\mathrm{e}^{\rho_1^*},-\mathrm{e}^{\rho_2^*}\right)^{\mathrm{T}},$$
$$\psi=\left(\mathrm{e}^{\rho_1},\mathrm{e}^{\rho_2};\mathrm{e}^{-\rho_1^*},\mathrm{e}^{-\rho_2^*}\right)^{\mathrm{T}}, \tag{89}$$

where ρ_j is defined by Eq. (59).

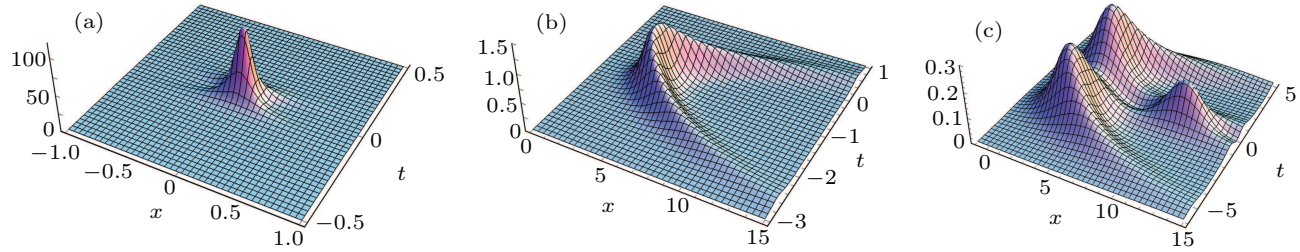


Fig. 7. Shape and motion of 1-soliton and 2-soliton solutions to the NNLSE-III: (a) a stationary rogue wave given by Eq. (86) for $k_1=0.1\mathrm{i}$ and $\rho_1^{(0)}=0$; (b) a moving rogue wave given by Eq. (86) for $k_1=1-\mathrm{i}$ and $\rho_1^{(0)}=2$; (c) scattering of two solitons given by Eqs. (88) and (89) for $k_1=2-2\mathrm{i}$, $k_2=-2-2\mathrm{i}$, and $\rho_1^{(0)}=\rho_2^{(0)}=1$.

When $\tilde{c}_1\cdot\tilde{c}_2\neq 0$ is combined with $a_1=a_2=a$ and $\tilde{c}_1/b_1=\tilde{c}_2/b_2=\tilde{\mu}$, we get two solitons traveling with the same velocity and along two parallel parabolic top traces

$$x=-2\tilde{\mu}(a+t)^2-2\tilde{\mu}b_j^2,\quad (j=1,2).$$

When $\tilde{c}_1=\tilde{c}_2=0$ we directly get two stationary solitons. In this case, the value of $|q_3|^2$ on the communal top trace $x=0$ is given by

$$|q_3(t,0)|^2=\frac{\mathcal{A}_{12}^2+\mathcal{A}_{21}^2+2\mathcal{A}_{12}\mathcal{A}_{21}\cos[2(\tilde{d}_1-\tilde{d}_2)]}{[(a_1-a_2)^2+b_1^2+b_2^2-2b_1b_2\cos[2(\tilde{d}_1-\tilde{d}_2)]][(a_1+t)^2+b_1^2]^2[(a_2+t)^2+b_2^2]^2}, \tag{90}$$

where

$$\mathcal{A}_{ij}=b_i[(a_j+t)^2+b_j^2][(a_i-a_j)^2+b_i^2+b_j^2]-2b_ib_j^2[(a_i+t)^2+b_i^2].$$

Therefore, there exists no periodic interaction for two stationary solitons of the NNLSE-III. Particularly, if $\tilde{d}_1=\tilde{d}_2=0$, equation (90) reduces to

$$|q(t,0)|^2=\frac{\{b_1[(a_2+t)^2+b_2^2]+b_2[(a_1+t)^2+b_1^2]\}^2}{[(a_1+t)^2+b_1^2]^2[(a_2+t)^2+b_2^2]^2}. \tag{91}$$

Figure 8 gives two different stationary 2-soliton interactions, which looks like a single rogue wave. In Fig. 8(b) the value of $|q_3|^2$ at the point $(t,x)=(0,0)$ is regarded as zero according

to Eq. (91).

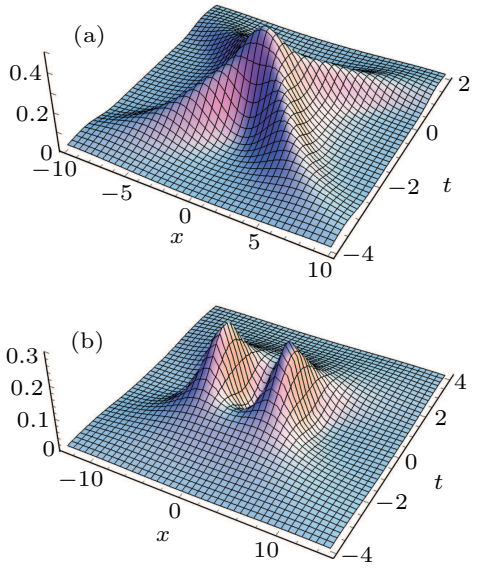


Fig. 8. Stationary 2-soliton interactions of the NNLSE-III: (a) stationary 2-soliton solution given by Eqs. (88) and (89) for $k_1 = 1+3\mathrm{i}$, $k_2 = 1-\mathrm{i}$, and $\rho_1^{(0)} = \rho_2^{(0)} = 0$; (b) stationary 2-soliton solution given by Eqs. (88) and (89) for $k_1 = -1-2\mathrm{i}$, $k_2 = 1+2\mathrm{i}$, and $\rho_1^{(0)} = \rho_2^{(0)} = 0$.

Finally, we consider the 2×2 Jordan block solution generated by

$$\begin{aligned}\phi &= (\mathcal{Q}_{1,0}^{-}, \mathcal{Q}_{1,1}^{-}; -\mathcal{Q}_{1,0}^{+*}, -\mathcal{Q}_{1,1}^{+*})^{\mathrm{T}},\\ \psi &= (\mathcal{Q}_{1,0}^{+}, \mathcal{Q}_{1,1}^{+}; \mathcal{Q}_{1,0}^{-*}, \mathcal{Q}_{1,1}^{-*})^{\mathrm{T}},\end{aligned} \tag{92}$$

where

$$\begin{aligned}\mathcal{Q}_{1,0}^{-} &= \mathrm{e}^{-\rho_1}, \quad \mathcal{Q}_{1,1}^{-} = \frac{\mathrm{i}}{2(t+k_1)^2}\mathrm{e}^{-\rho_1},\\ \mathcal{Q}_{1,0}^{+} &= \mathrm{e}^{\rho_1}, \quad \mathcal{Q}_{1,1}^{+} = \frac{-\mathrm{i}}{2(t+k_1)^2}\mathrm{e}^{\rho_1},\end{aligned} \tag{93}$$

and ρ_1 is defined as in Eq. (59).

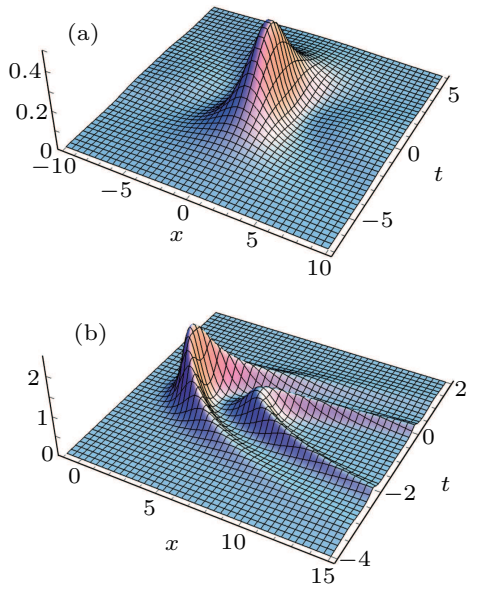


Fig. 9. Shape and motion of Jordan block solutions to the NNLSE-III: (a) a stationary generalized soliton given by Eqs. (88), (89), and (93) for $k_1 = 3\mathrm{i}$ and $\rho_1^{(0)} = 0$; (b) a moving generalized soliton given by Eqs. (88), (89), and (93) for $k_1 = 1-\mathrm{i}$ and $\rho_1^{(0)} = 2$.

Such solutions describe a limit result of two solitons in which $\rho_2 \to \rho_1$ due to $k_2 \to k_1$. When $c_1 = 0$, we get a stationary limit solution as shown in Fig. 9(a), like a rogue wave. When $c_1 \neq 0$, we get a limit case of two solitons moving along the same parabolic top trace equation (87), as shown in Fig. 9(b).

5. Conclusion

In this work, we presented the Lax representations of three different NNLSEs, and in particular, the gauge transformations between the NLSE, NNLSE-I, and NNLSE-II. The solutions and infinitely many conserved quantities for the NNLSE-I and NNLSE-II were provided from the known results of the NLSE by means of gauge transformations. We also derived solutions in double Wronskian forms for the three NNLSEs. Compared with the solitons of the NLSE, we investigated the dynamics of solitons of the three NNLSEs and studied the nonisospectral effects resulting from time-varying spectral parameters. For the NNLSE-I, the time dependency $\eta_t = \mathrm{i}\alpha/2$ does not change the shape and energy of solitons, but changes their velocities. It should be noted that there still exists periodic interactions in some 2-soliton solutions. For the NNLSE-II, the time dependency $\eta_t = -\beta\eta$ changes the shape, velocity, and energy of solitons. If $\beta > 0$, $|q_2|^2$ provides a damped solution, and there exist only quasiperiodic interactions in some 2-soliton solutions. For the NNLSE-III, the time dependency $\eta_t = -2\mathrm{i}\eta^2$ forces the wave energy of the 1-soliton solution equation (86) to be concentrated in a small area, and no periodic interaction exists between two stationary solitons. We also discussed the behaviors of Jordan block solutions, *i.e.*, limit solutions, to the three NNLSEs.

Finally, we end the paper with a comment on nonisospectral effects on rogue waves. It is known that, for many soliton equations, the amplitude of a soliton is governed by a spectral parameter that is constant in isospectral cases and time-dependent in nonisospectral cases. With a suitable time dependency of the spectral parameter, it is possible to get a soliton with a time-dependent amplitude that decays when $t \to \pm\infty$. This leads to a rogue wave that "appears from nowhere and disappears without a trace". Excluding solitons of the NNLSE-III, there exists more examples to illustrate the above mechanism, such as solitons of the nonisospectral Mavakov system[48] and lump solutions of the nonisospectral Kadomtsev–Petviashvili (I) equation.[49]

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