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Solitons in nonlinear systems and eigen-states in quantum wells*

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We study the relations between solitons of nonlinear Schrödinger equation and eigen-states of linear Schrödinger equation with some quantum wells. Many different non-degenerated solitons are re-derived from the eigen-states in the quantum wells. We show that the vector solitons for the coupled system with attractive interactions correspond to the identical eigen-states with the ones of the coupled systems with repulsive interactions. Although their energy eigenvalues seem to be different, they can be reduced to identical ones in the same quantum wells. The non-degenerated solitons for multi-component systems can be used to construct much abundant degenerated solitons in more components coupled cases. Meanwhile, we demonstrate that soliton solutions in nonlinear systems can also be used to solve the eigen-problems of quantum wells. As an example, we present the eigenvalue and eigen-state in a complicated quantum well for which the Hamiltonian belongs to the non-Hermitian Hamiltonian having parity–time symmetry. We further present the ground state and the first exited state in an asymmetric quantum double-well from asymmetric solitons. Based on these results, we expect that many nonlinear physical systems can be used to observe the quantum states evolution of quantum wells, such as a water wave tank, nonlinear fiber, Bose–Einstein condensate, and even plasma, although some of them are classical physical systems. These relations provide another way to understand the stability of solitons in nonlinear Schrödinger equation described systems, in contrast to the balance between dispersion and nonlinearity.

Keywords: non-degenerated soliton, eigen-states, quantum well, nonlinear Schrödinger equation

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1. Introduction

Solitons have been found in many different physical systems,[1–5] such as bright soliton, dark soliton, breather-like solitons, and abundant vector solitons. The prototypical model for the study of these solitons is the nonlinear Schrödinger (NLS) equation with attractive or repulsive interactions. Many experiments suggested that the NLS model described well the evolution dynamics of many different nonlinear systems, such as water wave tank,[6] optical fiber,[7] Bose–Einstein condensate,[8] etc. It is noted that the nonlinear terms can be understood as some certain quantum potentials.[9–12] If the density profiles depend on time, the NLS evolution corresponds to linear Schrödinger (LS) with time-dependent quantum wells. It is also difficult to solve LS with time-dependent quantum wells, therefore we focus on the eigen-problems of LS with time-independent potentials. If the density profile of nonlinear term is time-independent, the potentials will also become time-independent, and the NLS can be related with the eigen-problems of LS with the quantum potentials.[10,11] This provides possibilities to establish correspondence relations between soliton states and the eigen-states in quantum wells. The relations are nontrivial to be established because they enable us to obtain more abundant dynamics of localized waves in nonlinear systems,[13] and solve the eigen-problems in more complicated quantum wells. They also provide possibilities to investigate the evolution of quantum states in classical nonlinear systems.

In this paper, we study the quantitative relations between solitons of NLS equation with attractive or repulsive interactions and eigen-states of LS equation with quantum wells. The discussions are made on static solions for simplicity. It should be noted that soliton with velocities can be related with static soliton through Galilean transformation because the NLS equations admit Galilean symmetry. Therefore, the relations also hold well for solitons with velocities, just as the Galilean transformation holds. We start from the simplest scalar NLS with attractive interaction to establish the relation, and then generalize it to multi-component NLS cases with attractive interactions. Especially, we show that vector solitons in multi-component NLS with repulsive interaction can be generated from the eigen-problem in identical quantum wells for NLS with attractive interactions. We establish the explicit relations up to general N-component coupled case, and the results are summarized in Table 1. The non-degenerated soliton for M-component systems can be used to construct abundant degenerated solitons in more than M components cou-

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pled cases. Meanwhile, we also demonstrate that the soliton solution in nonlinear systems can be used to solve the eigen-problems of quantum wells. This paves the ways to solve the LS with complicated potentials with the aid of well-developed techniques for solving NLS equations.

The rest of this paper is organized as follows. In Section 2, we describe the simple idea for studying the relations between soliton of NLS and eigen-states of LS with certain quantum wells. In Section 3, we establish the relations between the non-degenerated soliton of vector NLS with attractive interactions and the eigen-states in quantum wells. Many different solitons presented previously are re-derived in much simpler way from the eigen-states in quantum wells. The eigenvalues of solitons are clarified clearly. In Section 4, we derive non-degenerated soliton solution of vector NLS with repulsive interactions from the identical quantum wells with the vector NLS with attractive interactions. They admit identical eigenvalues and eigen-states with the ones for attractive interaction cases. Then, we discuss how degenerated soliton for N-component coupled NLS with attractive or repulsive interactions can be generated from the non-degenerated solitons of M-component coupled NLS, with $N > M$, in Section 5. Meanwhile, we present eigenvalue and its eigen-state in two complicated quantum wells form the soliton solution of NLS in Section 6. This paves the ways to solve the LS with complicated potentials with the aid of well-developed techniques for solving NLS equations. Finally, we present our conclusion and discussion in Section 7.

Table 1. The correspondence relations between the eigenvalues of LS equation with a quantum well of $-\frac{N(N-1)}{2} f \operatorname{sech}^2[\sqrt{f}x]$ and non-degenerated solitons of N-component coupled NLS equation with attractive (AI) or repulsive (RI) interactions. It should be noted that the coefficients of these soliton solutions are different for the attractive and repulsive cases.

Eigenvalues	Coupled NLS with AI Coupled NLS with RI
0	$(N-1)$-valley dark soliton
$-f/2$	$(N-1)$-hump bright soliton
$-2f$	$(N-2)$-hump bright soliton
$-9f/2$	$(N-3)$-hump bright soliton
.	.
.	.
.	.
$-(N-2)^2 f/2$	double-hump bright soliton
$-(N-1)^2 f/2$	single-hump bright soliton

2. A simple idea for identifying the relations between solitons and eigen-states in quantum wells

With dimensionless unit, the simplest NLS equation can be written as follows:

$$\mathrm{i}\frac{\partial \psi(x,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 \psi(x,t)}{\partial x^2} + \gamma|\psi(x,t)|^2\psi(x,t) = 0, \qquad (1)$$

which has been studied widely in soliton fields.[2–5] If $\gamma = 0$, the equation will become a LS equation with no potentials. It is not possible to obtain a bound state in this case. If $\gamma = 1$, the NLS will admit bright soliton which is a hump density on zero background; if $\gamma = -1$, the NLS will admit dark soliton which is a defect on plane wave background. If the soliton solutions admit $\psi(x,t) = \phi(x)\mathrm{e}^{-\mathrm{i}\mu t}$, the NLS will be transformed to an eigen-problem of LS with a potential term

$$-\frac{1}{2}\frac{\partial^2 \phi(x)}{\partial x^2} + U(x)\phi(x) = \mu\phi(x). \qquad (2)$$

The potential term $U(x) = -\gamma|\phi(x)|^2$. Especially, if there are some spatial-independent terms in $-\gamma|\phi(x)|^2$, the constant terms can be transferred to the other side of the equation and be absorbed to the eigen-energy value. This can be used to discuss the explicit relations between the soliton solution of NLS and the eigenstate of LS with a potential well. This idea has been contained in some published papers,[9] which has been used to derive some striking solitons in NLS described systems from the solutions of LS equation.[10,11,13,14] But the relations have not been discussed systemically, and the eigenvalues of solitons seemed to be different for attractive interactions and repulsive interactions in previous studies. In this paper, we would like to demonstrate the relations more clearly, and show that more abundant solitons can be obtained from the relations. Many different vector solitons reported before can also be obtained from the relations directly, such as bright-bright, bright-dark, dark-bright, and dark-dark solitons.[15,16] Meanwhile, we will show that the soliton solution of NLS can also be used to obtain the eigen-solution of LS with complicated potential wells, which is hard to solve directly. First, we discuss how to generate soliton solution of NLS from eigen-solutions of LS.

We will then discuss two cases: the attractive interaction and repulsive interaction cases. For N-component coupled NLS with attractive interactions, we can derive many different soliton solutions from the eigen-states in some certain quantum wells. The discussion begins with the well-known bright soliton of the simplest scalar NLS equation with attractive interaction. For N-component coupled NLS with repulsive interactions, we can derive many different soliton solutions from the identical eigen-states with the attractive cases. We discuss them started with the well-known dark soliton of the simplest scalar NLS equation with repulsive interaction. Our results will show that vector solitons for coupled NLS with attractive interactions and repulsive interactions can be generated from identical quantum wells, and they admit identical eigenvalues of the quantum wells. It should be noted that soliton with velocities can be related with static soliton through Galilean transformation because the NLS equations admit Galilean symmetry. Therefore, we discuss the subject mainly based on static solitons.

It should be emphasized that many of the following soliton solutions have been obtained by some other methods (Darboux transformation, Hirota bilinear forms, and other methods), and even many similar solutions have been given in Refs. [10], [11], [13], and [14]. But we would like to re-derive them and describe how to generate these soliton solutions in details, to show the unified properties of solitons clearly. The parameters of soliton solutions are also written in a different way from the ones in Refs. [11], [13], and [14] to show their physical meaning more clearly and demonstrate the unified characters of solitons more conveniently.

3. Non-degenerated soliton of NLS with attractive interactions generated from eigen-states in quantum wells

For $\gamma = 1$ attractive case, the fundamental soliton solution of the NLS is

$$\psi(x,t) = \sqrt{f}\,\mathrm{sech}[\sqrt{f}x]\,\mathrm{e}^{\mathrm{i}f/2t}, \tag{3}$$

where $f > 0$ denotes the peak of soliton. This static soliton has been given for a long time.[15,16] From the above idea, we can see that the soliton solution corresponds to the eigen-state of LS as follows:

$$-\frac{1}{2}\frac{\partial^2\phi(x)}{\partial x^2} - f\,\mathrm{sech}^2[\sqrt{f}x]\phi(x) = -f/2\phi(x), \tag{4}$$

where $\phi(x) = \sqrt{f}\,\mathrm{sech}[\sqrt{f}x]$ is the corresponding eigen-state of the quantum well $-f\,\mathrm{sech}^2[\sqrt{f}x]$. From the general properties of bound state in one-dimensional potential in quantum theory, we know that the eigen-state is the ground state in the quantum well because there is no node for $\phi(x)$. The eigenvalue of energy for this eigen-state is $-f/2$. From the eigenvalues in the quantum well $-f\,\mathrm{sech}^2[\sqrt{f}x]$,[17,18] we know that there is another eigenvalue 0. We show them in Table 2.

Table 2. The eigenvalues and eigen-states in the quantum well $-f\,\mathrm{sech}^2[\sqrt{f}x]$. This can be used to construct the well-known bright-dark soliton and dark-bright soliton for two-component coupled nonlinear Schödinger equation.

Quantum well	Eigenvalues	Eigen-states
$-f\,\mathrm{sech}^2[\sqrt{f}x]$	0	$\tanh[\sqrt{f}x]$
	$-f/2$	$\mathrm{sech}[\sqrt{f}x]$

We have studied vector soliton in multi-component coupled NLS equations. The component number can be very large and marked by N. Then the N-component coupled NLS equation can be written as

$$\mathrm{i}\psi_t + \frac{1}{2}\psi_{xx} + \gamma\psi^\dagger\psi\psi = 0, \tag{5}$$

where $\psi = (\psi_1, \psi_2, \ldots, \psi_N)^{\mathrm{T}}$. If all components occupy the identical fundamental eigen-state $\mathrm{sech}[\sqrt{f}x]$, namely, $\psi_j = c_j\sqrt{f}\,\mathrm{sech}[\sqrt{f}x]\,\mathrm{e}^{\mathrm{i}f/2t}$, then the N-component coupled NLS will admit a vector bright soliton. For example, the well-known bright-bright soliton can be obtained directly. But there is an additional constrain condition on the coefficients of wave functions in components, namely, $\sum_{j=1}^{N}|c_j|^2 = 1$. When more than one component admit the same eigen-state, the corresponding soliton solutions are called degenerated solitons. Many previous reported vector solitons, such as bright-bright,[19] dark-dark soliton,[20,21] bright-bright-dark,[22] and dark-bright-bright soliton,[23] are all degenerated solitons.[24] Since the degenerated cases are just chosen from the non-degenerated soliton, we firstly focus on the non-degenerated solitons for which each component admits different eigen-state of quantum well. We will discuss degenerated soliton systemically in Section 5.

From the eigen-state $\tanh[\sqrt{f}x]$, we know that there is a dark soliton state, and it admits a constant background, and is a free state. Therefore, we introduce a parameter a to describe the plane wave background amplitude, and assume $|\psi_1|^2 + |\psi_2|^2 = a^2 + f\,\mathrm{sech}^2[\sqrt{f}x]$. Let $\psi_j(x,t) = \phi_j(x)\,\mathrm{e}^{-\mathrm{i}\mu_j t}$ and transfer the constant terms to eigenvalue terms, then the related eigen-equations can be simplified as

$$\begin{aligned} -\frac{1}{2}\frac{\partial^2\phi_1(x)}{\partial x^2} - f\,\mathrm{sech}^2[\sqrt{f}x]\phi_1(x) &= (\mu_1 + a^2)\phi_1(x), \\ -\frac{1}{2}\frac{\partial^2\phi_2(x)}{\partial x^2} - f\,\mathrm{sech}^2[\sqrt{f}x]\phi_2(x) &= (\mu_2 + a^2)\phi_2(x). \end{aligned} \tag{6}$$

The eigenvalues and eigen-states for the quantum well $-f\,\mathrm{sech}^2[\sqrt{f}x]$ are given in Table 2. We know that the fundamental eigenvalues for the fundamental state and the first excited state are $-f/2$ and 0, respectively. These characters can be used to identify the phases of dark-bright soliton, namely, $\mu_1 + a^2 = -f/2$ and $\mu_2 + a^2 = 0$.

The eigen-states correspond to $\mathrm{sech}[\sqrt{f}x]$ and $\tanh[\sqrt{f}x]$, respectively (Table 2). Therefore the eigen-state of vector soliton should have the same spatial distribution forms. But the coefficients of the soliton cannot be arbitrary because the superposition of their density should be $|\psi_1|^2 + |\psi_2|^2 = a^2 + f\,\mathrm{sech}^2[\sqrt{f}x]$, which is distinctive from linear cases for quantum well.

Therefore, we must introduce some new coefficients for them: namely, $\phi_1(x) = a_2\,\mathrm{sech}[\sqrt{f}x]$ and $\phi_2(x) = b_2\tanh[\sqrt{f}x]$. We can identify the values of a_2 and b_2 directly with the constrain condition $|\psi_1|^2 + |\psi_2|^2 = a^2 + f\,\mathrm{sech}^2[\sqrt{f}x]$. Then, the well-known static dark-bright soliton of the two-component NLS equation is obtained as follows:

$$\begin{aligned} \psi_1(x,t) &= \sqrt{f + a^2}\,\mathrm{sech}[\sqrt{f}x]\,\mathrm{e}^{\mathrm{i}(f/2 + a^2)t}, \\ \psi_2(x,t) &= a\tanh[\sqrt{f}x]\,\mathrm{e}^{\mathrm{i}a^2 t}, \end{aligned} \tag{7}$$

where a denotes the amplitude of plane wave background for dark soliton component. It is seen that the bright soliton still

corresponds to the eigenvalue $-f/2$ in the quantum well, and the dark soliton corresponds to eigenvalue zero in the quantum well. Especially, we can see that ϕ_2 admits one node in spatial distribution. From the general properties of bound state in one-dimensional potential,[17] we know that the eigen-state of dark soliton is the first-excited state in the quantum well and it is a free state. Therefore, we know that there is no other bound states among the ground state and the free state for the quantum well $-f\,\mathrm{sech}^2[\sqrt{f}x]$. Even for bright-bright-dark soliton presented before,[22,23] the eigenvalues of the two bright solitons are the same and there is no other bound state. This happens because all of the density superposition form is still a constant $-f\,\mathrm{sech}^2[\sqrt{f}x]$ form, the quantum well is identical with the one for the two-component case. Therefore bright solitons in the two components of the three-component case admit identical eigen-state of the quantum well.

From the knowledge that more bound states can emerge for much deeper potential well,[17] we can obtain more bound states though making the quantum well deeper. Therefore we introduce a multiply factor h to the quantum well, namely, $-hf\,\mathrm{sech}^2[\sqrt{f}x]$. Based on the results in Refs. [11], [18] and the textbook,[17] we know that the parameter h cannot be arbitrary, and $h = N(N-1)/2$ (where N is a positive integer larger than one), it is possible to present exact eigen-state analytically. Since we intend to establish the relations between soliton solution of NLS and eigen-problems of LS, therefore we focus on these integrable cases. Then, we investigate the $N = 3$ case. Let $\psi_j(x,t) = \phi_j(x)\,\mathrm{e}^{-\mathrm{i}\mu_j t}$ and $\sum_{j=1}^{3}|\phi_j(x)|^2 = a^2 + 3f\,\mathrm{sech}^2[\sqrt{f}x]$, the related eigen-equations can be simplified as

$$-\frac{1}{2}\frac{\partial^2\phi_j(x)}{\partial x^2} - 3f\,\mathrm{sech}^2[\sqrt{f}x]\phi_j(x) = (\mu_j + a^2)\phi_j(x), \quad (j = 1,2,3). \tag{8}$$

The eigenvalues and eigen-states for the quantum well $-3f\,\mathrm{sech}^2[\sqrt{f}x]$ are given in Table 3. We know that the eigenvalues for the ground state, the first-excited state, and the second-excited state are $-2f$, $-f/2$, and 0, respectively. These characters can be used to identify the phases of dark-bright-bright soliton, namely: $\mu_1 + a^2 = -2f$, $\mu_2 + a^2 = -f/2$, and $\mu_3 + a^2 = 0$.

Table 3. The eigenvalues and eigen-states in the quantum well $-3f\,\mathrm{sech}^2[\sqrt{f}x]$. This can be used to construct non-degenerated vector solitons for three-component coupled nonlinear Schödinger equation with attractive or repulsive interactions.

Quantum well	Eigenvalues	Eigen-states
$-3f\,\mathrm{sech}^2[\sqrt{f}x]$	0	$1-3\tanh^2[\sqrt{f}x]$
	$-f/2$	$\tanh[\sqrt{f}x]\,\mathrm{sech}[\sqrt{f}x]$
	$-2f$	$\mathrm{sech}^2[\sqrt{f}x]$

The eigen-states correspond to $\mathrm{sech}^2[\sqrt{f}x]$, $\mathrm{sech}[\sqrt{f}x]\tanh[\sqrt{f}x]$, and $(1-3\tanh^2[\sqrt{f}x])$, respectively (see the above bright-bright-dark soliton). Therefore the eigen-state of dark-bright soliton here should have the same spatial distribution forms. Similarly, we must introduce some new coefficients for them, namely, $\phi_1(x) = a_3\,\mathrm{sech}^2[\sqrt{f}x]$, $\phi_2(x) = b_3\,\mathrm{sech}[\sqrt{f}x]\tanh[\sqrt{f}x]$, and $\phi_3(x) = c_3(1-3\tanh^2[\sqrt{f}x])$. We can identify the values of a_3, b_3, and c_3 with the constrain condition $|\psi_1|^2 + |\psi_2|^2 + |\phi_3(x)|^2 = a^2 + 3f\,\mathrm{sech}^2[\sqrt{f}x]$. Then, the static vector soliton of the three-component NLS equations is obtained as follows:

$$\begin{aligned}
\psi_1(x,t) &= \sqrt{3}\sqrt{a^2/4+f}\,\mathrm{sech}^2[\sqrt{f}x]\,\mathrm{e}^{\mathrm{i}t(a^2+2f)},\\
\psi_2(x,t) &= \sqrt{3}\sqrt{a^2+f}\tanh[\sqrt{f}x]\,\mathrm{sech}[\sqrt{f}x]\,\mathrm{e}^{\mathrm{i}t(a^2+f/2)},\\
\psi_3(x,t) &= \frac{a}{2}(1-3\tanh^2[\sqrt{f}x])\,\mathrm{e}^{\mathrm{i}a^2t}.
\end{aligned} \tag{9}$$

We can see that there are a bright soliton with one peak in component ψ_1, a double-peak bright soliton in component ψ_2, and a double-valley dark soliton in component ψ_3, through plotting their density evolution. The corresponding eigen-state of the double-peak bright soliton admits one node in spatial distribution, and it is the first-exited state and is also a bound state. Its eigenvalue is $-f/2$. The corresponding eigen-state of the double-valley dark soliton admits two nodes in spatial distribution, therefore it is the second-excited state, and is a free state.

Table 4. The eigenvalues and eigen-states in the quantum well $-6f\,\mathrm{sech}^2[\sqrt{f}x]$. This can be used to construct non-degenerated vector solitons for four-component coupled nonlinear Schödinger equation with attractive or repulsive interactions.

Quantum well	Eigenvalues	Eigen-states
$-6f\,\mathrm{sech}^2[\sqrt{f}x]$	0	$\tanh[\sqrt{f}x](5\tanh^2[\sqrt{f}x]-3)$
	$-f/2$	$(5\tanh^2[\sqrt{f}x]-1)\,\mathrm{sech}[\sqrt{f}x]$
	$-2f$	$\tanh[\sqrt{f}x]\,\mathrm{sech}^2[\sqrt{f}x]$
	$-9f/2$	$\mathrm{sech}^3[\sqrt{f}x]$

Then, we investigate the $N = 4$ case. Let $\psi_j(x,t) = \phi_j(x)\,\mathrm{e}^{-\mathrm{i}\mu_j t}$ and $\sum_{j=1}^{4}|\phi_j(x)|^2 = a^2 + 6f\,\mathrm{sech}^2[\sqrt{f}x]$, the vector soliton solution of the four-component coupled NLS can be derived in a similar way. The eigenvalues and eigen-states are given in Table 4. It is seen that the eigenvalue of the fundamental state is $-9f/2$ for the quantum well $-6f\,\mathrm{sech}^2[\sqrt{f}x]$. The vector soliton solution is derived as follows:

$$\begin{aligned}
\psi_1(x,t) &= \frac{1}{2}\sqrt{\frac{5}{2}(a^2+9f)}\,\mathrm{sech}^3[\sqrt{f}x]\,\mathrm{e}^{\frac{1}{2}\mathrm{i}t(2a^2+9f)},\\
\psi_2(x,t) &= \frac{1}{2}\sqrt{15(a^2+4f)}\tanh[\sqrt{f}x]\\
&\quad\times\mathrm{sech}^2[\sqrt{f}x]\,\mathrm{e}^{\mathrm{i}t(a^2+2f)},\\
\psi_3(x,t) &= \frac{1}{2}\sqrt{\frac{3}{2}(a^2+f)}(5\tanh^2[\sqrt{f}x]-1)\\
&\quad\times\mathrm{sech}[\sqrt{f}x]\,\mathrm{e}^{\frac{1}{2}\mathrm{i}t(2a^2+f)},\\
\psi_4(x,t) &= \frac{1}{2}a\tanh[\sqrt{f}x](5\tanh^2[\sqrt{f}x]-3)\,\mathrm{e}^{\mathrm{i}a^2t}.
\end{aligned} \tag{10}$$

We can see that there are a bright soliton with one peak in component ψ_1, a double-peak bright soliton in component ψ_2, a

triple-peak bright soliton in component ψ_3, and a triple-valley dark soliton in component ψ_4, through plotting their density evolution. The results are shown in Fig. 1(a). The corresponding eigen-state of the double-peak bright soliton admits one node in spatial distribution, and it is the first-exited state and its eigenvalue is $-2f$. The corresponding eigen-state of the triple-peak bright soliton admits two nodes in spatial distribution, therefore it is the second-excited state and its eigenvalue is $-f/2$. The triple-valley dark soliton admits three nodes and it is the third-excited state and its eigenvalue is zero. The nodes characters are shown in Fig. 1(b).

It should be pointed that the above results are identical with the results in Ref. [11]. Here we demonstrate more clearly that the solitons of NLS can be related with the eigen-state in $-hf\,\mathrm{sech}^2[\sqrt{f}x]$ type quantum wells. And we further show that the eigenvalues and eigen-state can also be used to construct soliton solutions of NLS with repulsive interactions. As far as we know, this has not been discussed systemically before.

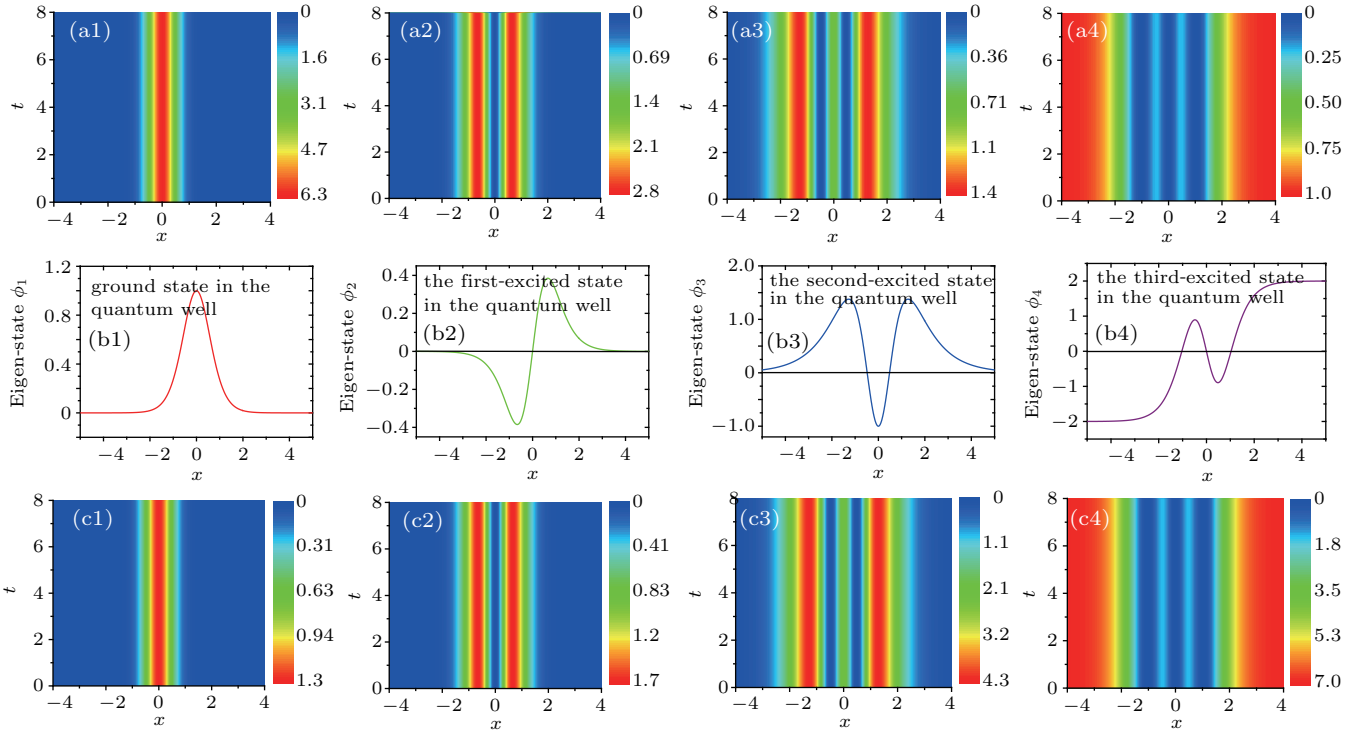


Fig. 1. The correspondence between eigen-states in the quantum well and non-degenerated solitons in 4-component coupled NLS with attractive interactions or repulsive interactions. Panels (a1)–(a4) show the evolution of solitons in the four components with attractive interactions, respectively. Panels (b1)–(b4) show the first four eigen-states in the quantum well $-6f\,\mathrm{sech}^2[\sqrt{f}x]$. Panels (c1)–(c4) show the evolution of solitons in the four components with repulsive interactions, respectively. It is shown that the density profiles of soliton correspond to the eigen-states precisely. The solitons in attractive case have different amplitudes from the ones in repulsive cases, but they admit identical eigenvalues in the quantum well. The parameters in soliton solutions are $a = 1$ and $f = 1$.

4. Non-degenerated soliton of NLS with repulsive interactions generated from eigen-states in quantum wells

For $\gamma = -1$ repulsive case, the fundamental soliton solution of the NLS is

$$\psi(x,t) = \sqrt{f}\tanh[\sqrt{f}x]\,\mathrm{e}^{-\mathrm{i}ft}, \tag{11}$$

where $f > 0$ denotes the amplitude of plane wave background for dark soliton. This static dark soliton has been given for a long time,[25] and it has been observed in many experiments.[26,27] Substituting it to NLS, we can obtain a similar eigen-problem in a quantum potential

$$-\frac{1}{2}\frac{\partial^2\phi(x)}{\partial x^2} + f\tanh^2[\sqrt{f}x]\phi(x) = f\phi(x). \tag{12}$$

It looks like that the eigenvalue of dark soliton is $f > 0$ in this form. The potential is $f\tanh^2[\sqrt{f}x]$. But the potential form can be rewritten as $f - f\,\mathrm{sech}^2[\sqrt{f}x]$ with the aid of $\tanh^2(x) = 1 - \mathrm{sech}^2(x)$. In this way, we can also make a correspondence between the dark soliton in repulsive case with the eigen-problem in a identical quantum well $-f\,\mathrm{sech}^2[\sqrt{f}x]$ with the attractive case, namely,

$$-\frac{1}{2}\frac{\partial^2\phi(x)}{\partial x^2} - f\,\mathrm{sech}^2[\sqrt{f}x]\phi(x) = 0\phi(x). \tag{13}$$

Therefore, the dark soliton in repulsive case also corresponds to zero eigenvalue of energy in the quantum well $-f\,\mathrm{sech}^2[\sqrt{f}x]$. Then, we can find the fundamental state with the help of dark-bright soliton solutions in two-component coupled NLS with repulsive interactions. This is different from the case with attractive interactions because we obtain

the first excited state and try to find the ground state for the repulsive case.

Comparing with the $|\psi_1|^2+|\psi_2|^2=a^2+f\,\mathrm{sech}^2[\sqrt{f}x]$ form in two-component coupled NLS with attractive case, we introduce a form to describe potential wells, $a^2+f\tanh^2[\sqrt{f}x]=|\psi_1|^2+|\psi_2|^2$, to find the ground state in the repulsive cases. With the help of $\tanh^2[x]=1-\mathrm{sech}^2[x]$, we can rewrite the potential form as $a^2+f\tanh^2[\sqrt{f}x]=a^2+f-f\,\mathrm{sech}^2[\sqrt{f}x]$. Similar to the attractive case, letting $\psi_j(x,t)=\phi_j(x)\mathrm{e}^{-\mathrm{i}\mu_j t}$ and transferring the constant terms to eigenvalue terms, the related eigen-equations can be simplified as

$$\begin{aligned}&-\frac{1}{2}\frac{\partial^2\phi_1(x)}{\partial x^2}-f\,\mathrm{sech}^2[\sqrt{f}x]\phi_1(x)=(\mu_1-a^2-f)\phi_1(x),\\&-\frac{1}{2}\frac{\partial^2\phi_2(x)}{\partial x^2}-f\,\mathrm{sech}^2[\sqrt{f}x]\phi_2(x)=(\mu_2-a^2-f)\phi_2(x).\end{aligned}\tag{14}$$

Form the results for the quantum well $-f\,\mathrm{sech}^2[\sqrt{f}x]$, we know that the eigenvalues for the ground state and the first-excited state are $-f/2$ and 0, respectively. These characters can be used to identify the phases of dark-bright soliton, namely, $\mu_1-a^2-f=-f/2$ and $\mu_2-a^2-f=0$. The eigen-states correspond to $\mathrm{sech}[\sqrt{f}x]$ and $\tanh[\sqrt{f}x]$, respectively (see the above bright-dark soliton). Therefore the eigen-state of dark-bright soliton here should have the same spatial distribution forms. But the coefficients cannot be the same because the superposition of their density should be $|\psi_1|^2+|\psi_2|^2=a^2+f\tanh^2[\sqrt{f}x]$ which is distinctive from the attractive cases.

Therefore we must introduce some new coefficients for them, namely, $\phi_1(x)=a_2\,\mathrm{sech}[\sqrt{f}x]$ and $\phi_2(x)=b_2\tanh[\sqrt{f}x]$. We can identify the values of a_2 and b_2 directly with the constrain condition $|\psi_1|^2+|\psi_2|^2=a^2+f\tanh^2[\sqrt{f}x]$. Then, the well-known static dark-bright soliton of the two-component NLS equation is obtained as follows:

$$\begin{aligned}\psi_1(x,t)&=a\,\mathrm{sech}[\sqrt{f}x]\,\mathrm{e}^{\mathrm{i}(f/2-a^2-f)t},\\\psi_2(x,t)&=\sqrt{f+a^2}\tanh[\sqrt{f}x]\,\mathrm{e}^{-\mathrm{i}(a^2+f)t},\end{aligned}\tag{15}$$

where $\sqrt{f+a^2}$ denotes the amplitude of plane wave background for dark soliton component. It is shown that the dark-bright soliton form for repulsive interactions is distinctive from the bright-dark soliton for attractive cases. However, they are related with the eigenvalue in the same quantum well $-f\,\mathrm{sech}^2[\sqrt{f}x]$. This provides a possible way to construct the static vector soliton with repulsive case from the vector soliton with attractive interactions.

Similar to the attractive case for more eigen bound states, we further deep the potential well $a^2+3f\tanh^2[\sqrt{f}x]=|\psi_1|^2+|\psi_2|^2+|\psi_3|^2$. Letting $\psi_j(x,t)=\phi_j(x)\mathrm{e}^{-\mathrm{i}\mu_j t}$ and transferring the constant terms to eigenvalue terms, the related eigen-equations can be simplified as $-\frac{1}{2}\frac{\partial^2\phi_j(x)}{\partial x^2}-3f\,\mathrm{sech}^2[\sqrt{f}x]\phi_1(x)=(\mu_j-a^2-3f)\phi_j(x)$ $(j=1,2,3)$. Then the static vector soliton of the three-component NLS equations is obtained as follows:

$$\begin{aligned}\psi_1(x,t)&=\frac{1}{2}\sqrt{3(a^2-f)}\,\mathrm{sech}^2[\sqrt{f}x]\,\mathrm{e}^{-\mathrm{i}t(a^2+f)},\\\psi_2(x,t)&=\sqrt{3(a^2+2f)}\tanh[\sqrt{f}x]\,\mathrm{sech}[\sqrt{f}x]\,\mathrm{e}^{-\mathrm{i}t(a^2+\frac{5f}{2})},\\\psi_3(x,t)&=\frac{1}{2}\sqrt{(a^2+3f)}\\&\quad\times(1-3\tanh^2[\sqrt{f}x])\,\mathrm{e}^{-\mathrm{i}t(a^2+3f)}.\end{aligned}\tag{16}$$

We emphasize that this solution is quite distinctive from the ones reported previously.[22,23] The solutions are similar to the ones in Ref. [28].

In a similar way, we can obtain the vector soliton in the four-component coupled NLS with repulsive case as follows from further deepened potential well $a^2+6f\tanh^2[\sqrt{f}x]=|\psi_1|^2+|\psi_2|^2+|\psi_3|^2+|\psi_4|^2$:

$$\begin{aligned}\psi_1(x,t)&=\frac{1}{2}\sqrt{\frac{5}{2}(a^2-3f)}\,\mathrm{sech}^3[\sqrt{f}x]\,\mathrm{e}^{-\mathrm{i}(a^2+3f/2)t},\\\psi_2(x,t)&=\frac{1}{2}\sqrt{15(a^2+2f)}\tanh[\sqrt{f}x]\\&\quad\times\mathrm{sech}^2[\sqrt{f}x]\,\mathrm{e}^{-\mathrm{i}(a^2+4f)t},\\\psi_3(x,t)&=\frac{1}{2}\sqrt{\frac{3}{2}(a^2+5f)}(5\tanh^2[\sqrt{f}x]-1)\\&\quad\times\mathrm{sech}[\sqrt{f}x]\,\mathrm{e}^{-\mathrm{i}(a^2+11f/2)t},\\\psi_4(x,t)&=\frac{1}{2}\sqrt{a^2+6f}\tanh[\sqrt{f}x]\\&\quad\times(5\tanh^2[\sqrt{f}x]-3)\,\mathrm{e}^{-\mathrm{i}(a^2+6f)t}.\end{aligned}\tag{17}$$

The density evolution is shown in Fig. 1(c). The corresponding eigen-state is shown in Fig. 1(b). The correspondence between soliton excitation and eigenvalues of quantum well is summarized in Table 5.

Table 5. The correspondence between solions in 4-component coupled NLS with repulsive interactions and the eigenvalues in quantum well $-6f\,\mathrm{sech}^2[\sqrt{f}x]$. This can be used to construct degenerated vector solitons for N-component ($N>4$) coupled nonlinear Schödinger equation with repulsive interactions.

Quantum well	Eigenvalues	Soliton in NLS
$-6f\,\mathrm{sech}^2[\sqrt{f}x]$	0	triple-valley dark soliton
	$-f/2$	triple-hump bright soliton
	$-2f$	double-hump bright soliton
	$-9f/2$	single-hump bright soliton

Furthermore, we establish the correspondence between non-degenerated solitons and eigen-states in the quantum well $-N(N-1)/2f\,\mathrm{sech}[\sqrt{f}x]$ to generalized N-component coupled NLS equation with attractive or repulsive interactions. The results are summarized in Table 1. This is the main result in this paper.

It should be pointed out that many soliton solutions re-derived from the eigen-states have been presented by

Darboux-transformation, Hirota bilinear method, which usually admits $\mathrm{sech}^1(x)$ form even for the multi-component coupled cases. Also we emphasize that many multi-soliton complexes have been given in Refs. [11] and [14], which were also derived from the solutions of a LS with adding some constrain conditions. But the correspondence between eigen-states of LS and NLS with attractive or repulsive interactions were not addressed systemically. In particular, we show that solitons of the arbitrary N-component NLS with repulsive interactions have identical eigenvalues with the ones of N-component NLS with attractive interactions. Furthermore, there is a special character for dark solitons of N-component NLS with both attractive and repulsive interactions, namely, they always correspond to the zero energy eigenvalue in the quantum well, with a proper reference for which the free plane wave admits zero kinetic energy. These underlying characters have not been uncovered clearly in the previous studies, and they enable us to obtain soliton solutions of N-component coupled NLS with repulsive interactions from the ones in NLS with attractive interactions. It is well known that Darboux transformation forms are different for the NLS with repulsive and attractive interactions.[20] Some authors have even tried to derive dark solitons by Hirota method,[21,23] or the KP-hierarchy reduction method.[24] Therefore, these underlying characters are non-trivial and enable us to derive vector solitons more conveniently. Based on the non-degenerated soliton presented above, we can obtain very abundant vector solitons directly; such as degenerated solitons and beating solitons.

5. Degenerated vector solitons generated from non-degenerated solitons

From the above non-degenerated soliton solution, we know that the soliton solution of M-component coupled NLS equation can be used to construct much abundant vector solitons in N-component coupled cases with $N > M$. Obviously, larger M component number will generate more abundant vector soliton types. As an example, we show explicitly how many vector solitons for 4-component coupled cases can be generated from the 3-component cases.

There are mainly three different cases for degenerated soliton in 4-component NLS which are generated from the non-degenerated solitons in three-component NLS because there are at most two degenerated components in this case. First, if there are two components degenerated in the fundamental state, then the vector solitons for the repulsive interaction case can be written as

$$\psi_j = c_j \frac{1}{2}\sqrt{3(a^2-f)}\,\mathrm{sech}^2[\sqrt{f}x]\,\mathrm{e}^{-\mathrm{i}t(a^2+f)}, \quad (j=1,2),$$
$$\psi_3 = \sqrt{3(a^2+2f)}\tanh[\sqrt{f}x]\,\mathrm{sech}[\sqrt{f}x]\,\mathrm{e}^{-\mathrm{i}t(a^2+\frac{5f}{2})},$$
$$\psi_4 = \frac{1}{2}\sqrt{(a^2+3f)}(1-3\tanh^2[\sqrt{f}x])\,\mathrm{e}^{-\mathrm{i}t(a^2+3f)}.$$

The constrain condition is $|c_1|^2+|c_2|^2=1$ because the superposition of them must be $a^2+3f\,\mathrm{sech}^2[\sqrt{f}x]$. There are three bright solitons in three components separately and one dark soliton in the fourth component, which is similar to the case generated from the dark-bright soliton in the two-component case. But the soliton profiles are different from the ones obtained above, namely, there are double-hump soliton and double-valley dark soliton in this case. Second, if there are two components degenerated in the first excited state, then the vector solitons for the repulsive interaction case can be written as

$$\psi_1 = \frac{1}{2}\sqrt{3(a^2-f)}\,\mathrm{sech}^2[\sqrt{f}x]\,\mathrm{e}^{-\mathrm{i}(a^2+f)t},$$
$$\psi_j = c_j\sqrt{3(a^2+2f)}\tanh[\sqrt{f}x]\,\mathrm{sech}[\sqrt{f}x]\,\mathrm{e}^{-\mathrm{i}(a^2+\frac{5f}{2})t},$$
$$(j=2,3),$$
$$\psi_4 = \frac{1}{2}\sqrt{(a^2+3f)}\,(1-3\tanh^2[\sqrt{f}x])\,\mathrm{e}^{-\mathrm{i}(a^2+3f)t},$$

with the constrain condition $|c_2|^2+|c_3|^2=1$. Third, if there are two components degenerated in the second excited state, then the vector solitons for the repulsive interaction case can be written as

$$\psi_1 = \frac{1}{2}\sqrt{3(a^2-f)}\,\mathrm{sech}^2[\sqrt{f}x]\,\mathrm{e}^{-\mathrm{i}(a^2+f)t},$$
$$\psi_2 = \sqrt{3(a^2+2f)}\tanh[\sqrt{f}x]\,\mathrm{sech}[\sqrt{f}x]\,\mathrm{e}^{-\mathrm{i}(a^2+\frac{5f}{2})t},$$
$$\psi_j = c_j\frac{1}{2}\sqrt{(a^2+3f)}\,(1-3\tanh^2[\sqrt{f}x])\,\mathrm{e}^{-\mathrm{i}(a^2+3f)t},$$
$$(j=3,4),$$

with the constrain condition $|c_3|^2+|c_4|^2=1$. The third case corresponds to that there are one bright soliton with a single hump in one component, one bright soliton with two humps in one component, and two dark solitons with two valleys in the other two components separately. As far as we know, these types of solitons have not been derived by Darboux transformation and other methods,[19–21,23,24] and even in Refs. [11] and [14].

It is well-known that the arbitrary linear superposition of eigen-states of LS equation with the quantum well is still the solution of the LS equation. Therefore the linear superposition of the above non-degenerated and degenerated soliton solutions will also be used to generated localized waves. However, there are some constrain conditions on the superposition coefficients for the NLS equation because the effective potential in NLS depends on the wave functions. The linear superposition forms are generated from these eigen-states, but they demonstrate striking dynamics characters. The beating dark-dark soliton was obtained by the similar $SU(2)$ symmetry.[29,30] Based on the above results, we know that the beating effects are induced by the eigenvalue difference between the solitons.[31] In fact, this simple idea can be used to generated very abundant localized waves with beating effects.

Explicitly, N-component coupled NLS with equal nonlinear coefficients admits $SU(N)$ symmetry, and the related unitary matrix can be used to construct beating solitons. Especially, the forms of unitary matrix can be chosen to admit $SU(M)$ ($M \leq N$) symmetry, which enables us to obtain very abundant different beating patterns.[31]

These solutions are all derived from the eigen-states in quantum wells. The corresponding relations deep our understanding on the eigenvalues of solitons and the relations between solitons in attractive and repulsive interaction cases. Then, we would like to show that the soliton solution of NLS could also help us to obtain eigen-state and eigenvalues in more complicated quantum wells.

6. Eigen-state in a complicated quantum well generated from soliton excitation of nonlinear equations

We consider the following LS equation with a more complicated well

$$-\frac{1}{2}\frac{\partial^2\phi(x)}{\partial x^2}+[U(x)+\mathrm{i}V(x)]\phi(x)=\mu\phi(x), \qquad (18)$$

where $U(x) = -f\,\mathrm{sech}^2[\sqrt{f}x](2\beta^2 f\,\mathrm{sech}^2[\sqrt{f}x]+1)$ and $V(x) = -4\beta f^{3/2}\tanh[\sqrt{f}x]\,\mathrm{sech}^2[\sqrt{f}x]$. Their profiles are shown in Fig. 2. Obviously, the corresponding Hamiltonian belongs to the non-Hermitian Hamiltonian having parity–time symmetry as defined in Refs. [32]–[34]. It is hard to solve it directly because the quantum well is much more complicated than the ones discussed above. We note that the eigenvalue problem can be related with a NLS with some high-order effects (the well-known Kundu–Eckhaus equation),[35,36] namely,

$$\mathrm{i}\frac{\partial\psi(x,t)}{\partial t}+\frac{1}{2}\frac{\partial^2\psi(x,t)}{\partial x^2}+|\psi(x,t)|^2\psi(x,t)$$
$$+2\beta^2|\psi(x,t)|^4\psi(x,t)-2\mathrm{i}\beta|\psi(x,t)|_x\psi(x,t)=0.$$

Based on the soliton solution derived by Darboux transformation method,[37] we can represent the soliton as

$$\psi(x,t) = \sqrt{f}\,\mathrm{sech}[\sqrt{f}x]$$
$$\times\exp\left[\mathrm{i}2\beta\sqrt{f}\exp(\sqrt{f}x)\,\mathrm{sech}(\sqrt{f}x)+\mathrm{i}ft/2\right].$$

Then, we can obtain the eigenvalue of the parity-time symmetry potential as $\mu = -f/2$, and the corresponding eigen-state is

$$\phi(x) = \sqrt{f}\,\mathrm{sech}[\sqrt{f}x]$$
$$\times\exp\left[\mathrm{i}2\beta\sqrt{f}\exp(\sqrt{f}x)\,\mathrm{sech}[\sqrt{f}x]\right]. \qquad (19)$$

We can see clearly that the non-Hermitian Hamiltonian having parity–time symmetry admits a real eigenvalue. The soliton state is the fundamental eigen-state in the quantum well. The excited states need more systematic analysis with coupled models. We will discuss on this subject in a separated paper. The similar discussions can be made to obtain more complicated quantum wells, even for NLS with parity–time symmetry potential cases.[38–40]

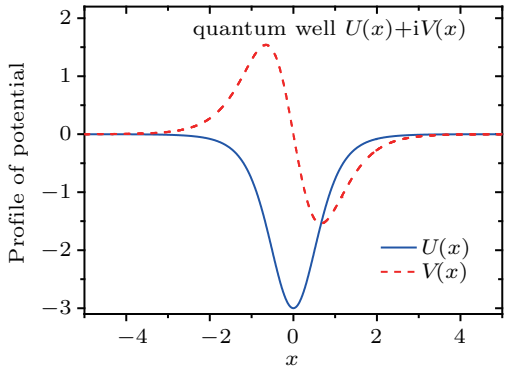


Fig. 2. The profiles of quantum well which makes the Hamiltonian belong to the non-Hermitian Hamiltonian having parity–time symmetry. The quantum well admits the profile $U(x) = -f\,\mathrm{sech}^2[\sqrt{f}x](2\beta^2 f\,\mathrm{sech}^2[\sqrt{f}x]+1)$ and $V(x) = -4\beta f^{3/2}\tanh[\sqrt{f}x]\,\mathrm{sech}^2[\sqrt{f}x]$, for which it is hard to solve the Hamiltonian directly. But the corresponding eigen-state and eigenvalue can be obtained from the soliton solution of a nonlinear partial equation which can be solved by Darboux transformation. The parameters are $\beta = 1$ and $f = 1$.

Moreover, the asymmetric soliton solution of the coupled NLS reported in Ref. [41] can be used to construct eigen-states of an asymmetric quantum double-well. For example, we show a simple case to present the ground state and the first exited state in an asymmetric quantum double-well. The corresponding quantum well admits the following form:

$$-\frac{1}{2}\frac{\partial^2\phi(x)}{\partial x^2}+[V_{\mathrm{dw}}(x)]\phi(x)=\mu\phi(x), \qquad (20)$$

where $V_{\mathrm{dw}}(x) = -\frac{2(\sqrt{f}+\sqrt{f+1})}{(\sqrt{f+1}-\sqrt{f})H^2(x)}[f\cosh[2\sqrt{f+1}(x-h)]+(f+1)\cosh[2\sqrt{f}(h+x)]+1]$ describes a double-well with variable profiles, and $H(x) = \cosh[\sqrt{f+1}(x-h)+\sqrt{f}(h+x)]+\sqrt{\frac{\sqrt{f}+\sqrt{f+1}}{\sqrt{f+1}-\sqrt{f}}}\cosh[\sqrt{f+1}(x-h)-\sqrt{f}(h+x)]$. When $f = 1$ and $h = 2$, the quantum potential form will admit an asymmetric double-well, and is shown in Fig. 3. It is hard to solve the LS with this potential. However, the eigen-states can be given from soliton states in Ref. [41], namely,

$$\phi_1(x) = \frac{2\sqrt{f+1}\sqrt{\frac{\sqrt{f}+\sqrt{f+1}}{\sqrt{f+1}-\sqrt{f}}}\cosh\left(\sqrt{f}(h+x)\right)}{H(x)}, \qquad (21)$$

$$\phi_2(x) = \frac{2\sqrt{f}\sqrt{\frac{\sqrt{f}+\sqrt{f+1}}{\sqrt{f+1}-\sqrt{f}}}\sinh\left(\sqrt{f+1}(x-h)\right)}{H(x)}, \qquad (22)$$

which are the ground state and the first-excited state in the quantum well. The eigenvalues of the eigen-states are $-(f+1)/2$ and $-f/2$, respectively. These results show clearly that the simple relations between solitons and eigen-states in quantum well indeed can help us to solve some complicate problems.

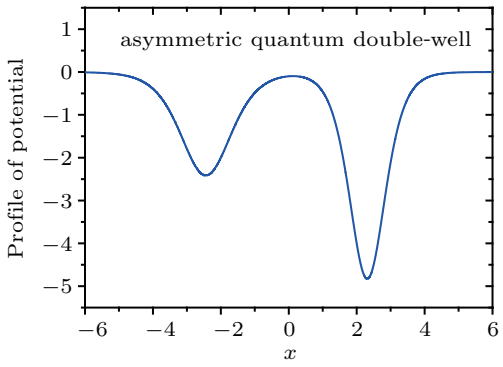


Fig. 3. The profiles of an asymmetric quantum double-well, for which it is hard to solve the Hamiltonian directly. But the corresponding eigen-state and eigenvalue can be obtained from the soliton solution of a nonlinear partial equation which is solved by Darboux transformation. The parameters are $f = 1$ and $h = 2$.

7. Discussion and conclusion

A systemic relation is established between the eigen-state in quantum wells and the soliton solution of multi-component coupled NLS with attractive or repulsive interactions. It is shown that static soliton excitation of N-component coupled nonlinear Schödinger equation corresponds to the first N eigen-states in a $-\frac{N(N-1)}{2} f \operatorname{sech}^2(fx)$ type potential. The corresponding energy eigenvalues are calculated exactly. Explicitly, $N-1$ soliton states on zero background have negative energy eigenvalues and are all bound states, the dark solitons always correspond to the eigen-states with zero energy eigenvalue in the quantum wells. The characters holds well for both attractive and repulsive nonlinear interaction cases. The above results are summarized in Table 1. These results enable us to understand the soliton solution of NLS with attractive and repulsive interactions in a unified way.

Abundant degenerated solitons can be derived directly from the non-degenerated soliton solutions. Degenerated soliton for N-component coupled NLS with attractive or repulsive interactions can be generated from the non-degenerated solitons of M-component coupled NLS, even with $N > M$. Moreover, the linear superposition principle in quantum mechanics can be used to construct very abundant vector solitons with beating effects in the systems with some certain constrain conditions on the superposition coefficients. The beating behavior of solitons can also be extended to the high-dimensional systems.[42–44]

Although these soliton solutions are all static, the related solitons with velocities can be obtained directly by Galilean transformation for the NLS equations admit Galilean symmetry. We emphasize that the analytical solution describing interactions between these solutions can not be derived from the eigen-states in quantum wells. But these solitons with different velocities can be chosen as the initial states to simulate soliton interaction numerically because when solitons are well separated, they can be seen as the linear superposition of solitons with different velocities approximately.[45,46] For example, the collisions between solitons with nodes were investigated in Ref. [47], which demonstrated that they were robust to collisions and even noises. Additionally, a complicated method was proposed to study the collision of these solitons with nodes analytically,[41] with the aid of results in Ref. [10]. However, we have not discussed this subject in this paper. As far as we know, the solitons with more than one nodes have not been derived by the well-known Darboux transformation method,[15,16,20] Hirota bilinear method,[19,21,23] and even other methods.[24] Some of our results are similar to the soliton solutions with more than one nodes presented before,[11,14] which were also generated from the eigen-states of LS. However, we focus on the correspondence relations between solitons in nonlinear systems and eigen-states in quantum wells. The degenerated solitons or beating solitons generated from the eigen-states have not been addressed systemically before either.

We also demonstrate that soliton solution of nonlinear partial equation can be used to solve the eigen-problems of quantum wells. As an example, we present an eigenvalue and its eigen-state in a complicated quantum well for which the Hamiltonian belongs to the non-Hermitian Hamiltonian having parity–time symmetry. We further present the ground state and the first exited state in an asymmetric quantum double-well from asymmetric solitons. This paves the ways to solve the LS with complicated potentials with the aid of well-developed techniques for solving NLS equations. This is very meaningful for LS with more complicated quantum wells which is hard to be solved directly because the exact solutions of LS equation with different potentials play an important role in mathematics and physics.[48–50] These relations suggest that the stability of soliton states is related with the evolution of eigen-states in quantum wells directly. The stability of moving solitons can also be understood by the eigen-states in moving localized quantum wells. This provides another way to understand the stability of solitons in NLS described systems.

As an example, we discuss the possibilities to observe these quantum states in BEC systems.[51] Soliton interactions in BEC systems have been demonstrated widely in real experiments.[52–55] The experimental techniques can be used to produce the initial conditions for observing these soliton states, and the bound states or quasi-bound states have been widely shown to be robust. Very recently, a dark-bright-bright soliton was experimentally demonstrated in spinor BEC systems.[56] It is noted that the dark soliton component admits one node and the two bright soltion components admit identical eigen-state and they are degenerated. Therefore the observed solitons correspond to the degenerated soliton case here. In fact, the vector soliton in the three-component coupled systems can admit non-degenerated solitons which are

presented above. The dark soliton in one component can admit a double-valley structure, and one bright soliton component can also admit a node which makes the bright soliton admit double-hump. These are quite different from the ones observed in Ref. [56]. Therefore, the results here predict much more abundant soliton profiles in the coupled NLS described systems.[57] Many different types of beating solitons were also demonstrated and more beating patterns were expected.[31] The results here can be used to construct more exotic beating solitons with the help of symmetry properties. The evolution of quantum states could be observed in some classical systems, such as nonlinear fiber, water wave tank, etc. because the NLS describe many different classical systems [58] and a generalized hydrodynamics could provide a remarkable quantum–classical equivalence.[59]

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