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Average fidelity estimation of twirled noisy quantum channel using unitary $2t$-design*

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We propose a method to estimate the average fidelity using the unitary $2t$-design of a twirled noisy channel, which is suitable for large-scale quantum circuits. Compared with the unitary 2-design in randomized benchmarking, the unitary $2t$-design for the twirling of noisy channels is more flexible in construction and can provide more information. In addition, we prove that the proposed method provides an efficient and reliable estimation of the average fidelity in benchmarking multistage quantum gates and estimating the weakly gate- and time-dependent noise. For time-dependent noise, we provide a scheme of moment superoperator to analyze the noise in different experiments. In particular, we give a lower bound on the average fidelity of a channel with imperfect implementation of benchmarking and state preparation and measurement errors (SPAM).

Keywords: quantum channel, average fidelity, unitary $2t$-design

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1. Introduction

The fault-tolerant quantum error correction code (FTQEC)[1,2] is necessary for a quantum computer to reliably operate and obtain an exponential speed-up over the classical algorithm in some fields. The FTQEC requires that the operation of the quantum process is of high quality and that the error is below a certain threshold.[3] One of the main challenges for FTQEC is to characterize the noise that affects a quantum system. Particularly, quantum process tomography (QPT)[4–6] is a method to analyze the noise of a quantum channel.[7] However, QPT is infeasible for large quantum systems. Fortunately, the evaluation of the average fidelity[8,9] is an efficient method to partially characterize the noise of quantum channels or gates.[10,11] The advantage of the average fidelity estimation method[12] is that it does not require large amounts of resources and is insensitive to state preparation and measurement errors (SPAM).[13]

Randomized benchmarking[14–23] is an experimental scheme to measure the average fidelity. In this scheme, the average fidelity of a completely positive trace-preserving (CPTP) noise channel is measured by using a set of unitary operators that approximate the entire Clifford group, which is called a unitary 2-design.[24] A unitary 2-design is a special case of the unitary t-design[25] for $t = 2$. For $t = 1$, it corresponds to the case of a private quantum channel,[26] whereas $t = 4$ corresponds to the conditions of the state-distinction problem.[27,28]

In this paper, we present the first systematic analysis of the average fidelity estimation using the unitary $2t$-design for a twirled t-tensor product noisy channel. Then, we extend this method to the scenario of $(m+1)$-stage noise channels. Furthermore, we analyze the $(m+1)$-stage sequence of quantum gates under their corresponding noise, such as gate-independent and time-independent noise, weakly gate-dependent and time-independent noise and gate-independent and weakly time-dependent noise. It can be recognized as the estimation of average gateset fidelity for quantum circuits. In particular, we give a lower bound on the average fidelity of the twirled channel under the unitary error caused by nonideal experimental implementations and SPAM.

The method of using unitary $2t$-design for the twirling of noises has the following advantages. First, compared to the unitary 2-design in randomized benchmarking, the unitary $2t$-design for the twirling of noises is more general to evaluate the average fidelity and provides more information about the noise. Here, t corresponds to the high dimensionality of a channel or the experiment times using the method of moment superoperator.[29] Second, using the method of moment superoperator, it is convenient to provide the analysis of the time-dependent noise. Finally, a unitary design is similar to a spherical design in terms of Jamiolkowski isomorphism[30,31] and the frame potential.[32,33] Some constructions of a spherical $2t$-design can be used in the unitary $2t$-design to estimate the average fidelity. Compared with the Clifford group in the previous randomized benchmarking, the construction of the unitary $2t$-design for the twirling of noises is more flexible

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and extensible.

The paper is organized as follows. In Section 2, we show the process of average fidelity estimation using the unitary $2t$-design to twirl a t-tensor product noisy channel. Then, we introduce the method of moment superoperator to estimate the time-dependent noise estimation. In Section 3, we extend the fidelity estimation of an $(m+1)$-stage noise channel using unitary $2t$-design for twirling of channels. We also analyze the $(m+1)$-stage sequence of quantum gates under their corresponding noise, such as gate-independent and time-independent noise, weakly gate-dependent and time-independent noise and gate-independent and weakly time-dependent noise. In Section 4, we provide a bound on the unitary error caused by nonideal experimental implementations and SPAM. Finally, we conclude the paper in Section 5.

2. Unitary $2t$-design for twirling of a channel

Consider a completely positive trace-preserving (CPTP) noise channel as shown in Fig. 1.

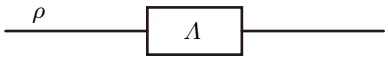


Fig. 1. A CPTP quantum channel for any d^n-dimensional input density operator $\rho = |\psi\rangle\langle\psi|$.

We assume that the CPTP noise channel can be written as

$$\Lambda(\rho) = \sum_{k=1}^{d^n} A_k \rho A_k^\dagger, \tag{1}$$

with $A_k \in L(\mathbb{C}^{d^n})$ being the linear operator for any d^n-dimensional input density operator $\rho = |\psi\rangle\langle\psi|$, where $L(\mathbb{C}^{d^n})$ is the space of d^n-dimensional complex linear operators. We define the superoperator representation of the CPTP noise channel Λ based on Ref. [34], i.e.,

$$\hat{\Lambda} = \sum_{j,k=1}^{d^n} A_k \otimes A_j^*, \tag{2}$$

where the superoperator $\hat{\Lambda}$ has a d^{2n}-dimensional matrix representation and $\mathrm{Tr}(\hat{\Lambda})$ denotes the usual sum over the diagonal elements of the matrix. The advantage of such representation is that the cascade of two arbitrary d^n-dimensional channels, Λ_1 and Λ_2, corresponds to their matrix multiplication, i.e.,

$$\Lambda_1 \circ \Lambda_2 \to \hat{\Lambda}_1 \hat{\Lambda}_2. \tag{3}$$

Then, the average channel fidelity[8] can be defined by

$$\begin{aligned} \overline{F}(\Lambda) &= \mathbb{E}_\psi \left[F\left(|\psi\rangle, \Lambda, E_{|\psi\rangle}\right)\right] \\ &= \int \langle\psi|\Lambda(|\psi\rangle\langle\psi|)|\psi\rangle \mu_{\mathrm{Haar}}(\mathrm{d}\psi), \end{aligned} \tag{4}$$

where $F(|\psi\rangle, \Lambda, E_{|\psi\rangle})$ is with respect to the fidelity with input state $|\psi\rangle$, CPTP noise channel Λ and positive-operator valued measure (POVM) element $E_{|\psi\rangle} = |\psi\rangle\langle\psi|$; $\mu_{\mathrm{Haar}}(\cdot)$ is a uniform distribution; and the integrals are with respect to the unitarily invariant Haar measure. The above average channel fidelity can be recognized as the probability that the output state $\Lambda(|\psi\rangle\langle\psi|)$ passes the test of being the input state $|\psi\rangle$ averaged over all input states. Note that it is insensitive to the SPAM, if the POVM element $E_{|\psi\rangle} = |\psi\rangle\langle\psi|$ is perfect.

Now, we define the unitary t-design as follows. Let t be a natural number and $\mathcal{U}(d^n)$ be the set of unitary operators in d^n-dimensional Hilbert space. A finite subset $\{U_i\}_{i=1}^D \subset \mathcal{U}(d^n)$ is called a unitary t-design if

$$\frac{1}{D}\sum_{i=1}^{D} p_{(t,t)}(U_i) = \int_{\mathcal{U}(d^n)} p_{(t,t)}(U)\mu_{\mathrm{Haar}}(\mathrm{d}U) \tag{5}$$

holds for any polynomial $p_{(t,t)}(U)$ homogeneous of degree t in the matrix elements of $U \in \mathcal{U}(d^n)$ and their complex conjugates of U^*. We can use the superoperator representation of unitary operators to redefine the unitary t-design $\{U_i\}_{i=1}^D$, i.e.,

$$\frac{1}{D}\sum_{i=1}^{D} \hat{U}_i^{\otimes t} = \int_{\mathcal{U}(d^n)} \hat{U}^{\otimes t}\mu_{\mathrm{Haar}}(\mathrm{d}U), \tag{6}$$

where $\hat{U}^{\otimes t} = U^{\otimes t} \otimes (U^*)^{\otimes t}$.

Then, we consider a t-tensor product CPTP noise channel model using the unitary $2t$-design to estimate the corresponding average fidelity in Fig. 2. The noise channel $\Lambda^{\otimes t}$ is a $d^{tn} \times d^{tn}$ matrix composed of t d^n-dimensional CPTP noise channels. In addition, if there exists a d^l-dimensional CPTP noise channel Λ_j for $l < n$, we can apply a d^{n-l}-dimensional ancillary quantum system to fit the model. Then, we set t-copies of the d^n-dimensional unitary operations U and $U^\dagger$ randomly chosen from a unitary $2t$-design at both ends of the noise channels. Note that the above model can be used to estimate the average fidelity of a non-local noise channel, when $t = 1$.

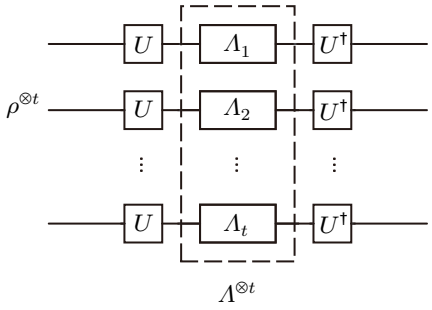


Fig. 2. A t-tensor product CPTP noise channel model using unitary $2t$-design is composed of three linear mappings $\Lambda^{\otimes t}$, $U^{\otimes t}$, and $(U^\dagger)^{\otimes t}$. For any d^{tn}-dimensional input density operator $\rho^{\otimes t}$, the representation of the unitary operation $\hat{U}^{\otimes t} = U^{\otimes t} \otimes (U^\dagger)^{\otimes t}$ and the noise channel $\hat{\Lambda}^{\otimes t} = \sum_{j,k=1}^{d^n} A_k^{\otimes t} \otimes (A_j^{\otimes t})^* = (\sum_{k=1}^{d^n} \bigotimes_{i=1}^t A_{k,i}) \otimes (\sum_{j=1}^{d^n} \bigotimes_{m=1}^t A_{j,m})^*$ are $d^{2tn} \times d^{2tn}$ matrices, where $\bigotimes_{i=1}^t A_{k,i} = A_{k,1} \otimes A_{k,2} \otimes \cdots \otimes A_{k,t}$.

If a set of unitary operators $\{U_i\}_{i=1}^D$ is a unitary $2t$-design to twirl a CPTP noise channel $\Lambda^{\otimes t}(\rho^{\otimes t}) =$

$\sum_{k=1}^{d^n} A_k^{\otimes t}\rho^{\otimes t}(A_k^{\otimes t})^\dagger$, it satisfies

$$\frac{1}{D}\sum_{i=1}^{D}(U_i^\dagger)^{\otimes t}\Lambda^{\otimes t}(U_i^{\otimes t}\rho^{\otimes t}(U_i^\dagger)^{\otimes t})U_i^{\otimes t} = \int_{\mathcal{U}(d^n)}(U^\dagger)^{\otimes t}\Lambda^{\otimes t}(U^{\otimes t}\rho^{\otimes t}(U^\dagger)^{\otimes t})U^{\otimes t}\mu_{\rm Haar}(\mathrm{d}U), \quad (7)$$

where $A_k^{\otimes t}\in L(\mathbb{C}^{d^{tn}})$ is a linear operator in the d^{tn}-dimensional Hilbert space satisfying $A_k^{\otimes t}=\bigotimes_{i=1}^t A_{k,i}=A_{k,1}\otimes A_{k,2}\otimes\cdots\otimes A_{k,t}$. Let $\Delta_{\mu_{\rm Haar},2t}(\cdot)$ be the Haar-averaged mapping using unitary $2t$-design for a twirled noise channel, i.e.,

$$\Delta_{\mu_{\rm Haar},2t}(\rho_0^{\otimes t})=\int_{\mathcal{U}(d^n)}(U^\dagger)^{\otimes t}\Lambda^{\otimes t}(U^{\otimes t}\rho_0^{\otimes t}(U^\dagger)^{\otimes t})U^{\otimes t}\mu_{\rm Haar}(\mathrm{d}U), \quad (8)$$

where $\rho_0^{\otimes t}=|\psi_0\rangle^{\otimes t}\langle\psi_0|^{\otimes t}$ is a fixed density operator in d^{tn}-dimensional Hilbert space. Then, the Haar-average superoperator representation can be presented as

$$\begin{aligned}\hat{\Delta}_{\mu_{\rm Haar},2t} &= \int_{\mathcal{U}(d^n)}\hat{U}^{\otimes t}\hat{\Lambda}^{\otimes t}(\hat{U}^\dagger)^{\otimes t}\mu_{\rm Haar}(\mathrm{d}U)\\ &= \int_{\mathcal{U}(d^n)}\left[\sum_{k=1}^{d^n}\bigotimes_{i=1}^{t}UA_{k,i}U^\dagger\right]\\ &\quad\otimes\left[\sum_{j=1}^{d^n}\bigotimes_{m=1}^{t}UA_{j,m}U^\dagger\right]^*\mu_{\rm Haar}(\mathrm{d}U), \quad (9)\end{aligned}$$

where the superoperator representation of t-tensor product noise channel is represented as $\hat{\Lambda}^{\otimes t}=\sum_{j,k=1}^{d^n}A_k^{\otimes t}\otimes(A_j^{\otimes t})^*=(\sum_{k=1}^{d^n}\bigotimes_{i=1}^t A_{k,i})\otimes(\sum_{j=1}^{d^n}\bigotimes_{m=1}^t A_{j,m})^*$ and $\bigotimes_{i=1}^t A_{k,i}=A_{k,1}\otimes A_{k,2}\otimes\cdots\otimes A_{k,t}$.

Therefore, the average fidelity of the twirled channel in Fig. 2 is given by

$$\begin{aligned}\overline{F}(\Delta_{\mu_{\rm Haar},2t}) &= \mathbb{E}_\psi\left[F\left(|\psi\rangle^{\otimes t},\Lambda^{\otimes t},E_{|\psi\rangle^{\otimes t}}\right)\right]\\ &= \int\langle\psi|^{\otimes t}\Lambda^{\otimes t}\left(|\psi\rangle^{\otimes t}\langle\psi|^{\otimes t}\right)|\psi\rangle^{\otimes t}\mu_{\rm Haar}(\mathrm{d}\psi)\\ &= \mathbb{E}_U\left[F\left(|\psi_0\rangle^{\otimes t},U^{\otimes t}\circ\Lambda^{\otimes t}\circ(U^\dagger)^{\otimes t},E_{|\psi_0\rangle^{\otimes t}}\right)\right]\\ &= \int_{\mathcal{U}(d^n)}\langle\psi_0|^{\otimes t}(U^\dagger)^{\otimes t}\Lambda^{\otimes t}\left[U^{\otimes t}|\psi_0\rangle^{\otimes t}\langle\psi_0|^{\otimes t}(U^\dagger)^{\otimes t}\right]\\ &\quad\times U^{\otimes t}|\psi_0\rangle^{\otimes t}\mu_{\rm Haar}(\mathrm{d}U)\\ &= \frac{1}{D}\sum_{i=1}^{D}\mathrm{Tr}\left[U_i^{\otimes t}|\psi_0\rangle^{\otimes t}\langle\psi_0|^{\otimes t}(U_i^\dagger)^{\otimes t}\Lambda^{\otimes t}\right.\\ &\quad\left.\times\left(U_i^{\otimes t}|\psi_0\rangle^{\otimes t}\langle\psi_0|^{\otimes t}(U_i^\dagger)^{\otimes t}\right)\right], \quad (10)\end{aligned}$$

with the perfect POVM element $E_{|\psi\rangle^{\otimes t}}=|\psi\rangle^{\otimes t}\langle\psi|^{\otimes t}$ for any input state $=|\psi\rangle^{\otimes t}$ and $E_{|\psi_0\rangle^{\otimes t}}=|\psi_0\rangle^{\otimes t}\langle\psi_0|^{\otimes t}$ for a fixed input state $|\psi_0\rangle^{\otimes t}$ in a d^{tn}-dimensional Hilbert space. The above equality indicates that we can also obtain the average channel fidelity by uniformly averaging over all unitary operators with a fixed input state instead of averaging over all input states.

In addition, we introduce a method of moment superoperator,[29] which is derived from the action of t-copies of the random circuit on nt-qubit density operators. It will be used to analyze the time-dependent noise in Section 3. Using Liouville representation,[19,22] we define moment superoperator kets shown in Fig. 3 as

$$|M^{\otimes t}\rangle\rangle=\sum_j\langle B_j^{\otimes t},M^{\otimes t}\rangle|B_j^{\otimes t}\rangle\rangle, \quad (11)$$

in a d^{2nt}-dimensional space, where $\{B_j^{\otimes t}\}$ is an orthonormal basis and $j\in\{1,\ldots,d^{2tn}\}$ with respect to the Hilbert Schmidt inner product. Let $\langle\langle M^{\otimes t}|=|M^{\otimes t}\rangle\rangle^\dagger$, we have

$$\langle\langle M^{\otimes t}|N^{\otimes t}\rangle\rangle=\langle M^{\otimes t},N^{\otimes t}\rangle. \quad (12)$$

For a quantum channel, we have

$$|\Lambda^{\otimes t}(\rho^{\otimes t})\rangle\rangle=\sum_j|\Lambda^{\otimes t}(B_j^{\otimes t})\rangle\rangle\langle\langle B_j^{\otimes t}|\rho^{\otimes t}\rangle\rangle. \quad (13)$$

Thus, the representation of the linear mapping $\Lambda^{\otimes t}(\cdot)$ is given by

$$\hat{\Lambda}^{\otimes t}=\sum_j|\Lambda^{\otimes t}(B_j^{\otimes t})\rangle\rangle\langle\langle B_j^{\otimes t}|. \quad (14)$$

The previous estimation can only analyze the time-dependent noise channel in one experiment. However, the noises in different experiments are assumed to have the identical change, which obviously affects the estimation result. The advantage of the moment superoperator is that it provides more information to characterize the time-dependent noise for t-copies of experiments.

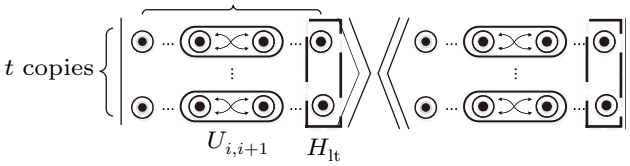


Fig. 3. Moment superoperator of a d^{2nt}-dimensional space can be visualized as an array of $2nt$ qubits, so that t copies support a ket in the state space on nt qubits, and the remaining t copies support the corresponding bra. The ovals $U_{i,i+1}$ correspond to a unitary operator U that acts nontrivially on two neighboring qubits i and $i+1$. The dashed rectangles indicate that the $2t$ qubits that correspond to a local moment space $H_{\rm lt}$ indicate the random circuit of two neighboring qubits that correspond to a d^2-dimensional space.

3. Average fidelity estimation of $(m+1)$-stage twirled noisy channels using unitary $2t$-design

If the set of unitary operators $\{U_i\}_{i=1}^D$ construct a unitary $2t$-design, the average fidelity using the unitary $2t$-design to twirl a channel is given by

$$\begin{aligned}\overline{F}(\Delta_{\mu_{\rm Haar},2t}) &= \int_{\mathcal{U}(d^n)}\mathrm{Tr}\left[U^{\otimes t}\rho_0^{\otimes t}(U^\dagger)^{\otimes t}\Lambda^{\otimes t}\left(U^{\otimes t}\rho_0^{\otimes t}(U^\dagger)^{\otimes t}\right)\right]\\ &\quad\times\mu_{\rm Haar}(\mathrm{d}U)\\ &= \frac{1}{D}\sum_{i=1}\mathrm{Tr}\left[U_i^{\otimes t}\rho_0^{\otimes t}(U_i^\dagger)^{\otimes t}\Lambda^{\otimes t}\right.\\ &\quad\left.\times\left(U_i^{\otimes t}\rho_0^{\otimes t}(U_i^\dagger)^{\otimes t}\right)\right] \quad (15)\end{aligned}$$

for a fixed input density operator $\rho_0^{\otimes t} = |\psi_0\rangle^{\otimes t}\langle\psi_0|^{\otimes t}$ and a perfect POVM element $E_{|\psi_0\rangle^{\otimes t}} = |\psi_0\rangle^{\otimes t}\langle\psi_0|^{\otimes t}$ in d^{tn}-dimensional Hilbert space. Following this method, if we can implement an exact unitary $2t$-design to twirl a noise channel, we can directly obtain the average channel fidelity using the normalized sum D times of the results.

From Schur's lemma,[35] the Haar-averaged superoperator $\Delta_{\mu_{\text{Haar}},2t}$ can be expressed as a depolarizing channel

$$\Delta_{\mu_{\text{Haar}},2t}(\rho_0^{\otimes t}) = p\rho_0^{\otimes t} + (1-p)\,\text{tr}\left(\rho_0^{\otimes t}\right)\frac{\mathbb{I}_{d^{tn}}}{d^{tn}}, \tag{16}$$

with strength parameter

$$p = \frac{\text{Tr}\left(\hat{\Delta}_{\mu_{\text{Haar}},2t}\right) - 1}{d^{2tn} - 1}. \tag{17}$$

However, the size of a unitary $2t$-design grows exponentially with the increase of parameters d and n. Hence, we generally use an $(m+1)$-stage sequence of unitary operators (selected from a unitary $2t$-design) to twirl the noise channels to estimate the relationship between the average channel fidelity and parameter m in Fig. 4.

We assume that each $\Lambda_{s_j,j}^{\otimes t}$ for $j \in \{1,\ldots,m+1\}$ is a t-tensor product gate-dependent and time-dependent CPTP noise channel. If the fixed input density operator is $\rho_0^{\otimes t}$, we can obtain the corresponding output density operator, i.e.,

$$\begin{aligned}
&\Lambda_{U_{s(m+1)}}^{\otimes t}\left(\rho_0^{\otimes t}\right)\\
&= \left(U_{s_{m+1}}^{\dagger}\right)^{\otimes t} \circ \Lambda_{s_{m+1},m+1}^{\otimes t} \circ U_{s_{m+1}}^{\otimes t} \circ \left(U_{s_m}^{\dagger}\right)^{\otimes t} \circ \Lambda_{s_m,m}^{\otimes t} \circ U_{s_m}^{\otimes t}\\
&\quad \circ \cdots \circ \left(U_{s_1}^{\dagger}\right)^{\otimes t} \circ \Lambda_{s_1,1}^{\otimes t} \circ U_{s_1}^{\otimes t}\left(\rho_0^{\otimes t}\right),
\end{aligned} \tag{18}$$

where $\Lambda_{U_{s(m+1)}}^{\otimes t}(\cdot)$ is the sequence $s(m+1)$ mapping of $(m+1)$-stage CPTP noise channels each with $U_{s_j}^{\otimes t}$ and $(U_{s_j}^{\dagger})^{\otimes t}, j \in \{1,\ldots,m+1\}$ from the unitary $2t$-design at both ends. Its superoperator representation can be given by

$$\begin{aligned}
\hat{\Lambda}_{U_{s(m+1)}}^{\otimes t} &= \left(\hat{U}_{s_{m+1}}^{\dagger}\right)^{\otimes t} \hat{\Lambda}_{s_{m+1},m+1}^{\otimes t} \hat{U}_{s_{m+1}}^{\otimes t}\\
&\quad \times \left(\hat{U}_{s_m}^{\dagger}\right)^{\otimes t} \hat{\Lambda}_{s_m,m}^{\otimes t} \hat{U}_{s_m}^{\otimes t} \cdots \left(\hat{U}_{s_1}^{\dagger}\right)^{\otimes t} \hat{\Lambda}_{s_1,1}^{\otimes t} \hat{U}_{s_1}^{\otimes t}.
\end{aligned} \tag{19}$$

If U_{s_j} is selected from a unitary $2t$-design $\{U_i\}_{i=1}^{D}$, then we obtain the average fidelity of $(m+1)$-stage noise channel using the unitary $2t$-design to twirl the noisy channels, i.e.,

$$\begin{aligned}
&\overline{F}\left(\Delta_{\mu_{\text{Haar}},2t,U_{s(m+1)}}\right)\\
&= \int_{\mathcal{U}(d^n)} \text{Tr}\left[\rho_0^{\otimes t}\Lambda_{U_{s(m+1)}}^{\otimes t}\left(\rho_0^{\otimes t}\right)\right]\mu_{\text{Haar}}\left(\prod_{j=1}^{m+1}\text{d}U_{s_j}\right)\\
&= \frac{1}{D^{m+1}}\prod_{j=1}^{m+1}\left\{\sum_{i=1}^{D}\text{Tr}\left[U_{s_{j,i}}^{\otimes t}\rho_0^{\otimes t}\left(U_{s_{j,i}}^{\dagger}\right)^{\otimes t}\Lambda_{s_j,j}^{\otimes t}U_{s_{j,i}}^{\otimes t}\rho_0^{\otimes t}\left(U_{s_{j,i}}^{\dagger}\right)^{\otimes t}\right]\right\}.
\end{aligned} \tag{20}$$

Following Eq. (16), the Haar averaged superoperator of each noise channel is as follows:

$$\Delta_{\mu_{\text{Haar}},2t,s_j,j}(\rho_0^{\otimes t}) = p_{s_j,j}\rho_0^{\otimes t} + \left(1-p_{s_j,j}\right)\frac{\mathbb{I}_{d^{tn}}}{d^{tn}}, \tag{21}$$

with

$$p_{s_j,j} = \frac{\text{Tr}\left(\hat{\Delta}_{\mu_{\text{Haar}},2t,s_j,j}\right) - 1}{d^{2tn} - 1}, \tag{22}$$

where we assume that $\text{tr}(\rho_0^{\otimes t}) = 1$. Then, the Haar-averaged superoperator of the $(m+1)$-stage noise channel using the unitary $2t$-design can be written as

$$\Delta_{\mu_{\text{Haar}},2t,U_{s(m+1)}}\left(\rho_0^{\otimes t}\right) = \prod_{j=1}^{m+1}p_{s_j,j}\rho_0^{\otimes t} + \left(1-\prod_{j=1}^{m+1}p_{s_j,j}\right)\frac{\mathbb{I}_{d^{tn}}}{d^{tn}}. \tag{23}$$



Fig. 4. $(m+1)$-stage twirled noisy channels using unitary $2t$-design.

3.1. The case of gate-dependent and time-dependent noise

We redraw $(m+1)$-stage noise circuits using the unitary $2t$-design for an actual experiment by inductively defining new gates, which are uniformly selected from the unitary $2t$-design shown in Fig. 5.

(1) Define $U_{s_1,1} = \mathbb{I}_{d^n}$;

(2) Define $G_{s_j,j}^{\otimes t} = (U_{s_j,j}^{\dagger})^{\otimes t} \circ U_{s_{j+1},j+1}^{\otimes t}$ for $j \in \{1,\cdots,m\}$. The superoperator representation can be written as $\hat{G}_{s_j,j}^{\otimes t} = (\hat{U}_{s_j,j}^{\dagger})^{\otimes t}\hat{U}_{s_{j+1},j+1}^{\otimes t}$;

(3) Define $G_{s_{m+1},m+1}^{\otimes t} = (U_{s_{m+1},m+1}^{\dagger})^{\otimes t}$. Then, we have $\prod_{j=1}^{m+1}\hat{G}_{s_j,j}^{\otimes t} = (\hat{U}_{s_1,1}^{\dagger})^{\otimes t} = \mathbb{I}_{d^{2tn}}$.

Note that if $i \neq k$, $U_{s_i,i}$ and $U_{s_k,k}$ are independent chosen from a unitary $2t$-design for each $i,k \in \{2,\ldots,m+1\}$. Hence, for each $j \in \{1,\ldots,m\}$, $G_{s_j,j}^{\otimes t}$ is independent from each other. However, $G_{s_{m+1},m+1}^{\otimes t}$ depends on the first m gates $G_{s_j,j}^{\otimes t}$ for $j \in \{1,\ldots,m\}$.

Therefore, we can use the unitary $2t$-design to estimate the average gateset fidelity of $(m+1)$-stage noise circuits by considering a perturbative expansion of each noise $\Lambda_{s_j,j}^{\otimes t}$ about the ideal average noise $\Lambda^{\otimes t}$. We define the difference between $\Lambda_{s_j,j}^{\otimes t}$ and $\Lambda^{\otimes t}$ for $j \in \{1,\ldots,m+1\}$ as

$$\delta\Lambda_{s_j,j}^{\otimes t} = \Lambda_{s_j,j}^{\otimes t} - \Lambda^{\otimes t}. \tag{24}$$

Note that each $\delta\Lambda_{s_j,j}^{\otimes t}$, $j \in \{1,\ldots,m+1\}$ is a Hermiticity-preserving, trace-annihilating linear superoperator. Then, its surperoperator representation is

$$\delta\hat{\Lambda}_{s_j,j}^{\otimes t} = \hat{\Lambda}_{s_j,j}^{\otimes t} - \hat{\Lambda}^{\otimes t}. \tag{25}$$

The ideal average strength parameter can be written as

$$p = \frac{\mathrm{Tr}(\hat{\Delta}_{\mu_{\mathrm{Haar}},2t}) - 1}{d^{2tn} - 1}, \tag{26}$$

where $\hat{\Delta}_{\mu_{\mathrm{Haar}},2t}$ is the Haar-averaged superoperator of $\hat{\Lambda}^{\otimes t}$ using the unitary $2t$-design for twirling of noise. Then, we obtain the difference between $p_{s_j,j}$ and p, i.e.,

$$\delta p_{s_j,j} = p_{s_j,j} - p = \frac{\mathrm{Tr}(\delta\hat{\Delta}_{\mu_{\mathrm{Haar}},2t,s_j,j})}{d^{2tn} - 1}, \tag{27}$$

where

$$\delta\hat{\Delta}_{\mu_{\mathrm{Haar}},2t,s_j,j} = \hat{\Delta}_{\mu_{\mathrm{Haar}},2t,s_j,j} - \hat{\Delta}_{\mu_{\mathrm{Haar}},2t}. \tag{28}$$

Under the above conditions, this approach enables us to fit the experimental fidelity decay sequence to a model with fit parameters. The noise channel in Fig. 5 can be written as

$$\Lambda^{\otimes t}_{G_{s(m+1)}}(\rho_0^{\otimes t}) = G^{\otimes t}_{s_{m+1},m+1} \circ \Lambda^{\otimes t}_{s_{m+1},m+1} \circ \cdots \circ G^{\otimes t}_{s_1,1} \circ \Lambda^{\otimes t}_{s_1,1}(\rho_0^{\otimes t}). \tag{29}$$

Using the unitary $2t$-design for the twirling of noise, we obtain

$$\Delta_{\mu_{\mathrm{Haar}},2t,G_{s(m+1)}}\left(\rho_0^{\otimes t}\right) = \prod_{j=2}^{m+1} p_{s_j,j} \Lambda^{\otimes t}_{s_1,1}\left(\rho_0^{\otimes t}\right) + \left(1 - \prod_{j=2}^{m+1} p_{s_j,j}\right) \frac{\mathbb{I}_{d^{tn}}}{d^{tn}}. \tag{30}$$

The average gateset fidelity for any input density operator $\rho^{\otimes t}$ can be written as

$$\begin{aligned}
&\overline{F}\left(\Delta_{\mu_{\mathrm{Haar}},2t,G_{s(m+1)}}\right) \\
&= \mathrm{Tr}\left[\rho_0^{\otimes t} \Delta_{\mu_{\mathrm{Haar}},2t,G_{s(m+1)}}\left(\rho_0^{\otimes t}\right)\right] \\
&= \prod_{j=2}^{m+1} p_{s_j,j} F\left(|\psi_0\rangle^{\otimes t}, \Lambda^{\otimes t}_{s_1,1}, E_{|\psi_0\rangle^{\otimes t}}\right) \\
&\quad + \frac{1}{d^{tn}}\left(1 - \prod_{j=2}^{m+1} p_{s_j,j}\right) \\
&= \frac{d^{tn} F\left(|\psi_0\rangle^{\otimes t}, \Lambda^{\otimes t}_{s_1,1}, E_{|\psi_0\rangle^{\otimes t}}\right) - 1}{d^{tn}} \\
&\quad \times \prod_{j=2}^{m+1} \left(p + \delta p_{s_j,j}\right) + \frac{1}{d^{tn}},
\end{aligned} \tag{31}$$

where $F(|\psi_0\rangle^{\otimes t}, \Lambda^{\otimes t}_{s_1,1}, E_{|\psi_0\rangle^{\otimes t}})$ is the fidelity of channel $\Lambda^{\otimes t}_{s_1,1}$ with the input state $|\psi_0\rangle^{\otimes t}$, and the perfect POVM is $E_{|\psi_0\rangle^{\otimes t}} = |\psi_0\rangle^{\otimes t}\langle\psi_0|^{\otimes t}$.

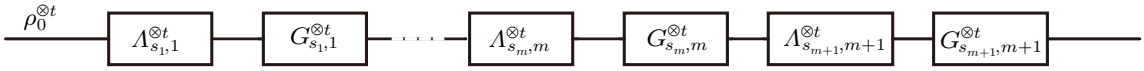


Fig. 5. $(m+1)$-stage noise circuits with each gate uniformly chosen from the unitary $2t$-design for an actual experiment.

3.2. The case of gate-independent and time-independent noise

If $\Lambda^{\otimes t}_{s_j,j}$ is a gate-independent and time-independent noise channel, the average channel fidelity $\overline{F}(\Delta_{\mu_{\mathrm{Haar}},2t,G_{s(m+1)}})$ is the zeroth order fitting model $\overline{F}^{(0)}(\Delta_{\mu_{\mathrm{Haar}},2t,G_{s(m+1)}})$ for $p_{s_j,j} = p$. Then, we obtain

$$\begin{aligned}
&\overline{F}^{(0)}\left(\Delta_{\mu_{\mathrm{Haar}},2t,G_{s(m+1)}}\right) \\
&= \frac{d^{tn} F(|\psi_0\rangle^{\otimes t}, \Lambda^{\otimes t}_{s_1,1}, E_{|\psi_0\rangle^{\otimes t}}) - 1}{d^{tn}} p^m + \frac{1}{d^{tn}} \\
&= A_0 p^m + B_0,
\end{aligned} \tag{32}$$

with

$$A_0 = \mathrm{Tr}\left[E_{|\psi_0\rangle^{\otimes t}} \Lambda^{\otimes t}_{s_1,1}\left(\rho_0^{\otimes t}\right)\right] - \frac{1}{d^{tn}}, \tag{33}$$

$$B_0 = \frac{1}{d^{tn}}. \tag{34}$$

Hence, the ideal case where the average gateset fidelity of each gate-independent and time-independent noise exponentially decays in p is shown in Fig. 6.

3.3. The case of weakly gate-dependent and time-independent noise

If $\Lambda^{\otimes t}_{s_j,j}$ is a weakly gate-dependent and time-independent noise channel, which is abbreviated $\Lambda^{\otimes t}_{s_j}$, the average gateset fidelity $\overline{F}_{\mathrm{gd}}(\Delta_{\mu_{\mathrm{Haar}},2t,G_{s(m+1)}})$ is the sum of the zeroth- and first-order fitting model. We neglect the higher-order perturbation terms for small δp_{s_j}. Then, we obtain

$$\begin{aligned}
&\overline{F}_{\mathrm{gd}}\left(\Delta_{\mu_{\mathrm{Haar}},2t,G_{s(m+1)}}\right) \\
&= \overline{F}^{(0)}\left(\Delta_{\mu_{\mathrm{Haar}},2t,G_{s(m+1)}}\right) + \overline{F}^{(1)}\left(\Delta_{\mu_{\mathrm{Haar}},2t,G_{s(m+1)}}\right) \\
&= \frac{d^{tn} F\left(|\psi_0\rangle^{\otimes t}, \Lambda^{\otimes t}_{s_1}, E_{|\psi_0\rangle^{\otimes t}}\right) - 1}{d^{tn}} p^m \\
&\quad + \frac{d^{tn} F\left(|\psi_0\rangle^{\otimes t}, \Lambda^{\otimes t}_{s_1}, E_{|\psi_0\rangle^{\otimes t}}\right) - 1}{d^{tn}} \sum_{j=2}^{m+1} \delta p_{s_j} p^{m-1} + \frac{1}{d^{tn}} \\
&= A_1 p^m + B_1 m p^{m-1} + C_1,
\end{aligned} \tag{35}$$

with

$$A_1 = \mathrm{Tr}\left[E_{|\psi_0\rangle^{\otimes t}} \Lambda^{\otimes t}_{s_1}\left(\rho_0^{\otimes t}\right)\right] - \frac{1}{d^{tn}}, \tag{36}$$

$$\begin{aligned}
B_1 &= \frac{\left(d^{tn} F\left(|\psi_0\rangle^{\otimes t}, \Lambda^{\otimes t}_{s_1}, E_{|\psi_0\rangle^{\otimes t}}\right) - 1\right) \sum_{j=2}^{m+1} \mathrm{Tr}\left(\delta\hat{\Delta}_{\mu_{\mathrm{Haar}},2t,s_j}\right)}{m d^{tn}\left(d^{2tn} - 1\right)} \\
&= \left(\mathrm{Tr}\left[E_{|\psi_0\rangle^{\otimes t}} \Lambda^{\otimes t}_{s_1}\left(\rho_0^{\otimes t}\right)\right] - \frac{1}{d^{tn}}\right) \delta p,
\end{aligned} \tag{37}$$

and

$$C_1 = \frac{1}{d^{tn}}, \tag{38}$$

where $\delta p = 1/m \sum_{j=2}^{m+1} \delta p_{s_j}$.

The standard fitting model fixes $B_1 = 0$. However, $B_1 \neq 0$ is more appropriate when the noise may be weakly gate-dependent, instead of perfectly gate-independent, as shown in Fig. 6.

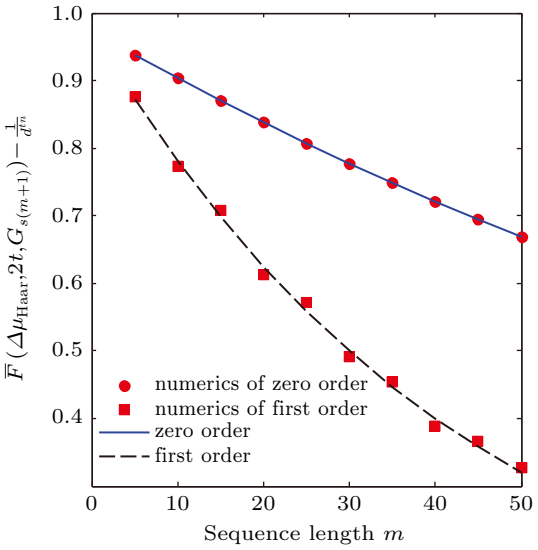


Fig. 6. Average fidelity estimation of $(m+1)$-stage quantum circuits using unitary $2t$-designs. In these examples, we assume that we implement perfect quantum devices and have on SPAM, where $d = 2$, $n = 1$, and $t = 6$. We assume that the fidelity of the first noise channel is 0.99. The strength parameter is $p^{(0)} = 0.9925$ in the zeroth-order case and $p^{(1)} = 0.9775$ in the first-order case. In the first-order case, the perturbation of the strength parameter $\delta p^{(1)}$ is from $[10^{-4}, 10^{-3}]$.

3.4. Bounding higher-order perturbation terms

To bound higher-order perturbation terms, we have

$$\begin{aligned}&\left|\overline{F}^{(k+1)}\left(\Delta_{\mu_{\rm Haar},2t,G_{s(m+1)}}\right)-\overline{F}^{(k)}\left(\Delta_{\mu_{\rm Haar},2t,G_{s(m+1)}}\right)\right|\\ =&\left|\left[\frac{1}{p}\binom{m}{k-1}\sum_{j=2}^{m+1}\delta p_{s_j}-1\right]\overline{F}^{(k)}\left(\Delta_{\mu_{\rm Haar},2t,G_{s(m+1)}}\right)\right| \end{aligned} \tag{39}$$

for $k \in \{1, \cdots, m\}$.

Therefore, the $k+1$ order of the fidelity can be neglected if

$$\binom{m}{k-1}\sum_{j=2}^{m+1}\delta p_{s_j} \ll p. \tag{40}$$

For weakly gate-dependent and time-independent noise, if we assume that $\delta p_{s_j} \approx \delta p = 1/m\sum_{j=2}^{m+1}\delta p_{s_j}$, $j \in \{2, \ldots, m+1\}$, we can ignore the second-order terms when δp satisfies

$$\delta p \ll \frac{p}{m^2}. \tag{41}$$

3.5. The case of gate-independent and weakly time-dependent noise

If $\Lambda_{s_j,j}^{\otimes t}$, $j \in \{1, \ldots, m+1\}$ is a weakly time-dependent and gate-independent noise, we can use the method of moment superoperator to estimate the average channel fidelity. We take t copies of noise circuits abbreviated to $\Lambda_{s,j}$ for $j \in \{1, \ldots, m+1\}$ in a d^n-dimensional Hilbert space. The difference $\delta\Lambda_{s,j}$ between the practice noise $\Lambda_{s,j}$ and the ideal average noise Λ is a Hermiticity-preserving, trace-annihilating linear superoperator. By using the unitary 2-design to twirl noise, we define $q_{s,j}$ as the corresponding strength parameter, i.e.,

$$\Delta_{\mu_{\rm Haar},2,G_{s(m+1)}}(\rho_0) = \prod_{j=2}^{m+1} q_{s,j}\Lambda_{s,1}(\rho_0) + \left(1-\prod_{j=2}^{m+1} q_{s,j}\right)\frac{\mathbb{I}_{d^n}}{d^n}, \tag{42}$$

where the fixed input density operator is $\rho_0 = |\psi_0\rangle\langle\psi_0|$ in a d^n-dimensional Hilbert space. The average fidelity $\overline{F}(\Delta_{\mu_{\rm Haar},2t,G_{s(m+1)}})$ for t copies of noise circuits can be written as

$$\begin{aligned}\overline{F}\left(\Delta_{\mu_{\rm Haar},2t,G_{s(m+1)}}\right) &= \prod_{s=1}^{t}{\rm Tr}\left[\rho_0\Delta_{\mu_{\rm Haar},2,G_{s(m+1)}}(\rho_0)\right]\\ &= \prod_{s=1}^{t}\left[\frac{d^nF\left(|\psi_0\rangle,\Lambda_{s,1},E_{|\psi_0\rangle}\right)-1}{d^n}\right.\\ &\quad\left.\times\prod_{j=2}^{m+1}(q+\delta q_{s,j})+\frac{1}{d^n}\right], \end{aligned} \tag{43}$$

where $F(|\psi_0\rangle,\Lambda_{s,1},E_{|\psi_0\rangle})$ is the fidelity of noise $\Lambda_{s,1}$ with the fixed input state $|\psi_0\rangle$, and the perfect POVM $E_{\psi_0} = |\psi_0\rangle\langle\psi_0|$; q is the strength parameter of the ideal average noise Λ using the unitary 2-design. When t, d, and n are sufficiently large that $1/d^{tn} \ll \delta q_{s,j}$ for all $s \in \{1, \ldots, t\}$ and $j \in \{2, \ldots, m+1\}$, the first-order fitting model of the weakly time-dependent and gate-independent noise channel can be written as

$$\begin{aligned}&\overline{F}_{td}\left(\Delta_{\mu_{\rm Haar},2t,G_{s(m+1)}}\right)\\ =&\ \overline{F}^{(0)}\left(\Delta_{\mu_{\rm Haar},2t,G_{s(m+1)}}\right)+\overline{F}^{(1)}\left(\Delta_{\mu_{\rm Haar},2t,G_{s(m+1)}}\right)\\ \approx&\prod_{s=1}^{t}\left(\frac{d^nF\left(|\psi_0\rangle,\Lambda_{s,1},E_{|\psi_0\rangle}\right)-1}{d^n}\right)q^{tm}\\ &+\prod_{s=1}^{t}\left(\frac{d^nF\left(|\psi_0\rangle,\Lambda_{s,1},E_{|\psi_0\rangle}\right)-1}{d^n}\right)\sum_{s=1}^{t}\sum_{j=2}^{m+1}\delta q_{s,j}q^{tm-1}\\ =&\ A_1q^{tm}+B_1tmq^{tm-1}, \end{aligned} \tag{44}$$

with

$$A_1 = \prod_{s=1}^{t}\left\{{\rm Tr}\left[E_{|\psi_0\rangle}\Lambda_{s,1}(\rho_0)\right]-\frac{1}{d^n}\right\}, \tag{45}$$

$$B_1 = \prod_{s=1}^{t}\left\{{\rm Tr}\left[E_{|\psi_0\rangle}\Lambda_{s,1}(\rho_0)\right]-\frac{1}{d^n}\right\}\delta q, \tag{46}$$

where $\delta q = 1/(tm)\sum_{s=1}^{t}\sum_{j=1}^{m+1}\delta q_{s,j}$.

Here, we use the method of moment superoperator to characterize t-copies of experiments. When t, d, and n are sufficiently large, the constant item of the fitting model can be ignored. Therefore, the weakly time-dependent and gate-independent noise decays exponentially in q^t if we assume each noise strength parameter $q_{s,j} = (1+\eta)q$, where $\eta = \delta q/q$.

4. Analysis of average fidelity under unitary error

A unitary $2t$-design is similar to a spherical $2t$-design in terms of the Jamiolkowski isomorphism and the frame potential.[32,33] The unitary operators in a unitary $2t$-design are elements of the unitary group instead of points on a unit sphere. Therefore, we can use the partial results of a spherical $2t$-design to construct a unitary $2t$-design to estimate the average channel fidelity. If we can implement an exact unitary $2t$-design $\{U_i\}_{i=1}^{D}$ to twirl the noise, we must only take D times of the experiment to estimate the average fidelity.

However, the size of the unitary $2t$-design D grows exponentially with the increase in d and n. We generally take an $(m+1)$-stage sequence of gates with their corresponding noise using a benchmarking protocol, which requires selecting a sequence of gates at random from the unitary $2t$-design and repeating for many sequences to take the average of the results.

In an actual experiment, we take k times of the experiments to estimate the average fidelity, i.e.,

$$\begin{aligned}&\overline{F}\left(|\psi_0\rangle^{\otimes t},\Lambda_{G_{s(m+1)}}^{\otimes t},E_{|\psi_0\rangle}^{\otimes t}\right)\\&=\frac{1}{k}\sum_{s=1}^{k}F\left(|\psi_0\rangle^{\otimes t},\Lambda_{G_{s(m+1)}}^{\otimes t},E_{|\psi_0\rangle}^{\otimes t}\right),\end{aligned}\tag{47}$$

where $|\psi_0\rangle^{\otimes t}$ is the fixed input state and $E_{|\psi_0\rangle^{\otimes t}}=|\psi_0\rangle^{\otimes t}\langle\psi_0|^{\otimes t}$ is the perfect POVM. For an exact unitary $2t$-design $\{U_i\}_{i=1}^{D}$, the ideal average fidelity is

$$\begin{aligned}&\overline{F}_{\text{ideal}}\left(|\psi_0\rangle^{\otimes t},\Lambda_{G_{s(m+1)}}^{\otimes t},E_{|\psi_0\rangle^{\otimes t}}\right)\\&=\frac{1}{D^m}\sum_{s=1}^{D^m}F\left(|\psi_0\rangle^{\otimes t},\Lambda_{G_{s(m+1)}}^{\otimes t},E_{|\psi_0\rangle^{\otimes t}}\right).\end{aligned}\tag{48}$$

Note that the gate $G_{s_{m+1}}$ depends on the first m random gates. By Hoeffding's inequality,[36] we obtain

$$\begin{aligned}&P\left(\left|\overline{F}\left(|\psi_0\rangle^{\otimes t},\Lambda_{G_{s(m+1)}}^{\otimes t},E_{|\psi_0\rangle^{\otimes t}}\right)\right.\right.\\&\left.\left.-\overline{F}_{\text{ideal}}\left(|\psi_0\rangle^{\otimes t},\Lambda_{G_{s(m+1)}}^{\otimes t},E_{|\psi_0\rangle^{\otimes t}}\right)\right|\geq\varepsilon\right)\\&\leq 2\mathrm{e}^{-\frac{2(k\varepsilon)^2}{k(b-a)^2}},\end{aligned}\tag{49}$$

where the resulted fidelity of every experiment lies in $[a,b]\subset(0,1)$. We assume that $1-\varepsilon\geq 1-2\mathrm{e}^{-\frac{2(k\varepsilon)^2}{k(b-a)^2}}$ is the desired confidence level. Then, we can obtain

$$k\geq\left\lceil\frac{\ln\left(\frac{2}{\varepsilon}\right)(b-a)^2}{2\varepsilon^2}\right\rceil.\tag{50}$$

From the above benchmarking analysis, two main deficiencies will affect the final result. The first one is the pseudo-random selection of quantum gates, which will cause the deviation of the experimental average fidelity from the ideal value mainly because of the incorrect estimation of the number of experiments k caused by the inaccuracy of $b-a$. The other deficiency is the unitary error. Imperfect quantum gate $G_{s_{m+1},m+1}$ preparation and undesirable POVM will lead to errors in estimation results. Here, we neglect the effect of nontrace-preserving mappings.

The unitary error has an important effect on the estimation of average fidelity. It is mainly caused by undesirable experimental implementations and SPAM. Our approach is to bound the fidelity of unitary error using the method of moment superoperator. We use the moment superoperator because the unitary error in each gate is different from one another.

The quantum gate $G_{s_{m+1},m+1}$ in $(m+1)$-stage sequence of random gates with corresponding noise depends on other m gates that are randomly selected from the unitary 2-design, where $G_{s_{m+1},m+1}=\prod_{j=1}^{m}G_{s_j,j}^{\dagger}$. For undesirable experimental implementations, there is a d^n-dimensional unitary operator U_{G_s} with eigenvalues $\mathrm{e}^{\mathrm{i}\theta_{G_s,k}}$ for $\theta_{G_s,k}\in[-\pi,\pi]$ and $k\in\{1,\ldots,d^n\}$, which satisfy

$$G_{s_{m+1},m+1}^{ue}=U_{G_s}\prod_{j=1}^{m+1}G_{s_j,j}^{\dagger},\tag{51}$$

where $\sum_{k=1}^{d^n}\theta_{C_s,k}=0$. For SPAM, there is a d^n-dimensional unitary operator U_{SPAM_s} with eigenvalues $\mathrm{e}^{\mathrm{i}\theta_{\text{SPAM}_s,k}}$ for $\theta_{\text{SPAM}_s,k}\in[-\pi,\pi]$ and $k\in\{1,\ldots,d^n\}$, which satisfy

$$E_{|\psi_0\rangle,s}^{ue}=U_{\text{SPAM}_s}|\psi_0\rangle\langle\psi_0|U_{\text{SPAM}_s}^{\dagger},\tag{52}$$

where $E_{|\psi_0\rangle,s}^{ue}$ is the s-th sequence of measurement with a fixed input state $|\psi_0\rangle$ and $\sum_{k=1}^{d^n}\theta_{\text{SPAM}_s,k}=0$. Then, we obtain

$$\begin{aligned}&\mathrm{Tr}\left[E_{|\psi_0\rangle,s}^{ue}G_{s_{m+1},m+1}^{ue}\circ\Lambda_{s_{m+1},m+1}\circ\cdots\circ G_{s_1,1}\circ\Lambda_{s_1,1}\left(|\psi_0\rangle\langle\psi_0|\right)\right]\\&=\langle\psi_0|U_{\text{SPAM}_s}^{\dagger}U_{G_s}\prod_{i=m+1}^{1}G_{s_i,i}\Lambda_{s_i,i}\left(|\psi_0\rangle\langle\psi_0|\right)\\&\quad\times\prod_{j=1}^{1}\Lambda_{s_j,j}^{\dagger}G_{s_j,j}^{\dagger}U_{G_s}^{\dagger}U_{\text{SPAM}_s}|\psi_0\rangle.\end{aligned}\tag{53}$$

Let U_s be the total unitary error, such that

$$U_s=U_{\text{SPAM}_s}^{\dagger}U_{G_s},\tag{54}$$

with eigenvalues $\mathrm{e}^{\mathrm{i}\theta_{s,k}}=\mathrm{e}^{\mathrm{i}(\theta_{G_s,k}-\theta_{\text{SPAM}_s,k})}$ for $\theta_{s,k}\in[-\pi,\pi]$ and $k\in\{1,\ldots,d^n\}$. Thus, the corresponding average fidelity impact factor of unitary error γ_s[34,37] can be written as

$$\gamma_s=\frac{|\mathrm{Tr}\left(\hat{U}_s\right)|+d^n}{d^{2n}+d^n}=\frac{\left|\left(\sum_{k=1}^{d^n}\mathrm{e}^{\mathrm{i}\theta_{s,k}}\right)^2\right|+d^n}{d^{2n}+d^n}.\tag{55}$$

It is obvious that $\gamma_s\leq 1$. Our approach is to provide a lower bound on $(\sum_{k=1}^{d^n}\mathrm{e}^{\mathrm{i}\theta_{s,k}})^2$. We assume that $\theta_s=\max_{k\in\{1,\ldots,d^n\}}|\theta_{s,k}|$. Then, we assume that $\theta_{s,1}=\theta_s$ and $\theta_{s,k}=-\theta_s/(d^n-1)$ for all $k\in\{2,\ldots,d^n\}$. This assumption can minimize γ_s. We can verify this result by implementing a perturbation $\delta\theta$, i.e.,

$$\left[\mathrm{e}^{\mathrm{i}\theta_s}+(d^n-3)\,\mathrm{e}^{-\mathrm{i}\frac{\theta_s}{d^n-1}}+\mathrm{e}^{-\mathrm{i}\left(\frac{\theta_s}{d^n-1}+\delta\theta\right)}+\mathrm{e}^{-\mathrm{i}\left(\frac{\theta_s}{d^n-1}-\delta\theta\right)}\right]^2$$

$$= \left[e^{i\theta_s} + (d^n - 3)\, e^{-i\frac{\theta_s}{d^n-1}} + e^{-i\frac{\theta_s}{d^n-1}} \left(2 + \delta^2\theta\right) \right]^2$$
$$\geq \left[e^{i\theta_s} + (d^n - 1)\, e^{-i\frac{\theta_s}{d^n-1}} \right]^2. \quad (56)$$

For large d and n, we obtain

$$\left[e^{i\theta_s} + (d^n - 1)\, e^{-i\frac{\theta_s}{d^n-1}} \right]^2$$
$$\approx \left[e^{i\theta_s} + (d^n - 1)\left(1 - i\frac{\theta_s}{d^n - 1}\right) \right]^2$$
$$= \left[\cos\theta_s + d^n - 1 + i(\sin\theta_s - \theta_s)\right]^2$$
$$\geq \left[d^n - (1 - \cos\theta_s)\right]^2. \quad (57)$$

Then, we have

$$\gamma_s = \frac{\left| \left[e^{i\theta_s} + (d^n - 1)\, e^{-i\frac{\theta_s}{d^n-1}} \right]^2 \right| + d^n}{d^{2n} + d^n}$$
$$\geq \frac{\left[d^n - (1 - \cos\theta_s)\right]^2 + d^n}{d^{2n} + d^n}$$
$$= 1 - \frac{2(1 - \cos\theta_s)}{d^n} + \frac{(2 - \cos\theta_s)^2 - 1}{d^{2n} + d^n}. \quad (58)$$

Using the unitary 2-design to twirl the channels, we obtain

$$\Delta_{\mu_{\text{Haar}},2,G_{s(m+1)}}(\rho_0) = \prod_{j=2}^{m+1} q_{s_j,j}\gamma_s \Lambda_{s_1,1}(\rho_0) + \left(1 - \prod_{j=2}^{m+1} q_{s_j,j}\right) \frac{\mathbb{I}_{d^n}}{d^n}, \quad (59)$$

with the fixed input density operator $\rho_0 = |\psi_0\rangle\langle\psi_0|$ in a d^n-dimensional Hilbert space and strength parameters $q_{s_j,j}$. Then, the average fidelity for t-copies of sequences can be written as

$$\overline{F}\left(\Delta_{\mu_{\text{Haar}},2t,G_{s(m+1)}}\right) = \prod_{s=1}^{t} \left\{ \left\{ \text{Tr}\left[E_{|\psi_0\rangle}\Lambda_{s_1,1}(\rho_0)\right] + \frac{1}{d^n} \right\} \times \gamma_s \prod_{j=2}^{m+1} q_{s_j,j} + \frac{1}{d^n} \right\}. \quad (60)$$

We now numerically analyze the average fidelity of $(m+1)$-stage noise circuits with unitary errors using the unitary $2t$-design by considering t copies single-qubit noise models. We provide an example that the average fidelity of each single-qubit noise model is 0.99488 and its deviation range is $(10^{-5}, 10^{-4})$. We assume time-dependent unitary errors that correspond to rotations. The unitary error is Hamiltonian in d^n-dimensional Hilbert space with eigenvalues $e^{i\theta_k}$ for $\theta_k \in [-\pi, \pi]$ and $k \in \{1, \ldots, d^n\}$. For trace-preserving noise, the unitary error satisfies $\sum_{k=1}^{d^n} \theta_k = 0$.

For the convenience of comparison, we make the following assumptions.

(1) Assume $|\theta_k|$, $k \in \{1, \ldots, d^n - 1\}$ uniformly random chosen from $(10^{-5}, 10^{-4})$ with random positive and negative signs;

(2) Let $\theta_{d^n} = -\sum_{k=1}^{d^n-1} \theta_k$ and $\theta_{\max} = \max_{k\in\{1,\ldots,d^n\}} |\theta_k|$;

(3) Let $\theta'_k = 10^{-4}\theta_k/\theta_{\max}$.

Note that the above assumption ensures that the maximum eigenvalue of the unitary error is 10^{-4}. Numerical values for the average fidelity are shown in Fig. 7. Our first order average fidelity estimation of weakly time-dependent and gate-independent noise satisfies the results of data analysis.

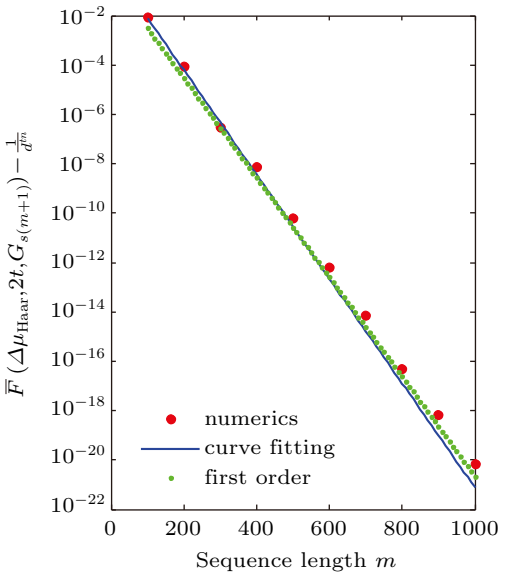


Fig. 7. Average fidelity of the $(m+1)$-stage noise circuit. In this example, we assume $d = 2$, $n = 1$, and $t = 6$ since the noise model is unital. The blue line is the curve fitting 1.054×0.9525^m. The green dotted line is the estimation of average fidelity in the case of gate-independent and weakly time-dependent noise with the lower bound on the average fidelity impact factor of the unitary error.

5. Conclusion

In this work, we have presented a first systematic analysis of the average fidelity estimation using the unitary $2t$-design for the twirling of noises. We provide a basic model using the unitary $2t$-design to twirl a noise channel. It efficiently estimates the average channel fidelity if we can implement an exact unitary $2t$-design. We also present a model of $(m+1)$-stage sequence of noise channels using the unitary $2t$-design to estimate the average channel fidelity. It can be recognized as the average gateset fidelity estimation for quantum circuit using a benchmarking protocol, which requires selecting a sequence of randomly selected gates from the unitary $2t$-design. We analyze such cases of the gate-independent and time-independent noise, weakly gate-dependent and time-independent noise, and gate-independent and weakly time-dependent noise. For the time-dependent noise, the method of moment superoperator plays an important role in noise analysis in different times of experiments. Finally, we provide a bound on the unitary error caused by non-ideal experimental implementations and SPAM.

Although much insight into the application of unitary designs has been obtained, many questions remain unresolved

as interesting open problems, such as the construction of unitary $2t$-designs or the errors produced by pseudo-randomness of the benchmarking protocol.

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