ISSN 1674-1056
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Citation: Chin. Phys. B . 2019, 28(1): 018104. **doi:** 10.1088/1674-1056/28/1/018104
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Influence of wall friction on granular column*

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(Received 28 September 2018; revised manuscript received 29 October 2018; published online 12 December 2018)

Granular packings under gravity in frictional and frictionless silos were simulated and the influence of the wall friction on the normal force distribution was investigated. Although there is an obvious Janssen effect in frictional silos, only a slight influence on the geometry of packing was found. The law of normal force distribution is different for frictional and frictionless walls, which is related to the pressure profile. A modified formula with consideration of the pressure profile was well fitted to the simulation results.

Keywords: Janssen effect, force distribution, wall friction, packing configuration

PACS: 81.05.Rm **DOI:** 10.1088/1674-1056/28/1/018104

1. Introduction

Granular packing is very important for the mining industry and civil engineering. It is a suitable model to study about how the microscopic randomness and nonlinear contacts influence the macroscopic properties. Although the ideal conditions, such as periodic boundary or weightlessness, are appropriate to be achieved in simulations, the influence of gravity and boundary effect are very crucial and also difficult to be removed in experiments and engineering. In Dorbolo *et al.*'s experiment, the gravity was changed by a rotation tumbler and the results showed that Beverloo scaling law is still valid.[1] In the last few decades, the influence of sidewalls on granular systems has been studied and the results have shown that the wall friction plays a significant role in granular packing and flowing.[2–4]

A universal feature of the confined granular packing is that there is a saturation of vertical stress with depth, which was observed by Hagen[5] in 1852 and explained by Janssen[6] in 1895. The explanation contains two a priori assumptions: the ratio of vertical stress to horizontal stress is a constant and the friction on the wall is fully mobilized.[7,8] The general formula of the profile of vertical stress is $\sigma_{zz}(z) = \rho g l\left[1-\exp\left(-\frac{z_0-z}{l}\right)\right]$, where the decay length $l = R/(2K\mu_s)$, μ_s is the static friction coefficient, $K = \sigma_{rr}/\sigma_{zz}$, and R is the diameter of the silo. This phenomenon will disappear and the profile will change into a linear one if the wall is frictionless or unfixed,[9,10] which shows the similarity with the fluid. Another cause of the discrepancy with the Janssen profile is the overload.[11] This 'overshoot effect' might be explained by anisotropy induced by the extra load.[12] It should be noted that in Cambau *et al.*'s experiment, a slight decrease of σ_{rz} on the sidewall close to the bottom was observed,[13] which can be related with the peak of σ_{zz} in the center.[14] Besides the static packing, during the discharging from a silo, there is still a dynamical Janssen profile for vertical stress,[7,10] which is less than the Janssen profile in static packing.[10] In Mehandia *et al.*'s experiment, it was found that the Janssen profile exists in a shearing flow.[15]

The normal force distribution is considered to be an important statistical characteristic in the granular system. In the last 20 years, the normal force distributions in granular packing and flows have been broadly investigated by both numerical simulations and experimental researches.[16–19] For a packing without external compression, Lovoll *et al.* found that the distribution function is consistent with a power law with an exponent 0.3 for small forces and an exponent decay with a constant 1.8 for larger forces.[20] In the static compressed system, there is a peak below the mean force with an exponential tail for forces larger than the mean force.[21] Meanwhile, van Eerd *et al.* simulated the packing of frictionless granules and fitted the parameters of the exponential tail in their force distribution. They suggested the tail as $P_f \propto \exp(-bf^{\alpha})$, with $\alpha \approx 2.0$ for two-dimensional (2D) systems and $\alpha \approx 1.7$ for three-dimensional (3D) systems.[22] The value of α in 3D packings is consistent with some previous results (see Table 1 for details).[23,24] Besides normal force, tangential force distribution in 2D packing also has an exponential tail.[25] The bimodal character of force distribution is observed in experiments employing dynamic loading and slow shear.[26,27]

Several formulas of the normal force distributions have been developed from empirical data or theoretical models (see Table 1). Moreover, many influential factors of this distribu-

*Project supported by the National Natural Science Foundation of China (Grant Nos. 11705256, 11772095, and 11572091) and the National Key Research and Development Program of China (Grant No. 2016YFB0700103).

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tion have been proposed. For example, Makse *et al.* simulated strongly compressed random packing of grains and the results showed that the shape of the force distribution changes when different stresses are applied.[28] The degree of deformation of an individual grain can also play a key role in determining the form of the distribution.[29] Blair *et al.* measured force distributions in granular packing and suggested that the distribution does not depend strongly on the packing structure, as well as on the inter-particle friction.[21] For shear flows, Estrada *et al.* tested the effects of sliding and rolling frictions using disks and found that these frictions have obvious influences on the force distribution.[30] The simulation results obtained by Silbert *et al.* indicated that the contact models have little influence on the force distribution except at large forces.[31]

Although several fitting formulas exist, we wonder if they are still valid when the packing is confined and under gravity. In this paper, we simulated packings of 200000 spheres in a 3D cylinder silo. The effect of wall friction on the force distribution was investigated. It was found that the laws of force distribution vary with different wall frictions and the discrepancy is related with the Janssen effect.

Table 1. Literature review of the formulas of force distribution.

Author	Formula	Ref.
Radjai *et al.*	$P_f \propto \begin{cases} f^{-\alpha}, & f < f_c, \\ \mathrm{e}^{-\beta f}, & f > f_c, \end{cases}$	[32]
Coppersmith *et al.*	$P_f = (N^N/(N-1)!) f^{N-1}\mathrm{e}^{-Nf}$	[33]
Mueth *et al.*	$P_f = a(1-b\mathrm{e}^{-f^2})\mathrm{e}^{-\beta f}$	[34]
Corwin *et al.*	$P_f = \alpha\left(1+\frac{f^{2/3}\langle\Delta\rangle}{d}\right)^2 f^{-1/3}\mathrm{e}^{-(\beta/\beta_0)f^{5/3}}$	[23]
Ngan	$P_f = a\mathrm{e}^{-\beta f^{5/3}+\gamma f}$	[24]
Metzger	$P_f = \left(\frac{\pi a}{2}\right)\mathrm{e}^{-\beta f}\sum_{n=0}^{2} b_n\langle F\rangle^n f^n$	[35]
Chan and Ngan	$P_F = a\mathrm{e}^{-\beta(\hat{f}^{5/3}-\gamma\hat{f}^{2/3})}$	[36]

Note: $f = F/\langle F\rangle$, $\langle F\rangle$ is the average normal force; N is the number of neighboring particles; $\langle\Delta\rangle$ is the average deformation of a bead; d is the bead diameter; $1/\beta_0$ is a temperature scale; $\hat{f} = F/(\int F^{2/3}P_F\,\mathrm{d}F)^{3/2}$; a, b, α, β, γ are fitting constants.

2. Method

The discrete element method (DEM)[37] has been widely accepted to be an effective method to solve problems of granular materials including static packings and flows.[38–41] DEM simulation includes calculation of contact forces among particles and solving the motion equation. Simulations of this work were performed on GPUs by the code developed with the DEM, in which interactions between two spheres were given by the Hertz–Mindlin model[42,43] and the integration of the motional equation was carried out using velocity-verlet scheme.[44]

According to the Hertz–Mindlin model, the normal and tangential forces between two contacting mono-sized spheres are

$$\begin{cases} \boldsymbol{F}_{ijn} = \frac{4}{3}E\sqrt{2r\delta_{ijn}}\boldsymbol{\delta}_{ijn} - 2\sqrt{\frac{5}{6}}\beta\sqrt{2E\sqrt{\frac{r}{2}\delta_{ijn}}m\boldsymbol{v}_{ijn}}, \\ \boldsymbol{F}_{ijt} = 4G\sqrt{2r\delta_{ijn}}\boldsymbol{\delta}_{ijt} - 2\sqrt{\frac{5}{6}}\beta\sqrt{2G\sqrt{\frac{r}{2}\delta_{ijn}}m\boldsymbol{v}_{ijt}}, \end{cases} \tag{1}$$

where G is the shear modulus, E is the Young's Modulus, and r and m are the radius and mass of the spheres, respectively. $\boldsymbol{\delta}_{ijn}$ and $\boldsymbol{\delta}_{ijt}$ are normal and tangential displacement vectors, and δ_{ijn} and δ_{ijt} are their modules, respectively. $\boldsymbol{v}_{ijn}$ and $\boldsymbol{v}_{ijt}$ are normal and tangential relative velocities between spheres i and j. $\beta = \ln(e)/\sqrt{\ln^2(e)+\pi^2}$, which is related to restitutional coefficient e between the spheres. Considering sliding frictions, the Coulomb yield criterion $|\boldsymbol{F}_{ijt}| \le |\mu_s\boldsymbol{F}_{ijn}|$ is satisfied by truncating the magnitude of $\boldsymbol{F}_{ijt}$. As a result, if $|\boldsymbol{F}_{ijt}| > |\mu_s F_{ijn}|$, $|\boldsymbol{F}_{ijt}| = |\boldsymbol{F}_{ijn}|\boldsymbol{u}_{ij}/|\boldsymbol{u}_{ij}|$[37] and the direction of $\boldsymbol{F}_{ijt}$ stays the same. $\boldsymbol{F}_{ijt}$ is the friction between two contacting particles i and j, and μ_s is the sliding friction coefficient.

In a gravity field, the equations of motion for the spheres are

$$\begin{cases} m_i\boldsymbol{a}_i = \sum_j(\boldsymbol{F}_{ijn}+\boldsymbol{F}_{ijt}) + m_i\boldsymbol{g}, \\ I_i\dot{\boldsymbol{\omega}}_i = \sum_j\left[-\frac{r_i}{r_{ij}}\boldsymbol{r}_{ij}\times(\boldsymbol{F}_{ijn}+\boldsymbol{F}_{ijt})\right]. \end{cases} \tag{2}$$

In this work, 200000 spherical particles were randomly inserted into the space within the silo with a bottom diameter 0.012 m and the initial volume fraction was about 0.2. The spheres then fell freely and the packing process did not stop until the kinetic energy of the whole system was small enough. To accelerate the packing process, the drag force was added on each particle.[45] We then calculated the distribution of normalized normal force with various friction coefficients between particles and walls. The parameters of the spherical particles and the friction coefficients are listed in Tables 2 and 3.

To analyze the geometry of packing, Voronoi tessellation was used to measure the local volume fraction of each particle. The bond orientational orders Q_4 and Q_6 are calculated as[46]

$$Q_l = \sqrt{\frac{4\pi}{2l+1}\sum_{m=-l}^{l}|\langle Y_{l,m}(\theta(\boldsymbol{r}_l),\phi(\boldsymbol{r}_l))\rangle|^2}, \tag{3}$$

where $Y_{l,m}(\theta(\boldsymbol{r}_l)\,\phi(\boldsymbol{r}_l))$ is spherical harmonic to the vector $\boldsymbol{r}$ between couples of spheres and $\theta(\boldsymbol{r}_l)$, and $\phi(\boldsymbol{r}_l)$ are polar and azimuthal angles of $\boldsymbol{r}$. The threshold distance is chosen as $1.1d$ here.

For the preparation, the particles were inserted randomly into the space inside the silo and no overlap between any two particles was guaranteed. Then, the gravity was activated to force the particles to fall to the bottom. Here, we followed Silbert *et al.*[45] to add a viscous damping which is helpful for accelerate the packing process. This process was finished when the total kinetic energy of the particles was less than 10^{-9} J.

Table 2. Parameters of the particles in the simulation.

Quantity	Symbol	Value
Elastic modulus/GPa	E	206
Shear modulus/GPa	G	79
Bead diameter/m	d	0.001
Bead density/kg·m^{-3}	ρ	7850

Table 3. Different friction coefficients used in the simulations.

Friction coefficients	Symbol	Value
Sliding friction between particle and particle	$\mu_{\rm p}$	0.2
Sliding friction between particle and bottom	$\mu_{\rm b}$	0.2
Sliding friction between particle and wall	$\mu_{\rm w}$	0.0–0.5

3. Results and discussion

3.1. Influence of geometry

For static granular packing under gravity, the results (Figs. 1(a)–1(c)) show that both of the volume fraction and coordination number have slight discrepancies with different heights, radical distances, and wall frictions. Larger wall friction coefficient leads to denser packing (the global volume fraction changes from 0.613 when $\mu_{\rm w}=0$ to 0.605 when $\mu_{\rm w}=0.5$) but does not affect the radial distribution (see Fig. 1(d)). These results show that the influence of the wall friction is much smaller than that of the friction between particles.[47]

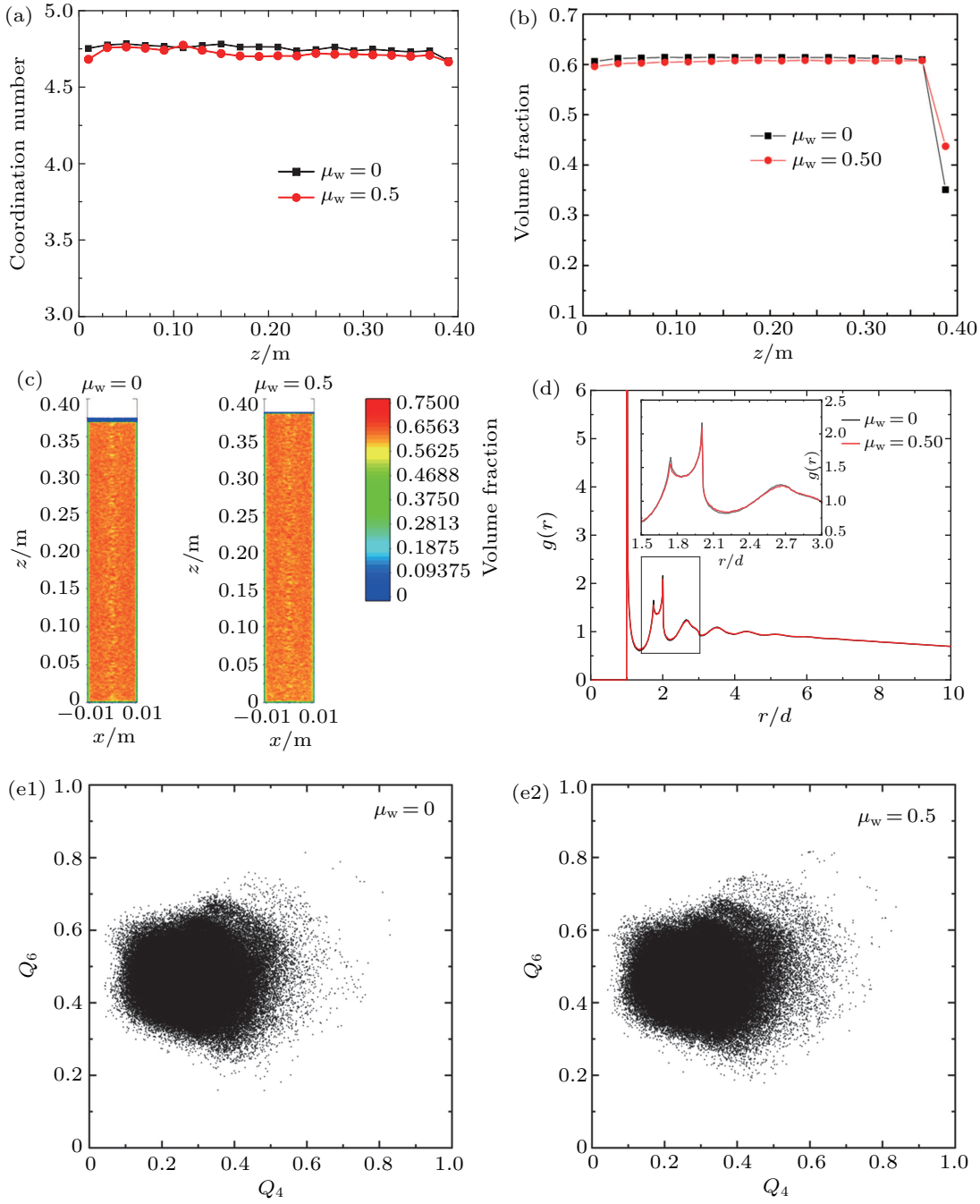


Fig. 1. (a) Average coordination number varies with height. (b) Volume fraction varies with height. (c) Distribution of volume fraction in vertical cross section. (d) The radial distribution in frictional and frictionless silos. (e) Distribution of Q_4 and Q_6 values.

We also analyze the distributions of local bond orientational orders Q_4 and Q_6 in the packing.[46] No obvious crystallization is observed in the packing. From Fig. 1(e), we find that the profiles are not influenced by the wall friction and are similar with those of sample C in Aste *et al.*'s experiments,[48] whose global volume fraction is 0.619.

3.2. Janssen effect

As introduced above, the packs of granular materials have a remarkable feature called the Janssen effect, which is the saturation of vertical stress at the bottom of the silo.[49,50] The appearance of the Janssen effect depends on the existence of the friction between grains and walls.[51] Figure 2(a) shows how the wall friction affects the profiles of the vertical stress. There is an obvious saturation value of the vertical stress, even when $\mu_w = 0.1$. It is noted that the friction coefficient μ_s in Janssen formula denotes the effective friction between particles and sidewall. It always has $\mu_s \leq \mu_w$.

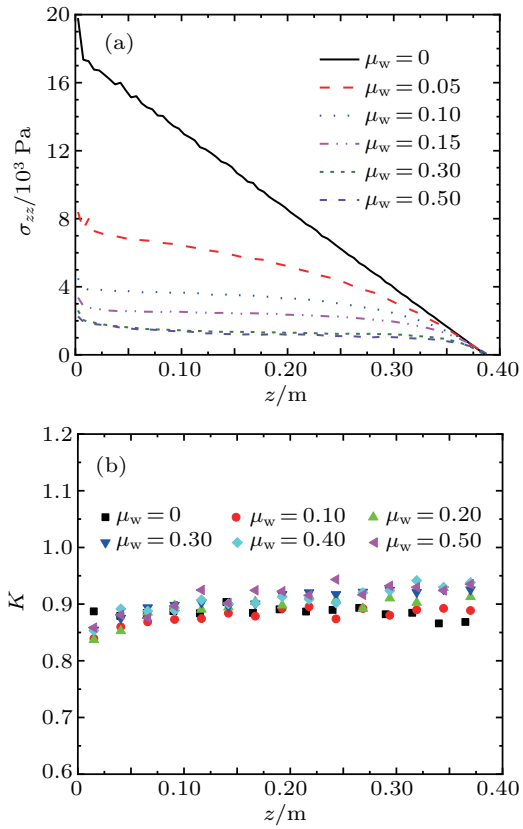


Fig. 2. (a) Diagram of Janssen effects. (b) Stress ratio K with height.

In our simulations, the stress ratio K has slight fluctuations with different heights and is not influenced by μ_w (see Fig. 2(b)). In Rankine's theory,[52] $\tan^2(45 - \theta_c/2) \leq K \leq \tan^2(45 + \theta_c/2)$, where θ_c denotes the internal friction angle of particles. In this work, $\theta_c = \tan^{-1}(\mu_p) = 11.3°$, that is, $0.67 \leq K \leq 1.48$. The value of K is about 0.9 here, which is slightly larger than the prediction value 0.804 from Jaky's simplified equation.[53] This discrepancy indicates that in the packing, the effective friction coefficient is less than μ_w.

3.3. Force distribution

We scanned wall friction coefficients μ_w from 0.0 to 0.5 and found that the distribution of normal forces is not invariant especially when the walls are frictionless enough (see Fig. 3(a)). When $\mu_w > 0.05$, the variance of the force distribution cannot be easily identified. Moreover, this figure shows that all these profiles of force distributions have the exponential tails with different slopes in the semi-logarithmic coordinate system.[22] We also calculated the angular distribution of normal force, which

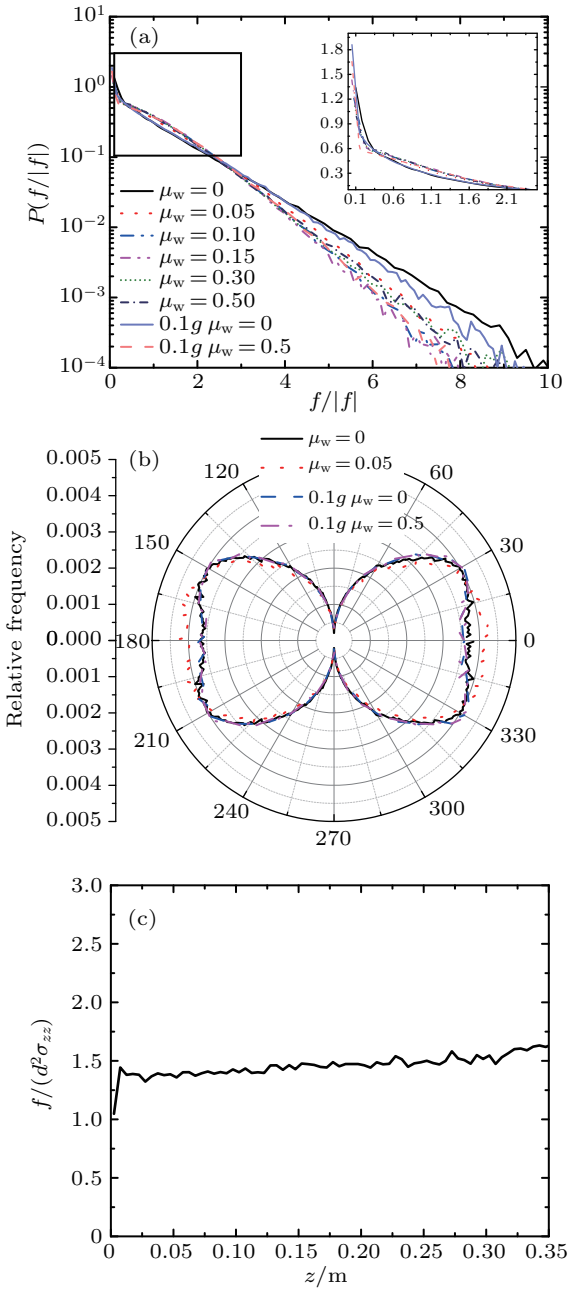


Fig. 3. (a) Force distribution in simulation results with different wall frictions. $|f|$ is the mean value of f. (b) Angular distribution of forces. (c) Dimensionless comparison of f and σ_{zz} with height.

is shown in Fig. 3(b) and there is a slight difference between $\mu_w = 0.0$ and $\mu_w = 0.5$. There are more forces in the vertical direction if the wall friction is larger, which is related with a looser packing (Fig. 1(c)). Similar discrepancy also appeared in previous study.[54]

For the normal force distribution, there are several formulas developed from previous investigations. The q model, developed by Coppersmith *et al.*, used the mean-field method to describe the force distribution in every layer (the whole space is vertically divided into layers and the thickness of the layers is equal to the diameter of the spheres) within the granular packing and considered the variances caused by gravity between the layers.[33] The formulas in most studies are more suitable to describe the granular packings when the influence of gravity is neglected. From Fig. 2(a), it is found that due to the Janssen effect, the vertical stress stays constant in most layers when $\mu_w > 0.05$. Here, it is assumed that the average normal force in every layer is proportional to the vertical stress since the volume fraction of the spheres is almost constant (see Fig. 1) and so is the coordination number.[50] Therefore, the total distribution of normal forces can be written as

$$P_{f_{\text{total}}} = \sum_{i=1}^{N} P_f(\langle F_i \rangle), \tag{4}$$

where N is the total number of layers and $\langle F_i \rangle$ is the average normal force in layer i. When $\mu_w > 0.05$, $\langle F_i \rangle$ stays constant in most layers, while there is a linear relationship between $\langle F_i \rangle$ and height z when $\mu_w = 0.0$. In order to verify our explanation, we used the formulas introduced in Table 1 as the force distribution for every layer and then added them using Eq. (4). Figure 4 shows the comparisons between the simulated and theoretical calculated results. In Fig. 4(a), the fitting curves show that formulas from Ngan, Chan and Ngan, Mueth *et al.*, Corwin *et al.*, and Radjai *et al.* describe the normal force distribution very well when $\mu_w = 0.5$ except the formula from Radjai *et al.*'s work when $f < 1$, which is perhaps due to different geometries used in this work. Moreover, from Fig. 4(b), it is found that the theoretical curves, which are calculated under our assumption provided by Eq. (4) and using the fitted parameters obtained from the case $\mu_w = 0.5$, are consistent with the simulation data when $\mu_w = 0.0$. This result supports our explanations; that is, the variances in force distributions between frictional and frictionless walls are linked to the Janssen effect. Additionally, the influence of gravity is studied by lowering the gravity acceleration to $0.1g$. It is found that the force distribution is similar with that under normal gravity acceleration except the tail (Fig. 3(a)). It is surprising that the influence of wall friction on angular distribution vanishes, which shows the complicated role of the gravity in packing process.

In statistics mechanics, the velocity distribution of the ideal gas obeys the classical Maxwell–Boltzmann distribution; that is, $f(\boldsymbol{v}) = \sqrt{\frac{m}{2\pi kT}}\, e^{-mv^2/2kT}$, where the uniform temperature T is a function of height z in the field of earth's gravity. The density of gas also varies with the height z; that is, $\rho(z) = \rho(z_0)\, e^{-mg(z-z_0)/kT}$.[55] In granular packings under gravity, the law of normal force distribution is also not uniform but relates with the pressure profile. When the gravity is not neglected, the normal force distribution within the rough silo is consistent with the uniform formulas, which is because of the Janssen effect, while the normal force distribution within the frictionless silo is not.

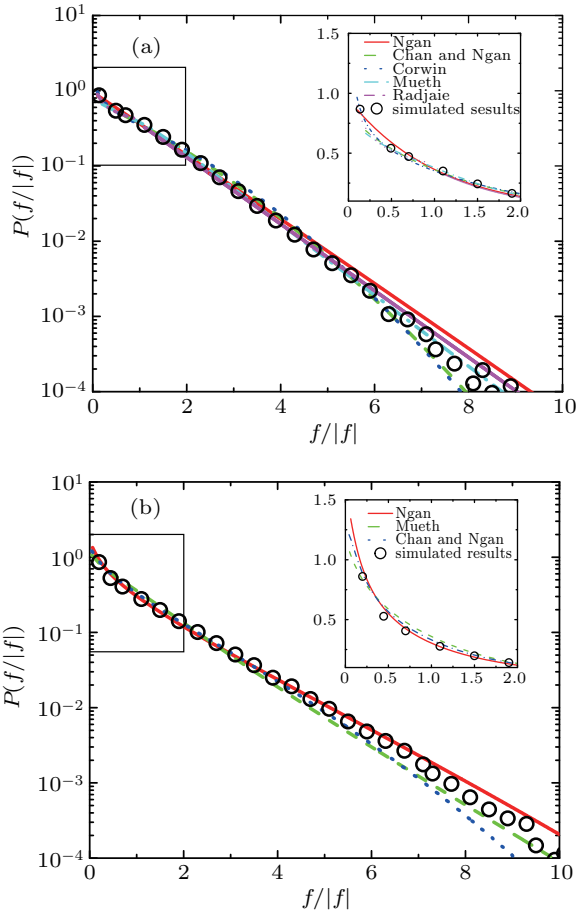


Fig. 4. (a) Simulation data is fitted with theories from Refs. [23], [24], [32], [34], and [36] In the simulations, the wall friction is set as 0.5. The enlarged graph shows the details of the fitting about normalized force from 0 to 2. (b) Simulation data is fitted with theories from Refs. [24], [34], and [36] along using Eq. (4). In the simulations, the wall friction is set as 0.0. The enlarged graph shows the details of the fitting about normalized force from 0 to 2.

4. Conclusion

This work studied the influence of wall frictions on the normal force distribution in granular packing with gravity. It was found that when the walls are frictionless, the distribution is different from that when the walls are not frictionless. This discrepancy can be explained using the Janssen effect; that is, the average normal force varies with the packing height when the walls are frictionless while it stays constant at most heights when the walls are frictional. We used the distribution function suggested in previous papers after considering

the pressure profile and the fitted results support our explanations. However, more properties need to be studied in the future, such as the tangential force distribution in packings, the force distribution in granular flows, and the angular distribution of the contact force.

Acknowledgments

We thank Prof. Xiaohui Cheng for his instructive comments. We thank Dr. Liang-wen Chen for revising the manuscript and for the discussion.

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