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$\mathscr{H}_\infty$ couple-group consensus of stochastic multi-agent systems with fixed and Markovian switching communication topologies*

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The paper addresses the issue of $\mathscr{H}_\infty$ couple-group consensus for a class of discrete-time stochastic multi-agent systems via output-feedback control. Both fixed and Markovian switching communication topologies are considered. By employing linear transformations, the closed-loop systems are converted into reduced-order systems and the $\mathscr{H}_\infty$ couple-group consensus issue under consideration is changed into a stochastic $\mathscr{H}_\infty$ control problem. New conditions for the mean-square asymptotic stability and $\mathscr{H}_\infty$ performance of the reduced-order systems are proposed. On the basis of these conditions, constructive approaches for the design of the output-feedback control protocols are developed for the fixed communication topology and the Markovian switching communication topologies, respectively. Finally, two numerical examples are given to illustrate the applicability of the present design approaches.

Keywords: multi-agent system, couple-group consensus, output feedback control, switching topologies

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1. Introduction

Generally, a multi-agent system is composed of a number of intelligent agents that communicate and interact with each other, in which each agent corresponds to an autonomous dynamic systems. Multi-agent systems can be employed to handle problems that are always difficult or even infeasible for an individual agent to figure out. Therefore, attention has increasingly been focused on the field of analysis and control design of multi-agent systems during the past few decades. In particular, considerable research activity has been dedicated to consensus study for multi-agent systems owing to its great application potential in various areas, including smart grids,[1] mobile robots,[2] sensor networks,[3] as well as multiple underwater vehicles.[4] In Ref. [5], Hong *et al.* investigated the consensus of a class of multi-agent systems with an active leader, whose state keeps changing and can be unmeasurable. It is shown in Ref. [5] that, by introducing a local control scheme, each agent is able to follow the leader asymptotically on connected undirected graphs. For directed networks, Ren considered multi-vehicle systems with bounded control inputs in Ref. [6], where both consensus algorithms and experiment platforms were developed; Olfati-Saber and Murray discussed continuous- and discrete-time multi-agent systems in Ref. [7], where several consensus conditions were proposed with the help of graph theory and Lyapunov theory for different cases, including the fixed topology without communication delays, the fixed topology with communication time delays, and the dynamical switching topologies. For recent results on the consensus of multi-agent systems, one can refer to Refs. [8]–[15] and the references therein.

To ensure the consensus of multi-agent systems, in most studies the communication topology is required to have a spanning tree. Sometimes, however, the whole topology does not meet the condition while it can be partitioned into several parts where each subgraph contains a spanning tree. In the light of this, many researchers have carried out studies on group consensus. For instance, Lu *et al.* discussed the group consensus of a family of interacting second-order integrators via pinning control in Ref. [16], where it is found that the system is able to attain group consensus by choosing pinned nodes according to the topological structure. The idea of pinning control was extended to multi-agent systems with nonlinear dynamics or general connected topology in Refs. [17] and [18]. Feng and Zheng considered a family of discrete-time agents with heterogeneous dynamics in Ref. [19], in which two algorithms were presented for ensuring the group consensus by means of matrix and graph theory. Note that references [16]–[19] are only concerned with the fixed topology case. In some applications, the communication topology among the interacting agents could change considerably over time and it is often the case that the next topology only depends on the current one. It is thus not surprising that many efforts have been put into multi-agent systems with Markovian switching communication topologies in recent years and there are reports available on various group consensus control protocols; see, e.g., Refs. [20]–[22].

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Despite significant developments in the group consensus theory of multi-agent systems with fixed and switching topologies, most studies are unconcerned with the influence of disturbances on every agent. However, the agents of a realistic multi-agent system are usually located in an environment subject to external disturbances,[23] thereby affecting the consensus behavior to a certain extent. To reduce the effects of these disturbances on the system's output to a prescribed level, Qin *et al.*[24] adopted the $\mathscr{H}_\infty$ control concept[25] for single-integrator multi-agent systems with external disturbances. It was proven in Ref. [24] that the $\mathscr{H}_\infty$ group consensus can be guaranteed provided the intra-cluster coupling strengths are large enough. In Ref. [26], Wang and Jia considered a class of nonlinear multi-agent systems with both parameter uncertainties and external disturbances, and developed sufficient conditions for ensuring the leader-following $\mathscr{H}_\infty$ group consensus by means of robust control theory. Large-scale multi-agent systems consisting of heterogeneous interconnected subsystems were discussed by Cui *et al.* in Ref. [27], where an event-triggered pinning control strategy was employed to achieve the group consensus and a pre-scheduled $\mathscr{H}_\infty$ performance simultaneously under the Markovian switching communication topologies.

It is noted that the control protocols in both Refs. [26] and [27] were grounded in the assumption that all agent states are available, while in practice full measurement of the agent state variables is often inconvenient and can even be infeasible.[28,29] Besides, the $\mathscr{H}_\infty$ group consensus results available in the literature are limited to the deterministic setting, whereas, as pointed out in Ref. [1], in a number of engineering areas including maneuvering target tracking and unmanned aerial vehicle (UAV) cooperative control, there is a need to model the dynamics of agents by stochastic differential equations. Motivated by the above analysis, this paper is concerned with the issue of $\mathscr{H}_\infty$ couple-group consensus for a class of discrete-time multi-agent systems subject to both external disturbances and additive stochastic noises. Both fixed communication topology and Markovian switching communication topologies are discussed. The purpose of the current study is to determine output-feedback control protocols so that the multi-agent system reaches couple-group consensus asymptotically in mean square and possesses a prescribed $\mathscr{H}_\infty$ performance bound. It is shown that the closed-loop systems are able to be converted into reduced-order systems via linear transformations. With the aid of the Lyapunov stability theory and the properties of mathematical expectation, new criteria for the mean-square asymptotic stability and $\mathscr{H}_\infty$ performance of the reduced-order systems are proposed. On the basis of the criteria obtained and by means of linear matrix inequality (LMI) techniques, constructive approaches for the design of the output-feedback control protocols are developed for the fixed communication topology and the Markovian switching communication topologies, respectively. Finally, two numerical examples are employed to verify the applicability of the present theoretical results.

Notations We let $\mathscr{E}$ be the expectation operator and $\|\cdot\|$ be the Euclidean norm. Define $G=(V,E)$ as a directed graph with $n+m$ nodes in which $V=\{v_1,v_2,\ldots,v_{n+m}\}$ and E stand for the node set and the edge set, respectively. The adjacency matrix associated with G is represented by $A_G=[a_{ij}]\in R^{(n+m)\times(n+m)}$, where $a_{ij}>0$ for $i\neq j$ and $a_{ij}=0$ for $i=j$. The Laplacian matrix $\mathscr{L}$ associated with A_G is defined as $\mathscr{L}=[l_{ij}]\in R^{(n+m)\times(n+m)}$, where $l_{ii}=\sum\limits_{j=1,j\neq i}^{n+m}a_{ij}$ and $l_{ij}=-a_{ij}$ for $i\neq j$. Denote by $\mathfrak{L}_\Phi=[\Phi_{ij}]\in\mathbb{R}^{n\times n}$ the symmetric matrix with $\Phi_{ii}=(n-1)/n$ and $\Phi_{ij}=-1/n$ for all $i\neq j$, by $\bar{\mathfrak{L}}_\Phi=[\bar{\Phi}_{ij}]\in\mathbb{R}^{m\times m}$ the symmetric matrix with $\bar{\Phi}_{ii}=(m-1)/m$ and $\bar{\Phi}_{ij}=-1/m$ for all $i\neq j$, by $He(X)$ the sum of a matrix X and its transpose X^{T} (i.e., $He(X)=X+X^{\mathrm{T}}$), by e_p the $p\times 1$ column vector of all ones, and by $0_{p\times q}$ the $p\times q$-dimensional zero matrix. Furthermore, we use symbol $*$ to represent the block that can be clearly inferred by symmetry.

2. Preliminaries

Consider a discrete-time stochastic multi-agent system consisting of $n+m$ agents, each with dynamic model

$$\begin{aligned} x_i(k+1)&=Ax_i(k)+B_1u_i(k)+B_2v_i(k)+\delta(x_i(k))w(k),\\ y_i(k)&=C_1x_i(k)+Dv_i(k), \end{aligned}\tag{1}$$

where A, B_1, B_2, C_1, and D are real constant matrices; $x_i(k)\in\mathbb{R}^{m_1}$, $y_i(k)\in\mathbb{R}^q$, and $u_i(k)\in\mathbb{R}^{m_2}$ are the state, the measured output, and the control input, respectively; $v_i(k)\in\mathbb{R}^{m_3}$ stands for the disturbance input that belongs to $Ł_2[0,\infty)$, where $Ł_2[0,\infty)$ is the space of square-integrable vector functions over $[0,\infty]$; $\sigma(x_i(k))w(k)$ represents the stochastic perturbation, where $\sigma(x_i(k)):\mathbb{R}^{m_1}\to\mathbb{R}^{m_1}$ is nonlinear function and $w(k)$ is a scalar Brownian motion defined on a complete probability space $(\Omega,F,\{F\}_{t\geq 0})$ with

$$\mathscr{E}\{w(k)\}=0,\ \mathscr{E}\{w(k)w(k)\}=1,\ \mathscr{E}\{w(i)w(j)\}=0,\ i\neq j.$$

To ensure that there is only one solution, function $\sigma(x_i(k)):\mathbb{R}^{m_1}\to\mathbb{R}^{m_1}$ is required to satisfy the Lipschitz condition and the linear growth condition as Ref. [30]. That is, for all $\zeta_1,\zeta_2\in\mathbb{R}^m$, there are two positive scalars α, ε, such that

$$\|\delta(\zeta_1)-\delta(\zeta_2)\|\leq\alpha\|\zeta_1-\zeta_2\|,\tag{2}$$

$$\|\delta(\zeta_1)\|^2\leq\varepsilon\big(1+\|\zeta_1\|^2\big).\tag{3}$$

Suppose that the $n+m$ agents are divided into two groups V_1 and V_2 for the couple-group consensus purpose. Then, the interaction topology $G(k)$ can be divided into two sub-groups

$G_1(k)$ and $G_2(k)$ subsequently. Without loss of generality, assume that the first n agents (i.e., v_1, v_2, …, v_n) belong to V_1 and the last m agents (i.e., v_{n+1}, v_{n+2}, …, v_{n+m}) belong to the V_2.

We can now give the definition of mean-square couple-group consensus:

Definition 1 Multi-agent system equation (1) is said to reach couple-group consensus asymptotically in mean square with given $u_i(k)$, if, when $v_i(k)=0$, the states of agents satisfy

$$\lim \mathscr{E}\{\|x(k)-\tilde{x}(k)\|^2\}=0,$$

where

$$\tilde{x}(k)=\left[\frac{e_n^{\mathrm{T}}}{n}\otimes\sum_{j=1}^{n}x_j^{\mathrm{T}}(k),\frac{e_m^{\mathrm{T}}}{m}\otimes\sum_{j=n+1}^{n+m}x_j^{\mathrm{T}}(k)\right]^{\mathrm{T}}.$$

Under the effects of external disturbances, couple-group consensus of multi-agent system Eq. (1) is often difficult to attain. For quantitatively measuring the disturbance attenuation ability of multi-agent system Eq. (1) on consensus, we introduce a controlled output function

$$\begin{aligned}z_i(k) &= \mu_{1i}C_2\left(x_i(k)-\frac{1}{n}\sum_{j=1}^{n}x_j(k)\right)\\&\quad+\mu_{2i}C_2\left(x_i(k)-\frac{1}{m}\sum_{j=n+1}^{n+m}x_j(k)\right),\end{aligned}\tag{4}$$

where $\mu_{1i}=1$, $\mu_{2i}=0$ for $i=1,2,\ldots,n$, and $\mu_{1i}=0$, $\mu_{2i}=1$ for $i=n+1,n+2,\ldots,n+m$. Then, we give the following definition regarding $\mathscr{H}_\infty$ performance.

Definition 2[31] Given a scalar $\gamma>0$, system equation (1) is said to have an $\mathscr{H}_\infty$ performance bound γ, if, under zero initial condition,

$$\|z(k)\|_{\mathscr{E}_2}\le\gamma\|v(k)\|_2\tag{5}$$

holds for any non-zero $v(k)\in Ł_2[0,\infty)$, where $\|z(k)\|_{\mathscr{E}_2}=\sqrt{\mathscr{E}\{\sum_{k=0}^{\infty}z^{\mathrm{T}}(k)z(k)\}}$, and $\|v(k)\|_2=\sqrt{\sum_{k=0}^{\infty}v^{\mathrm{T}}(k)v(k)}$.

We are now in a position to explicitly state the $\mathscr{H}_\infty$ couple-group consensus issue. That is, design an output-feedback control protocol so that discrete-time stochastic multi-agent system equation (1) achieves asymptotical consensus in mean square and has a prescribed $\mathscr{H}_\infty$ performance bound .

At the end of this section, we prepare the following lemmas that will be useful in deriving the main results.

Lemma 1[32] There are two eigenvalues in matrix $\mathfrak{L}_\Phi$. One is 1 with multiplicity $n-1$ and the other is 0 with multiplicity 1. Vectors e_n^{T} and e_n are, respectively, the left and right eigenvectors of matrix $\mathfrak{L}_\Phi$ corresponding to the zero eigenvalue. There is also an orthogonal matrix $\mathscr{U}\in\mathbb{R}^{n\times n}$ with the last column $e_n/\sqrt{n}$ such that

$$\mathscr{U}^{\mathrm{T}}\mathfrak{L}_\Phi\mathscr{U}=\mathrm{diag}\{I_{n-1},0\}.$$

Moreover, if $\mathscr{L}\in\mathbb{R}^{n\times n}$ is the Laplacian matrix of a balanced directed graph, then one has

$$\mathscr{U}^{\mathrm{T}}\mathscr{L}\mathscr{U}=\mathrm{diag}\{L_1,0\},\ \ L_1\in\mathbb{R}^{(n-1)\times(n-1)}.$$

Lemma 2[33] The following equalities:

$$\Psi\mathfrak{L}_\Phi=\Psi,\ \ \bar{\Psi}\bar{\mathfrak{L}}_\Phi=\bar{\Psi}$$

hold for any matrices $\Psi=(\psi_{ij})\in\mathbb{R}^{n\times n}$ and $\bar{\Psi}=(\bar{\psi}_{ij})\in\mathbb{R}^{m\times m}$ with $\psi_{ii}=-\sum_{j=1,j\neq i}^{n}\psi_{ij}$ and $\bar{\psi}_{ii}=-\sum_{j=1,j\neq i}^{m}\bar{\psi}_{ij}$.

Lemma 3[29] Suppose that the following condition concerning a real constant p and real matrices $\boldsymbol{\Theta}$, U_i, V_i, and W_i $(i=1,\ldots,n)$ holds:

$$\begin{bmatrix}\boldsymbol{\Theta} & U_1+pV_1 & \cdots & U_n+pV_n\\ * & \mathrm{diag}\{He(-pW_1) & \cdots & He(-pW_n)\}\end{bmatrix}<0.$$

Then one can obtain

$$\boldsymbol{\Theta}+\sum_{i=1}^{n}He(U_iW_i^{-1}V_i^{\mathrm{T}})<0.$$

3. Main results

In this section, we consider the issue of $\mathscr{H}_\infty$ couple-group consensus for multi-agent system Eq. (1) under the fixed communication topology and the Markovian switching communication topologies, respectively.

3.1. Fixed topology case

When the communication topology is fixed (i.e, $G(k)\equiv G$), the control protocol to be designed is given by

$$\begin{aligned}u_i(k) &= \mu_{1i}\mathscr{K}\sum_{j=1}^{n}a_{ij}(y_j(k)-y_i(k))\\&\quad+\mu_{2i}\mathscr{K}\sum_{j=n+1}^{n+m}a_{ij}(y_j(k)-y_i(k)),\end{aligned}\tag{6}$$

where $\mathscr{K}$ is the control gain to be solved, μ_{1i} and μ_{2i} are the same as those given in Eq. (4). The control protocol considered herein is in the form of static output feedback.[34] For dynamic output-feedback control protocol, one may refer to Refs. [35]–[37].

As Ref. [20], the two subgraphs G_1 and G_2 are assumed to satisfy a condition as follows.

Assumption 1 Each of the subgraphs G_1 and G_2 contains a spanning tree.

Denote

$$\begin{aligned}x(k)&=[x_1^{\mathrm{T}}(k)\ x_2^{\mathrm{T}}(k)\ \cdots\ x_{n+m}^{\mathrm{T}}(k)]^{\mathrm{T}}\in\mathbb{R}^{(n+m)m_1},\\u(k)&=[u_1^{\mathrm{T}}(k)\ u_2^{\mathrm{T}}(k)\ \cdots\ u_{n+m}^{\mathrm{T}}(k)]^{\mathrm{T}}\in\mathbb{R}^{(n+m)m_2},\\v(k)&=[v_1^{\mathrm{T}}(k)\ v_2^{\mathrm{T}}(k)\ \cdots\ v_{n+m}^{\mathrm{T}}(k)]^{\mathrm{T}}\in\mathbb{R}^{(n+m)m_3},\\g(x(k))&=[\sigma^{\mathrm{T}}(x_1(k))\ \sigma^{\mathrm{T}}(x_2(k))\ \cdots\ \sigma^{\mathrm{T}}(x_{n+m}(k))]^{\mathrm{T}}\in\mathbb{R}^{(n+m)m_1}.\end{aligned}$$

Then, by substituting Eq. (6) into system Eq. (1), one can get a closed-loop system as follows.

$$\begin{aligned} x(k+1) &= \begin{bmatrix} I_n\otimes A-\mathscr{L}_1\otimes(B_1\mathscr{K}C_1) & 0_{n\times m}\otimes I_{m_1} \\ 0_{m\times n}\otimes I_{m_1} & I_m\otimes A-\mathscr{L}_2\otimes(B_1\mathscr{K}C_1)\end{bmatrix}x(k) \\ &\quad+\begin{bmatrix} I_n\otimes B_2-\mathscr{L}_1\otimes(B_1\mathscr{K}D) & 0_{n\times m}\otimes I_{m_3} \\ 0_{m\times n}\otimes I_{m_3} & I_m\otimes B_2-\mathscr{L}_2(B_1\mathscr{K}D)\end{bmatrix}v(k)+I_{(n+m)m_1}g(x(k))w(k), \\ z(k) &= \left(\begin{bmatrix}\mathfrak{L}_\Phi & 0_{n\times m}\\ 0_{m\times n} & \bar{\mathfrak{L}}_\Phi\end{bmatrix}\otimes C_2\right)x(k), \end{aligned} \tag{7}$$

where

$$\mathscr{L}_1=\begin{bmatrix} l_{11} & l_{12} & \cdots & l_{1n}\\ l_{21} & l_{22} & \cdots & l_{2n}\\ \vdots & \cdots & \cdots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn}\end{bmatrix},$$

$$\mathscr{L}_2=\begin{bmatrix} l_{(n+1)(n+1)} & l_{(n+1)(n+2)} & \cdots & l_{(n+1)(n+m)}\\ l_{(n+2)(n+1)} & l_{(n+2)(n+2)} & \cdots & l_{(n+2)(n+m)}\\ \vdots & \cdots & \cdots & \vdots \\ l_{(n+m)(n+1)} & l_{(n+m)(n+2)} & \cdots & l_{(n+m)(n+m)}\end{bmatrix}.$$

Define the following linear transformations to $x(k)$, $v(k)$, and $g(x(k))$

$$\begin{aligned} \left(\begin{bmatrix}\mathscr{U}_1^{\mathrm{T}}\mathfrak{L}_\Phi & 0_{(n-1)\times m}\\ 0_{(m-1)\times n} & \bar{\mathscr{U}}_1^{\mathrm{T}}\bar{\mathfrak{L}}_\Phi\end{bmatrix}\otimes I_{m_1}\right)x(k) &= \xi(k), \\ \left(\begin{bmatrix}\mathscr{U}_1^{\mathrm{T}}\mathfrak{L}_\Phi & 0_{(n-1)\times m}\\ 0_{(m-1)\times n} & \bar{\mathscr{U}}_1^{\mathrm{T}}\bar{\mathfrak{L}}_\Phi\end{bmatrix}\otimes I_{m_3}\right)v(k) &= \theta(k), \\ \left(\begin{bmatrix}\mathscr{U}_1^{\mathrm{T}}\mathfrak{L}_\Phi & 0_{(n-1)\times m}\\ 0_{(m-1)\times n} & \bar{\mathscr{U}}_1^{\mathrm{T}}\bar{\mathfrak{L}}_\Phi\end{bmatrix}\otimes I_{m_1}\right)g(x(k)) &= f(\xi(k)). \end{aligned} \tag{8}$$

We then have

$$\begin{aligned} &\left(\begin{bmatrix}\mathfrak{L}_\Phi & 0_{n\times m}\\ 0_{m\times n} & \bar{\mathfrak{L}}_\Phi\end{bmatrix}\otimes I_{m_1}\right)x(k) \\ &=\left(\begin{bmatrix}\mathscr{U}_1 & 0_{n\times(m-1)}\\ 0_{m\times(n-1)} & \bar{\mathscr{U}}_1\end{bmatrix}\otimes I_{m_1}\right)\xi(k), \\ &\left(\begin{bmatrix}\mathfrak{L}_\Phi & 0_{n\times m}\\ 0_{m\times n} & \bar{\mathfrak{L}}_\Phi\end{bmatrix}\otimes I_{m_3}\right)v(k) \\ &=\left(\begin{bmatrix}\mathscr{U}_1 & 0_{n\times(m-1)}\\ 0_{m\times(n-1)} & \bar{\mathscr{U}}_1\end{bmatrix}\otimes I_{m_3}\right)\theta(k), \\ &\left(\begin{bmatrix}\mathfrak{L}_\Phi & 0_{n\times m}\\ 0_{m\times n} & \bar{\mathfrak{L}}_\Phi\end{bmatrix}\otimes I_{m_1}\right)g(x(k)) \\ &=\left(\begin{bmatrix}\mathscr{U}_1 & 0_{n\times(m-1)}\\ 0_{m\times(n-1)} & \bar{\mathscr{U}}_1\end{bmatrix}\otimes I_{m_1}\right)f(\xi(k)). \end{aligned} \tag{9}$$

By Lemma 1, there are orthogonal matrices $\mathscr{U}=[\mathscr{U}_1,\mathscr{U}_2]\in\mathbb{R}^{n\times n}$ with the last column $\mathscr{U}_2=e_n/\sqrt{n}$ and $\bar{\mathscr{U}}=[\bar{\mathscr{U}}_1,\bar{\mathscr{U}}_2]\in\mathbb{R}^{m\times m}$ with the last column $\bar{\mathscr{U}}_2=e_m/\sqrt{m}$, such that

$$\begin{aligned} &\mathscr{U}^{\mathrm{T}}\mathfrak{L}_\Phi\mathscr{U}=\mathrm{diag}\{I_{n-1},0\},\ \mathscr{U}^{\mathrm{T}}\mathscr{L}_1\mathscr{U}=\mathrm{diag}\{L_1,0\}, \\ &\bar{\mathscr{U}}^{\mathrm{T}}\bar{\mathfrak{L}}_\Phi\bar{\mathscr{U}}=\mathrm{diag}\{I_{m-1},0\},\ \bar{\mathscr{U}}^{\mathrm{T}}\mathscr{L}_2\bar{\mathscr{U}}=\mathrm{diag}\{L_2,0\}, \\ &\mathscr{U}_1^{\mathrm{T}}\mathscr{U}_1=I_{n-1},\ \bar{\mathscr{U}}_1^{\mathrm{T}}\bar{\mathscr{U}}_1=I_{m-1},\ \mathscr{U}_1\mathscr{U}_1^{\mathrm{T}}+\mathscr{U}_2\mathscr{U}_2^{\mathrm{T}}=I_n, \\ &\bar{\mathscr{U}}_1\bar{\mathscr{U}}_1^{\mathrm{T}}+\bar{\mathscr{U}}_2\bar{\mathscr{U}}_2^{\mathrm{T}}=I_m. \end{aligned} \tag{10}$$

In addition, from Lemma 2 one can easily write that

$$\begin{aligned} &\mathscr{L}_1\mathfrak{L}_\Phi=\mathscr{L}_1,\ \mathscr{L}_2\bar{\mathfrak{L}}_\Phi=\mathscr{L}_2, \\ &\mathfrak{L}_\Phi\mathfrak{L}_\Phi=\mathfrak{L}_\Phi,\ \bar{\mathfrak{L}}_\Phi\bar{\mathfrak{L}}_\Phi=\bar{\mathfrak{L}}_\Phi. \end{aligned} \tag{11}$$

Using Eqs. (7) and (11), the following holds true:

$$\begin{aligned} \xi(k+1) &= \left(\begin{bmatrix}\mathscr{U}_1^{\mathrm{T}}\mathfrak{L}_\Phi & 0_{(n-1)\times m}\\ 0_{(m-1)\times n} & \bar{\mathscr{U}}_1^{\mathrm{T}}\bar{\mathfrak{L}}_\Phi\end{bmatrix}\otimes I_{m_1}\right)x(k+1) \\ &=\begin{bmatrix}\Gamma_{A_1} & 0_{(n-1)\times m}\otimes I_{m_1}\\ 0_{(m-1)\times n}\otimes I_{m_1} & \Gamma_{A_2}\end{bmatrix}\left(\begin{bmatrix}\mathfrak{L}_\Phi & 0_{n\times m}\\ 0_{m\times n} & \bar{\mathfrak{L}}_\Phi\end{bmatrix}\otimes I_{m_1}\right)x(k) \\ &\quad+\begin{bmatrix}\Gamma_{B_1} & 0_{(n-1)\times m}\otimes I_{m_3}\\ 0_{(m-1)\times n}\otimes I_{m_3} & \Gamma_{B_2}\end{bmatrix}\left(\begin{bmatrix}\mathfrak{L}_\Phi & 0_{n\times m}\\ 0_{m\times n} & \bar{\mathfrak{L}}_\Phi\end{bmatrix}\otimes I_{m_3}\right)v(k) \\ &\quad+\begin{bmatrix}\mathscr{U}_1^{\mathrm{T}}\mathfrak{L}_\Phi\otimes I_{m_1} & 0_{(n-1)\times m}\otimes I_{m_1}\\ 0_{(m-1)\times n}\otimes I_{m_1} & \bar{\mathscr{U}}_1^{\mathrm{T}}\bar{\mathfrak{L}}_\Phi\otimes I_{m_1}\end{bmatrix}\begin{bmatrix}\mathfrak{L}_\Phi\otimes I_{m_1} & 0_{n\times m}\otimes I_{m_1}\\ 0_{m\times n}\otimes I_{m_1} & \bar{\mathfrak{L}}_\Phi\otimes I_{m_1}\end{bmatrix}g(x(k))w(k), \end{aligned} \tag{12}$$

where

$$\begin{aligned} \Gamma_{A_1}&=\mathscr{U}_1^{\mathrm{T}}\mathfrak{L}_\Phi\otimes A-\mathscr{U}_1^{\mathrm{T}}\mathfrak{L}_\Phi\mathscr{L}_1\otimes(B_1\mathscr{K}C_1), \\ \Gamma_{A_2}&=\bar{\mathscr{U}}_1^{\mathrm{T}}\bar{\mathfrak{L}}_\Phi\otimes A-\bar{\mathscr{U}}_1^{\mathrm{T}}\bar{\mathfrak{L}}_\Phi\mathscr{L}_2\otimes(B_1\mathscr{K}C_1), \\ \Gamma_{B_1}&=\mathscr{U}_1^{\mathrm{T}}\mathfrak{L}_\Phi\otimes B_2-\mathscr{U}_1^{\mathrm{T}}\mathfrak{L}_\Phi\mathscr{L}_1\otimes(B_1\mathscr{K}D), \\ \Gamma_{B_2}&=\bar{\mathscr{U}}_1^{\mathrm{T}}\bar{\mathfrak{L}}_\Phi\otimes B_2-\bar{\mathscr{U}}_1^{\mathrm{T}}\bar{\mathfrak{L}}_\Phi\mathscr{L}_2\otimes(B_1\mathscr{K}D). \end{aligned}$$

Now, by Eqs. (9), (12), and (10), one can convert system equation (1) into a reduced-order system, as follows:

$$\begin{aligned} \xi(k+1)&=\tilde{A}\xi(k)+\tilde{B}\theta(k)+h(\xi(k))w(k), \\ z(k)&=\tilde{\mathscr{U}}\xi(k), \end{aligned} \tag{13}$$

where

$$\begin{aligned} \tilde{A}&=I_{n+m-2}\otimes A+J\otimes B_1\mathscr{K}C_1, \\ \tilde{B}&=I_{n+m-2}\otimes B_2+J\otimes B_1\mathscr{K}D, \\ h(\xi(k))&=I_{(n+m-2)m_1}f(\xi(k)), \\ \tilde{\mathscr{U}}&=\begin{bmatrix}\mathscr{U}_1 & 0_{n\times(m-1)}\\ 0_{m\times(n-1)} & \bar{\mathscr{U}}_1\end{bmatrix}\otimes C_2, \\ J&=\begin{bmatrix}-L_1 & 0_{(n-1)\times(m-1)}\\ 0_{(m-1)\times(n-1)} & -L_2\end{bmatrix}. \end{aligned} \tag{14}$$

Remark 1 By the equalities in Eqs. (9), (11), and (10),

and noting the properties $\lambda_M(\mathfrak{L}_\Phi)=\lambda_M(\bar{\mathfrak{L}}_\Phi)=1$, we can obtain $\|x(k)-\tilde{x}(k)\|^2=|\xi(k)\|^2$, $\|\theta(k)|^2\le\|v(k)\|^2$, and write that

$$\lim_{k\to\infty}\mathscr{E}\|\xi(k)\|^2=0\Rightarrow\lim_{k\to\infty}\mathscr{E}\|x(k)-\tilde{x}(k)\|^2=0,$$
$$\|z(k)\|_{\mathscr{E}_2}\le\gamma\|\theta(k)\|_2\Rightarrow\|z(k)\|_{\mathscr{E}_2}\le\gamma\|v(k)\|_2.$$

Thus, with the help of linear transformations in Eq. (8), the $\mathscr{H}_\infty$ couple-group consensus issue of system Eq. (1) can be changed into a stochastic $\mathscr{H}_\infty$ control problem of the reduced-order system in Eq. (13); that is, to find a controller of the form in Eq. (6) so that system equation (13) is asymptotically stable in mean square and has a prescribed $\mathscr{H}_\infty$ performance bound γ.

Now, we are in a position to give our main results regarding the analysis and synthesis. The analysis result is given in the following theorem.

Theorem 1 System equation (13) is asymptotically stable in mean square and has a prescribed $\mathscr{H}_\infty$ performance bound γ provided there are a scalar $p>0$ and positive definite matrix $\mathscr{P}\in\mathbb{R}^{m_1\times m_1}$ such that

$$\mathscr{P}-pI_{m_1}<0, \tag{15}$$
$$\begin{bmatrix}\Upsilon+\Pi & \tilde{A}^{\mathrm{T}}\bar{\mathscr{P}}\tilde{B}\\ * & -\gamma^2 I_{(n+m-2)m_3}+\tilde{B}^{\mathrm{T}}\bar{\mathscr{P}}\tilde{B}\end{bmatrix}<0, \tag{16}$$

where

$$\Upsilon=I_{n+m-2}\otimes C_2^{\mathrm{T}}C_2,$$
$$\Pi=\tilde{A}^{\mathrm{T}}\bar{\mathscr{P}}\tilde{A}+a^2pI_{(n+m-2)m_1}-\bar{\mathscr{P}},$$
$$\bar{\mathscr{P}}=I_{n+m-2}\otimes\mathscr{P}.$$

Proof First, we prove the asymptotic stability of system Eq. (13). Define a Lyapunov function as follows:

$$V(\xi(k))=\xi^{\mathrm{T}}(k)\bar{\mathscr{P}}\xi(k).$$

Then, we have

$$\begin{aligned}&\mathscr{E}\{V(\xi(k+1))/\xi(k)\}-V(\xi(k))\\ =\;&\xi^{\mathrm{T}}(k)\tilde{A}^{\mathrm{T}}\bar{\mathscr{P}}\tilde{A}\xi(k)+2\xi^{\mathrm{T}}(k)\tilde{A}^{\mathrm{T}}\bar{\mathscr{P}}\tilde{B}\theta(k)\\ &+\theta^{\mathrm{T}}(k)\tilde{B}^{\mathrm{T}}\bar{\mathscr{P}}\tilde{B}\theta(k)\\ &+h^{\mathrm{T}}(\xi(k))\bar{\mathscr{P}}h(\xi(k))-\xi^{\mathrm{T}}(k)\bar{\mathscr{P}}\xi(k).\end{aligned}\tag{17}$$

It follows by Eqs. (8), (14), and (15) that

$$h^{\mathrm{T}}(\xi(k))\bar{\mathscr{P}}h(\xi(k))\le p\left\|\left\{\begin{bmatrix}U_1^{\mathrm{T}}\mathfrak{L}_\Phi & 0_{(n-1)\times m}\\ 0_{(m-1)\times n} & \bar{U}_1^{\mathrm{T}}\bar{\mathfrak{L}}_\Phi\end{bmatrix}\otimes I_{m_1}\right\}g(x(k))\right\|^2. \tag{18}$$

In the light of Eq. (10) and $\lambda_M(\mathfrak{L}_\Phi\mathfrak{L}_\Phi)=\lambda_M(\mathfrak{L}_\Phi)=1$, $\lambda_M(\bar{\mathfrak{L}}_\Phi\bar{\mathfrak{L}}_\Phi)=\lambda_M(\bar{\mathfrak{L}}_\Phi)=1$, one can obtain

$$\begin{aligned}&\left\|\left\{\begin{bmatrix}\mathscr{U}_1^{\mathrm{T}}\mathfrak{L}_\Phi & 0_{(n-1)\times m}\\ 0_{(m-1)\times n} & \bar{\mathscr{U}}_1^{\mathrm{T}}\bar{\mathfrak{L}}_\Phi\end{bmatrix}\otimes I_{m_1}\right\}g(x(k))\right\|\\ =\;&\left\|\left\{\begin{bmatrix}\mathscr{U}_1^{\mathrm{T}}\mathfrak{L}_\Phi & 0_{(n-1)\times m}\\ 0_{(m-1)\times n} & \bar{\mathscr{U}}_1^{\mathrm{T}}\bar{\mathfrak{L}}_\Phi\end{bmatrix}\otimes I_{m_1}\right\}[g(x(k))-e_{n+m}\otimes\sigma(x(k))]\right\|\\ \le\;&\left\|\begin{bmatrix}\mathscr{U}_1^{\mathrm{T}}\mathfrak{L}_\Phi & 0_{(n-1)\times m}\\ 0_{(m-1)\times n} & \bar{\mathscr{U}}_1^{\mathrm{T}}\bar{\mathfrak{L}}_\Phi\end{bmatrix}\otimes I_{m_1}\right\|\|g(x(k))-e_{n+m}\otimes\sigma(x(k))\|\\ \le\;&\|g(x(k))-e_{n+m}\otimes\sigma(x(k))\|.\end{aligned}\tag{19}$$

Based on Eqs. (2), (9), and (10), the following holds:

$$\begin{aligned}&\|g(x(k))-e_{n+m}\otimes\sigma(x(k))\|\\ \le\;&a\|x(k)-e_{n+m}\otimes x(k)\|\\ =\;&a\|\xi(k)\|.\end{aligned}\tag{20}$$

Substituting Eqs. (19) and (20) into Eq. (18) gives

$$h^{\mathrm{T}}(\xi(k))\bar{\mathscr{P}}h(\xi(k))\le a^2p\xi^{\mathrm{T}}(k)\xi(k).\tag{21}$$

By Eqs. (17) and (21) we can obtain

$$\begin{aligned}&\mathscr{E}\{V(\xi(k+1))/\xi(k)\}-V(\xi(k))\\ \le\;&\begin{bmatrix}\xi^{\mathrm{T}}(k) & \theta^{\mathrm{T}}(k)\end{bmatrix}\begin{bmatrix}\Pi & \tilde{A}^{\mathrm{T}}\bar{\mathscr{P}}\tilde{B}\\ * & \tilde{B}^{\mathrm{T}}\bar{\mathscr{P}}\tilde{B}\end{bmatrix}\begin{bmatrix}\xi(k)\\ \theta(k)\end{bmatrix}.\end{aligned}\tag{22}$$

When $\theta(k)=0$, it follows from Eq. (22) that

$$\mathscr{E}\{V(\xi(k+1))/\xi(k)\}-V(\xi(k))\le\xi^{\mathrm{T}}(k)\Pi\xi(k).$$

Noting that equation (16) implies $\Pi<0$, one can easily obtain the mean-square asymptotic stability of Eq. (13) via the same procedure as Ref. [31].

Next, attention is focused on the $\mathscr{H}_\infty$ performance for system Eq. (13). Toward this end, we introduce an index as follows:

$$J(l)=\mathscr{E}\left\{\sum_{k=0}^{l}\left(\|z(k)\|^2-r^2\|\theta(k)\|^2\right)\right\}.$$

Under the zero initial condition, using Eq. (22) and the law of total expectation yield

$$\begin{aligned}J(l)=\;&\mathscr{E}\left\{\sum_{k=0}^{l}\left(\|z(k)\|^2-r^2\|\theta(k)\|^2+\mathscr{E}\{V(\xi(k+1))/\xi(k)\}-V(\xi(k))\right)\right\}-\mathscr{E}\{\mathscr{E}\{V(\xi(l+1))/\xi(l)\}-V(\xi(0))\}\\ \le\;&\begin{bmatrix}\xi^{\mathrm{T}}(l) & \theta^{\mathrm{T}}(l)\end{bmatrix}\begin{bmatrix}\Upsilon+\Pi & \tilde{A}^{\mathrm{T}}\bar{\mathscr{P}}\tilde{B}\\ * & -\gamma^2I_{(n+m-2)m_3}+\tilde{B}^{\mathrm{T}}\bar{\mathscr{P}}\tilde{B}\end{bmatrix}\begin{bmatrix}\xi(l)\\ \theta(l)\end{bmatrix},\end{aligned}$$

which, in conjunction with Eq. (16), ensures $J(l)\le0$, $\forall l>0$. Hence, equation (5) holds true for any non-zero $v(k)\in Ł_2[0,\infty)$. By Definition 2, system equation (13) has a prescribed $\mathscr{H}_\infty$ performance bound γ. This completes the proof.

Based on Theorem 1, a constructive approach for the output-feedback control protocol design can be developed, which is

given in the following theorem.

Theorem 2 Let $\gamma > 0$ and $\beta > 0$ be given scalars. Then, under Assumption 1, the $\mathscr{H}_\infty$ couple-group consensus issue of system equation (1) subject to fixed topology is solvable, if, there are matrices $\mathscr{X} \in \mathbb{R}^{m_2 \times m_2}$, $\mathscr{Y} \in \mathbb{R}^{m_2 \times q}$, a scalar $p > 0$, and a positive definite matrix $\mathscr{P} \in \mathbb{R}^{m_1 \times m_1}$, $i \in N$ such that equation (15) and the following:

$$\begin{bmatrix} \Lambda_1 & * & * \\ \Lambda_2 & -\bar{\mathscr{P}} & * \\ \beta(I_{n+m-2} \otimes \mathscr{Y})\bar{C}_D & (J^{\mathrm{T}} \otimes B_1^{\mathrm{T}})\bar{\mathscr{P}} - (I_{n+m-2} \otimes \mathscr{X}^{\mathrm{T}})(J^{\mathrm{T}} \otimes B_1^{\mathrm{T}}) & -He(\beta I_{n+m-2} \otimes \mathscr{X}) \end{bmatrix} < 0 \tag{23}$$

hold true, where

$$\begin{aligned} \Lambda_1 &= \begin{bmatrix} \Upsilon + a^2 p I_{(n+m-2)m_1} - \bar{\mathscr{P}} & 0_{(n+m-2)m_1 \times (n+m-2)m_3} \\ 0_{(n+m-2)m_3 \times (n+m-2)m_1} & -\gamma^2 I_{(n+m-2)m_3} \end{bmatrix}, \\ \Lambda_2 &= \begin{bmatrix} \bar{\mathscr{P}}(I_{n+m-2} \otimes A) & \bar{\mathscr{P}}(I_{n+m-2} \otimes B_2) \end{bmatrix} \\ &\quad + (J \otimes B_1)(I_{n+m-2} \otimes \mathscr{Y})\bar{C}_D, \\ \bar{C}_D &= \begin{bmatrix} (I_{n+m-2} \otimes C_1) & (I_{n+m-2} \otimes D) \end{bmatrix}, \\ \bar{\mathscr{P}} &= I_{n+m-2} \otimes \mathscr{P}. \end{aligned}$$

In such a case, the desired gain is

$$\mathscr{K} = \mathscr{X}^{-1}\mathscr{Y}. \tag{24}$$

Proof Based on Theorem 1, one only need to prove that equation (16) is ensured by Eq. (23). Clearly, equation (16) can be rearranged as

$$\Lambda_1 + \bar{A}_B^{\mathrm{T}} \bar{\mathscr{P}}^{-1} \bar{A}_B < 0, \tag{25}$$

where

$$\bar{A}_B = \begin{bmatrix} \bar{\mathscr{P}}\tilde{A} & \bar{\mathscr{P}}\tilde{B} \end{bmatrix}.$$

By Schur's complements, equation (25) is equivalent to

$$\begin{bmatrix} \Lambda_1 & * \\ \bar{A}_B & -\bar{\mathscr{P}} \end{bmatrix} < 0. \tag{26}$$

According to Eq. (24), the following holds true:

$$\begin{aligned} &\bar{\mathscr{P}}(J \otimes B_1)(I_{n+m-2} \otimes \mathscr{K})\bar{C}_D \\ &= (J \otimes B_1)(I_{n+m-2} \otimes \mathscr{Y})\bar{C}_D \\ &\quad + \left\{\bar{\mathscr{P}}(J \otimes B_1) - (J \otimes B_1)(I_{n+m-2} \otimes \mathscr{X})\right\} \\ &\quad \times (I_{n+m-2} \otimes \mathscr{X}^{-1}\mathscr{Y})\bar{C}_D. \end{aligned} \tag{27}$$

Noting Eqs. (14) and (27), one can further rewrite Eq. (26) as

$$\begin{bmatrix} \Lambda_1 & * \\ \Lambda_2 & -\bar{\mathscr{P}} \end{bmatrix} + He\{U(I_{n+m-2} \otimes \mathscr{X}^{-1})V^{\mathrm{T}})\} < 0, \tag{28}$$

where

$$\begin{aligned} U &= \begin{bmatrix} 0_{(n+m-2)(m_1+m_3) \times (n+m-2)m_2} \\ \bar{\mathscr{P}}(J \otimes B_1) - (J \otimes B_1)(I_{n+m-2} \otimes \mathscr{X}) \end{bmatrix}, \\ V &= \begin{bmatrix} \bar{C}_D^{\mathrm{T}}(I_{n+m-2} \otimes \mathscr{Y})^{\mathrm{T}} \\ 0_{(n+m-2)m_1 \times (n+m-2)m_2} \end{bmatrix}. \end{aligned}$$

On the other hand, applying Lemma 3 to Eq. (23), one can get Eq. (28). Therefore, it can be concluded that equation (23) guarantees Eq. (16). This completes the proof of the theorem.

3.2. Markovian switching topology case

This section is concerned with a situation where the communication topologies switch between directed graphs $G^1(k), G^2(k), \ldots, G^r(k)$ and that the switching is governed by a discrete-time Markov chain $\{r_k, k \geq 0\}$, which takes values in a finite set $N = \{1, \ldots, r\}$ with transition probabilities

$$P\{r(k+1) = l | r(k) = i\} = \pi_{il}, \ \ \forall i, l \in N, \tag{29}$$

where π_{il} is the transition rate from mode i to mode l and $\pi_{il} \geq 0$, $\sum_{l=1}^{r} \pi_{il} = 1$.

The control protocol to be applied takes the form as

$$\begin{aligned} u_i(k) &= \mu_{1i}\mathscr{K} \sum_{j=1}^{n} a_{ij}^{r_k}(y_j(k) - y_i(k)) \\ &\quad + \mu_{2i}\mathscr{K} \sum_{j=n+1}^{n+m} a_{ij}^{r_k}(y_j(k) - y_i(k)), \end{aligned} \tag{30}$$

where μ_{1i} and μ_{2i} are the same as those given in Eq. (4), and $\mathscr{K}$ is the control gain to be solved.

For the case of Markovian switching topology, the following assumption is needed.[20]

Assumption 2 Each of the graphs $G_1^u(k)$ and $G_2^u(k)$ contains a spanning tree, where $G_\ell^u(k)$ is the union of subgraphs $G_\ell^1(k), G_\ell^2(k), \ldots, G_\ell^s(k)$ $(\ell = 1, 2)$.

Similarly to the case of fixed topologies, the following reduced-order system can be obtained:

$$\begin{aligned} \xi(k+1) &= \hat{A}_{r_k}\xi(k) + \hat{B}_{r_k}\theta(k) + h(\xi(k))w(k), \\ z(k) &= \tilde{\mathscr{U}}\xi(k), \end{aligned} \tag{31}$$

where

$$\begin{aligned} \hat{A}_{r_k} &= I_{n+m-2} \otimes A \\ &\quad + \begin{bmatrix} -L_{1,r_k} & 0_{(n-1)\times(m-1)} \\ 0_{(m-1)\times(n-1)} & -L_{2,r_k} \end{bmatrix} \otimes B_1\mathscr{K}C_1, \\ \hat{B}_{r_k} &= I_{n+m-2} \otimes B_2 \\ &\quad + \begin{bmatrix} -L_{1,r_k} & 0_{(n-1)\times(m-1)} \\ 0_{(m-1)\times(n-1)} & -L_{2,r_k} \end{bmatrix} \otimes B_1\mathscr{K}D, \\ h(\xi(k)) &= I_{(n+m-2)m_1} f(\xi(k)), \\ \tilde{\mathscr{U}} &= \begin{bmatrix} \mathscr{U}_1 & 0_{n\times(m-1)} \\ 0_{m\times(n-1)} & \mathscr{U}_1 \end{bmatrix} \otimes C_2, \end{aligned}$$

and the couple-group $\mathscr{H}_\infty$ consensus issue of system equation (1) with Markovian switching topologies can be changed into a stochastic $\mathscr{H}_\infty$ control problem: to find a control protocol of the form of Eq. (30) such that the reduced-order system

in Eq. (31) is mean square asymptotically stable and has a prescribed $\mathscr{H}_\infty$ performance γ.

We now give our analysis result for the $\mathscr{H}_\infty$ control problem.

Theorem 3 System equation (31) is mean-square asymptotically stable and has a prescribed $\mathscr{H}_\infty$ performance γ, if, there are scalars $p_i > 0$ and positive definite matrices $\mathscr{P}_i \in \mathbb{R}^{m_1\times m_1}$, $i \in N$ such that

$$\mathscr{P}_i - p_i I_{m_1} < 0, \tag{32}$$

$$\begin{bmatrix} \Upsilon+\Pi_i & \hat{A}_i^{\mathrm{T}}\sum_{l=1}^{r}\pi_{il}(I_{n+m-2}\otimes\mathscr{P}_l)\hat{B}_i \\ * & -\gamma^2 I_{(n+m-2)m_3}+\hat{B}_i^{\mathrm{T}}\sum_{l=1}^{n}\pi_{il}(I_{n+m-2}\otimes\mathscr{P}_l)\hat{B}_i \end{bmatrix} < 0, \tag{33}$$

where

$$\begin{aligned} \Upsilon &= I_{n+m-2}\otimes C_2^{\mathrm{T}}C_2, \\ \Pi_i &= \hat{A}_i^{\mathrm{T}}\sum_{l=1}^{r}\pi_{il}(I_{n+m-2}\otimes\mathscr{P}_l)\hat{A}_i \\ &\quad + a^2\sum_{l=1}^{r}\pi_{il}p_l I_{(n+m-2)m_1} - I_{n+m-2}\otimes\mathscr{P}_i. \end{aligned}$$

Proof We first prove the mean-square asymptotic stability of system Eq. (31). Define a Lyapunov function as follows:

$$V(\xi(k), r_k) = \xi^{\mathrm{T}}(k)(I_{n+m-2}\otimes\mathscr{P}_{r_k})\xi(k).$$

Then, one has

$$\begin{aligned} \Delta V(\xi(k)) &= \mathscr{E}\{V(\xi(k+1))|\xi(k), r_k = i\} - V(\xi(k), r_k = i) \\ &= \xi^{\mathrm{T}}(k)\hat{A}_i^{\mathrm{T}}\sum_{l=1}^{r}\pi_{il}(I_{n+m-2}\otimes\mathscr{P}_l)\hat{A}_i\xi(k) \\ &\quad + 2\xi^{\mathrm{T}}(k)\hat{A}_i^{\mathrm{T}}\sum_{l=1}^{r}\pi_{il}(I_{n+m-2}\otimes\mathscr{P}_l)\hat{B}_i\theta(k) \\ &\quad + \theta^{\mathrm{T}}(k)\hat{B}_i^{\mathrm{T}}\sum_{l=1}^{r}\pi_{il}(I_{n+m-2}\otimes\mathscr{P}_l)\hat{B}_i\theta(k) \\ &\quad + h^{\mathrm{T}}(\xi(k))\sum_{l=1}^{r}\pi_{il}(I_{n+m-2}\otimes\mathscr{P}_l)h(\xi(k)) \\ &\quad - \xi^{\mathrm{T}}(k)\left(I_{n+m-2}\otimes\mathscr{P}_i\right)\xi(k). \end{aligned} \tag{34}$$

Following the same procedure as in the proof of Theorem 1, one can get

$$h^{\mathrm{T}}(\xi(k))\sum_{l=1}^{r}\pi_{il}(I_{n+m-2}\otimes\mathscr{P}_l)h(\xi(k)) \le a^2\sum_{l=1}^{r}\pi_{il}p_l\xi^{\mathrm{T}}(k)\xi(k). \tag{35}$$

When $\theta(k) = 0$, then, from Eqs. (34) and (35) one obtain

$$\Delta V(\xi(k)) \le \xi^{\mathrm{T}}(k)\Pi_i\xi(k).$$

Note that equation (33) implies $\Pi_i < 0$, the mean-square asymptotic stability of system Eq. (31) is ensured by means of the stability theory in Ref. [30] for stochastic Markovian switching systems.

We next prove the $\mathscr{H}_\infty$ performance for system Eq. (31). Define

$$J(l) = \mathscr{E}\left\{\sum_{k=0}^{l}\left(\|z(k)\|^2 - r^2\|\theta(k)\|^2\right)\right\}.$$

Noting $V(\xi(0)) = 0$, using Eqs. (34), (35), and the law of total expectation results in

$$\begin{aligned} J(l) &= \mathscr{E}\left\{\sum_{0}^{l}\left(\|z(k)\|^2 - r^2\|\theta(k)\|^2 + \Delta V(\xi(k))\right)\right\} \\ &\quad - \mathscr{E}\{V(\xi(l))\} \\ &\le \mathscr{E}\{\sum_{0}^{l}\left(\|z(s)\|^2 - r^2\|\theta(s)\|^2 + \Delta V(\xi(k))\right)\}, \\ &= \begin{bmatrix}\xi^{\mathrm{T}}(l) & \theta^{\mathrm{T}}(l)\end{bmatrix} \\ &\quad \times \begin{bmatrix} \Upsilon+\Pi_i & \hat{A}_i^{\mathrm{T}}\sum_{j=1}^{r}\pi_{il}(I_{n+m-2}\otimes\mathscr{P}_l)\hat{B}_i \\ * & -\gamma^2 I_{(n+m-2)m_3}+\hat{B}_i^{\mathrm{T}}\sum_{j=1}^{r}\pi_{il}(I_{n+m-2}\otimes\mathscr{P}_l)\hat{B}_i \end{bmatrix} \\ &\quad \times \begin{bmatrix}\xi(l) \\ \theta(l)\end{bmatrix}. \end{aligned} \tag{36}$$

By Eqs. (33) and (36), it can be observed that $J(l) \le 0$, $\forall l > 0$, which infers that the $\mathscr{H}_\infty$ performance of system Eq. (31) is able to be established. This completes the proof.

Based on Theorem 3, we deal with the $\mathscr{H}_\infty$ couple-group consensus issue of system Eq. (1) with Markovian switching topologies and can deduce a result as follows.

Theorem 4 Let $\gamma > 0$ and $\beta > 0$ be given scalars. Then, under Assumption 2, the couple-group $\mathscr{H}_\infty$ consensus issue of system Eq. (1) subject to Markovian switching topologies is solvable, if, there are matrices $\mathscr{X} \in \mathbb{R}^{m_2\times m_2}$, $\mathscr{Y} \in \mathbb{R}^{m_2\times q}$, scalars $p_i > 0$, and positive definite matrices $\mathscr{P}_i \in \mathbb{R}^{m_1\times m_1}$, $i \in N$ such that equation (32) and the following:

$$\begin{bmatrix} \Lambda_{1_i} & * & * \\ \Lambda_{2_i} & \Sigma_{i_h} & * \\ \beta(I_{n+m-2}\otimes\mathscr{Y})\bar{C}_D & (J_i^{\mathrm{T}}\otimes B_1^{\mathrm{T}})\hat{\mathscr{P}}_h - e_r^{\mathrm{T}}\otimes[(I_{n+m-2}\otimes\mathscr{X}^{\mathrm{T}})(J_i^{\mathrm{T}}\otimes B_1^{\mathrm{T}})] & -He(\beta I_{n+m-2}\otimes\mathscr{X}) \end{bmatrix} < 0 \tag{37}$$

hold true, where

$$\begin{aligned} J_i &= \mathrm{diag}\{-L_{1i}, -L_{2i}\}, \bar{C}_D = \begin{bmatrix}(I_{n+m-2}\otimes C_1) & (I_{n+m-2}\otimes D)\end{bmatrix}, \\ \Lambda_{1_i} &= \begin{bmatrix} \Upsilon + a^2\sum_{l=1}^{r}\pi_{il}p_l I_{(n+m-2)m_1} - I_{n+m-2}\otimes\mathscr{P}_i & 0_{(n+m-2)m_1\times(n+m-2)m_3} \\ 0_{(n+m-2)m_3\times(n+m-2)m_1} & -\gamma^2 I_{(n+m-2)m_3} \end{bmatrix}, \\ \Lambda_{2i} &= \begin{bmatrix}\hat{\mathscr{P}}_i^{\mathrm{T}}(I_{n+m-2}\otimes A) & \hat{\mathscr{P}}_i^{\mathrm{T}}(I_{n+m-2}\otimes B_2)\end{bmatrix} + e_r\otimes\left[(J_i\otimes B_1)(I_{n+m-2}\otimes\mathscr{Y})\bar{C}_D\right], \\ \hat{\mathscr{P}}_h &= \begin{bmatrix} I_{n+m-2}\otimes\mathscr{P}_1 & \cdots & I_{n+m-2}\otimes\mathscr{P}_l & \cdots & I_{n+m-2}\otimes\mathscr{P}_h \end{bmatrix}, \end{aligned}$$

$$\Sigma_{i_h} = \mathrm{diag}\Big\{-\frac{1}{\pi_{i1}}\mathscr{P}_1,\cdots,-\frac{1}{\pi_{il}}\mathscr{P}_l,\cdots,-\frac{1}{\pi_{ih}}\mathscr{P}_h\Big\},\ \pi_{il}\neq 0,\ 1\le l\le h\le r.$$

In such a case, the desired gain is $\mathscr{K}=\mathscr{X}^{-1}\mathscr{Y}$.

Proof Define

$$\bar{A}_{B_{il}} = \big[\ (I_{n+m-2}\otimes\mathscr{P}_i)\hat{A}_i\ \ (I_{n+m-2}\otimes\mathscr{P}_l)\hat{B}_i\ \big].$$

Then, noting that equation (33) can be reorganized

$$\Lambda_{1_i}+\sum_{l=1}^{r}\pi_{il}\bar{A}_{B_{il}}^{\mathrm{T}}(I_{n+m-2}\otimes\mathscr{P}_l)^{-1}\bar{A}_{B_{il}}<0,$$

the reminder can be proven readily in a similar way as in the proof of Theorem 2 and, therefore, can be omitted here for the sake of brevity.

Remark 2 Based on the analysis results and by means of LMI techniques, Theorems 2 and 4 establish sufficient conditions for ensuring the $\mathscr{H}_\infty$ couple-group consensus of stochastic multi-agent systems with the fixed communication topology and the Markovian switching communication topologies, respectively. The desired control protocols can be constructed provided the LMIs in Theorems 2 and 4 are feasible, which can be checked readily via common mathematical software MATLAB.

4. Simulation examples

In this section, two numerical example are given to show the validity of the present control protocols.

Example 1 In this example, we consider a stochastic multi-agent system consisting of five agents and divide these agents into two groups. The parameters are set as follows:

$$A=1.2,\ B_1=0.7,\ B_2=0.1,\ C_1=-0.8,\ C_2=0.2,\ D=0.1,$$
$$v_i(k)=\mathrm{e}^{-ik},\ \sigma(x_i(k))=0.1\sin(x_i(k)),i=1,\ldots,5.$$

The topology of the multi-agent system is fixed and depicted in Fig. 1. One can find that $\sigma(x_i(k))$ satisfies Eqs. (2) and (3), where $\alpha=\varepsilon=0.1$.

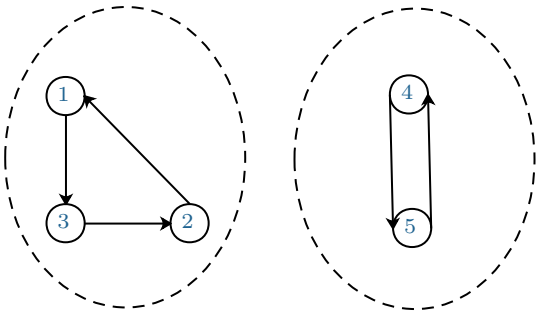


Fig. 1. Fixed communication topology.

Let $\beta=0.2$ and $\gamma=0.5$. Then, using MATLAB to solve the LMIs in Eqs. (15) and (23), the control gain matrix is obtained as $\mathscr{K}=-1.0089$. The state trajectories with control protocol are shown in Fig. 2. It is obvious to see that the agents achieve two different consistent states, respectively. Let us define

$$\gamma(k)=\sqrt{\frac{\mathscr{E}\{\sum_{s=0}^{k}z^{\mathrm{T}}(s)z(s)\}}{\sum_{s=0}^{k}v^{\mathrm{T}}(s)v(s)}} \tag{38}$$

to reflect the effect of disturbance $v(k)$ on the controlled output $z(k)$. Then, the trajectory of $\gamma(k)$ under zero initial conditions is shown in Fig. 3. It can be seen that $\gamma(k)$ converges to 0.029328, which is apparently less than the given $\mathscr{H}_\infty$ performance bound $\gamma=0.5$. Therefore, the example verifies the effectiveness of the design control protocol for the case of fixed communication topology.

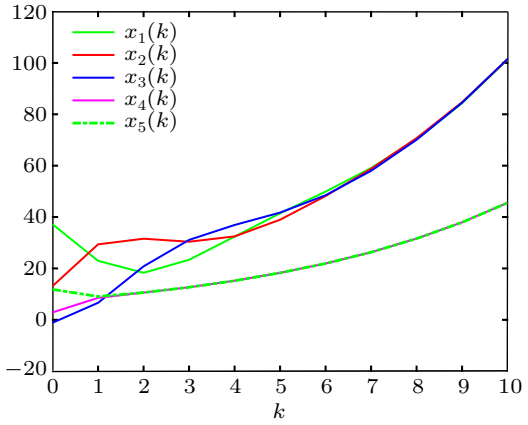


Fig. 2. State trajectories with control.

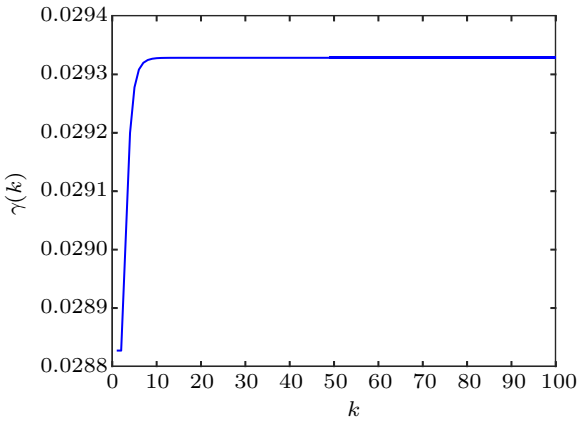


Fig. 3. Evolution of $\gamma(k)$.

Example 2 In this example, we consider a stochastic multi-agent system consisting of seven agents and divide these agents into two groups. The parameters are set as follows:

$$A=\begin{bmatrix}1&1\\0&1\end{bmatrix},\ B_1=\begin{bmatrix}-1&0\\0&-1\end{bmatrix},\ B_2=\begin{bmatrix}0\\-0.5\end{bmatrix},$$

$$C_1 = \begin{bmatrix} -1 & 0 \\ 0 & 0.3 \end{bmatrix},\ C_2 = \begin{bmatrix} 0 & 0.001 \\ 0.001 & 0 \end{bmatrix},\ D = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$
$$v_i(k) = i\mathrm{e}^{-0.1ik},$$
$$\sigma(x_i(k)) = 0.02[\cos(x_{i_1}(k))\ \cos(x_{i_2}(k))]^{\mathrm{T}},\ i = 1,\ldots,7.$$

One can find that $\sigma(x_i(k))$ satisfies Eqs. (2) and (3), where $\alpha = \varepsilon = 0.02$. Figure 4 shows the possible communication topologies of the multi-agent system. The Markov transition rate matrix of the graphs G_1, G_2, and G_3 is taken as

$$(\pi_{il})_{3\times 3} = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.5 & 0.3 & 0.2 \\ 0.5 & 0.5 & 0 \end{bmatrix}.$$

Let $\beta = 0.5$ and $\gamma = 0.98$. Then, by solving the LMIs in Eqs. (32) and (37), one can get

$$\mathscr{K} = \begin{bmatrix} 0.27041 & 1.2845 \\ -0.001078 & -0.73886 \end{bmatrix}.$$

The switching signal is depicted in Fig. 5. The state trajectories with control protocol are shown in Figs. 6 and 7, from which one can observe that the agents belonging to G_1, G_2, and G_3 achieve two different consistent states, respectively. The trajectory of $\gamma(k)$ defined in Eq. (38) under zero initial conditions is shown in Fig. 8. It can be seen that $\gamma(k)$ converges to 0.0015022, which is apparently less than the given $\mathscr{H}_\infty$ performance bound $\gamma = 0.98$. Therefore, the example verifies the effectiveness of the design control protocol for the Markovian switching communication topologies.

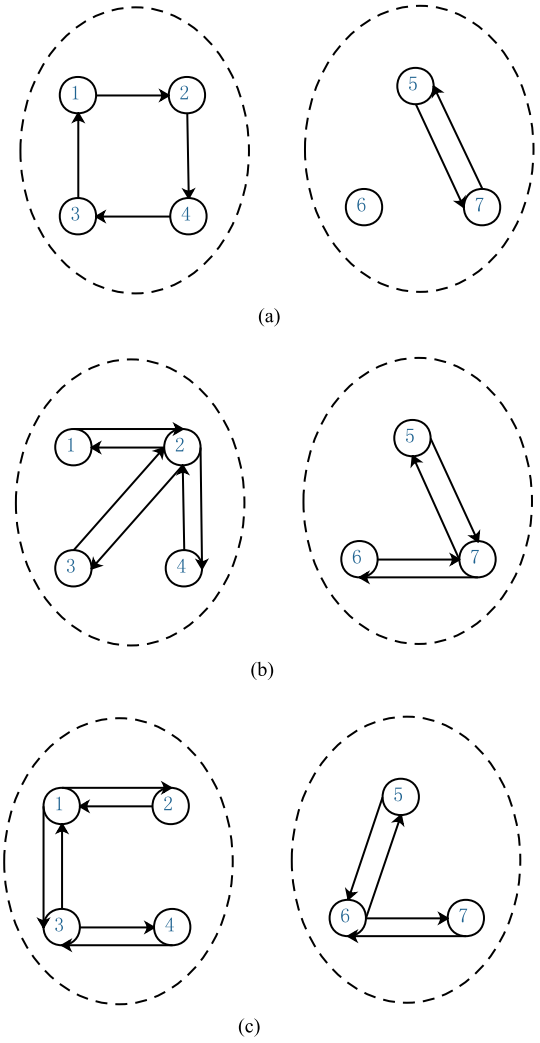


Fig. 4. Interaction graphs of 7 agents: (a) G_1, (b) G_2, and (c) G_3.

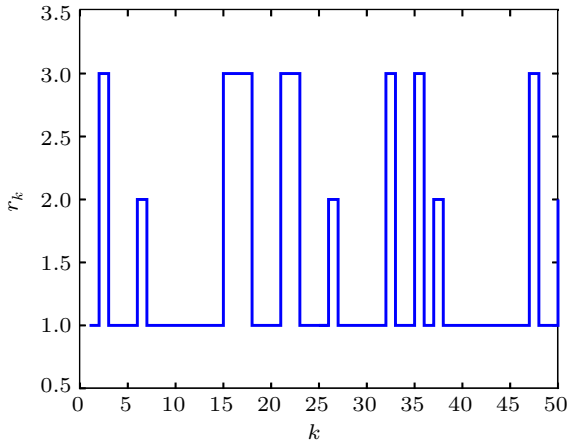


Fig. 5. Markovain switching signal.

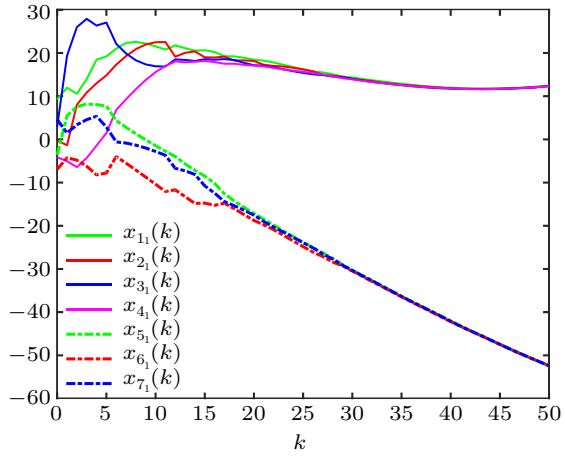


Fig. 6. State trajectories with control.

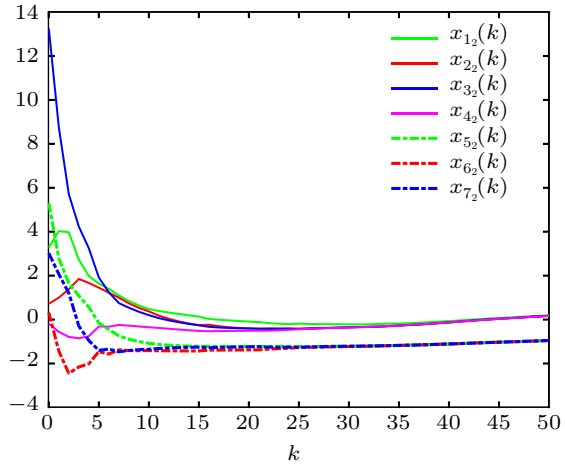


Fig. 7. State trajectories with control.

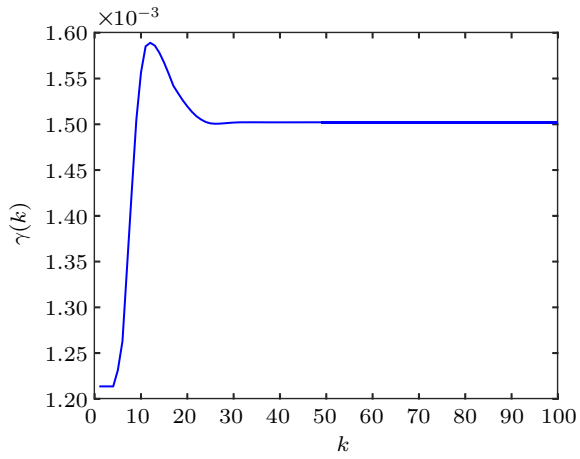


Fig. 8. Evolution of $\gamma(k)$.

5. Conclusion

The issue of $\mathscr{H}_\infty$ couple-group consensus for a class of discrete-time stochastic multi-agent systems via output-feedback control has been addressed in this paper. Both fixed communication topology and Markovian switching communication topologies have been discussed. It has been shown that via employing linear transformations, the closed-loop systems are able to be converted into reduced-order systems and, at the same time, the $\mathscr{H}_\infty$ couple-group consensus issue can be changed into a stochastic $\mathscr{H}_\infty$ control problem. New conditions for the mean-square asymptotic stability and $\mathscr{H}_\infty$ performance of the reduced-order system have been proposed. On the basis of these conditions, constructive approaches for the design of the output-feedback control protocols are developed for the fixed communication topology and the Markovian switching communication topologies, respectively. Finally, two numerical examples have been employed to illustrate the applicability of the present control protocols. Future work in the field of $\mathscr{H}_\infty$ group consensus for multi-agent systems will focus on finite-time control[38–40] with consideration of sensor saturations and sensor failures.[41–43]

References

[1] Khazaei J and Nguyen D H 2017 *IEEE Trans. Smart Grid*
[2] Peng Z, Wen G, Yang S and Rahmani A 2016 *Nonlinear Dyn.* **86** 605
[3] Yu W, Chen G, Wang Z and Yang W 2009 *IEEE Trans. Syst. Man Cybern. B* **39** 1568
[4] Yan Z, Liu Y, Zhou J, Zhang W and Wang L 2017 *Chin. Phys. B* **26** 040203
[5] Hong Y, Hu J and Gao L 2016 *Automatica* **42** 1177
[6] Ren W 2010 *IEEE Trans. Control Syst. Technol.* **18** 230
[7] Olfati-Saber R and Murray R M 2002 *IEEE Trans. Automat. Control*, **49** 1520
[8] Qin J, Ma Q, Shi Y and Wang L 2017 *IEEE Trans. Ind. Electron.* **64** 4972
[9] Yuan D, Ho D W C and Jiang G 2017 *IEEE Trans. Cybern.*
[10] Wang J, Su L, Shen H, Wu Z and Park J 2017 *J. Franklin Inst.* **354** 1302
[11] Shahamatkhah E and Tabatabaei M 2018 *Chin. Phys. B* **27** 010701
[12] Jin X, Wang S, Qin J, Zheng W and Kang Y 2018 *IEEE Trans. Circuits Syst. I. Regul. Pap.* **65** 2243
[13] Li X, Zhou C, Zhou J, Wang Z and Xia J 2018 *Int. J. Control Autom. Syst.* in press
[14] Wang Y, Tu L, Song S and Li K 2018 *Acta Phys. Sin.* **67** 050504 (in Chinese)
[15] Xie D, Zhang S and Xie J 2018 *Physica A* **509** 1195
[16] Lu X, Francis A and Chen S 2010 *Chin. Phys. B* **19** 120506
[17] Ma Q, Wang Z and Miao G 2014 *J. Franklin Inst.* **351** 1288
[18] Liao X and Ji L 2014 *Neurocomputing* **135** 262
[19] Feng Y and Zheng W 2018 *IET Control Theory Appl.* **12** 753
[20] Zhao H and Park J 2014 *Nonlinear Dyn.* **77** 1297
[21] Shang Y 2015 *J. Franklin Inst.* **352** 4826
[22] Xie D, Shi L and Jiang F 2017 *Neurocomputing* **281** 37
[23] Liu Y and Jia Y 2010 *Int. J. Control* **83** 527
[24] Qin J, Ma Q, Zheng W, Gao H and Kang Y 2017 *IEEE Trans. Automat. Contr.* **62** 3559
[25] Stoorvogel A 1992 *The $\mathscr{H}_\infty$ Control Problem: A State Space Approach* (New York: Prentice Hall)
[26] Wang P and Jia Y 2016 *Int. J. Syst. Sci.* **47** 1073
[27] Cui Y, Fei M and Du D 2016 *IET Gener. Transm. Dis.* **10** 2565
[28] Liang H, Zhang H, Wang Z and Wang J 2014 *Chin. Phys. B* **23** 018902
[29] Zhou J, Park J and Ma Q 2016 *Appl. Math. Comput.* **291** 69
[30] Mao X 1999 *Stoch. Process. Their App.* **l79** 45
[31] Xu S, Lam J and Chen T 2004 *Syst. Control. Lett.* **51** 203
[32] Lin P, Jia Y and Li L 2008 *Syst. Control. Lett.* **57** 643
[33] Zhou J, Sang C, Li X, Fang M and Wang Z 2018 *Appl. Math. Comput.* **325** 41
[34] Chang X, Yang C and Xiong J 2018 *IEEE Trans. Syst. Man Cybern. Syst.*
[35] Li S, Yang L and Li K 2015 *Chin. Phys. B* **24** 010503
[36] Ma Q, Xu S, Lewis F L, Zhang B and Zou Y 2016 *IEEE trans. Cybern.* **46** 1471
[37] Yan Z, Sang C, Fang M and Zhou J 2018 *Trans. Inst. Meas. Control*
[38] Xia J, Zhang J, Sun W, Zhang B and Wang Z 2018 *IEEE Trans. Syst. Man Cybern. Syst.*
[39] Wang Z, Shen L, Xia J, Shen H and Wang J 2018 *J. Franklin Inst.* **355** 6371
[40] Shen H, Li F, Yan H, Karimi H R and Lam H K 2017 *IEEE Trans. Fuzzy Syst.*
[41] Song X, Wang M, Song S and Tejado 2018 *IEEE Access* **6** 50066
[42] Zhuang G, Li Y and Li Z 2014 *Int J. Syst. Sci.* **47** 1514
[43] Liu Y, Guo B, Park J and Lee S 2018 *IEEE Trans. Fuzzy Syst.* **26** 2089