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COMMENT

A note on "Lattice soliton equation hierarchy and associated properties"*

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We demonstrate that the new hierarchy of integrable lattice equations in *Chin. Phys. B* **21** 090202 (2012) can be changed into the integrable lattice hierarchy in *Chin. Phys. B* **13** 1009 (2004) by using a very simple transformation.

Keywords: integrable lattice hierarchy, transformation between potentials, discrete zero curvature equation, Lax pair

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In Ref. [1], the authors claimed that a new hierarchy of integrable lattice equations is deduced, that is, the hierarchy (9). In this note, we will demonstrate that this 'new' hierarchy of integrable lattice equations is in fact not new and may be obtained from the integrable lattice hierarchy (12) in Ref. [2] through a very simple transformation.

For a lattice function $f_n = f(n)$, the shift operator E and the inverse of E are defined as

$$Ef_n = f_{n+1}, \quad E^{-1}f_n = f_{n-1}, \quad n \in \mathbf{Z}.$$

In Ref. [1], the authors introduced the discrete matrix spectral problem

$$\begin{aligned} &E\psi_n = U_n(u_n,\lambda)\psi_n,\\ &U_n(u_n,\lambda) = \begin{pmatrix} \lambda & \dfrac{q_n}{p_n} \\ p_n\lambda & 1+q_n \end{pmatrix}, \end{aligned} \tag{1}$$

where p_n, q_n are potentials, $p_n = p(n,t)$, and $q_n = q(n,t)$. For any integer m,

$$\Gamma_n^{(m)} = \sum_{i=0}^{m} \begin{pmatrix} a_n^{(i)}\lambda^{m-i} & b_n^{(i)}\lambda^{m-i} \\ c_n^{(i)}\lambda^{m-i+1} & -a_n^{(i)}\lambda^{m-i} \end{pmatrix}. \tag{2}$$

Using the discrete zero curvature representation

$$U_{n_{t_m}} = \Gamma_{n+1}^{(m)}U_n - U_n\Gamma_n^{(m)},$$

the following integrable lattice hierarchy is deduced:

$$u_{n_{t_m}} = \begin{pmatrix} p_n \\ q_n \end{pmatrix}_{t_m} = \begin{pmatrix} \dfrac{q_n b_n^{(m+1)}}{1+q_n} - \dfrac{p_n c_{n+1}^{(m+1)}}{q_n(1+q_n)} \\ -c_{n+1}^{(m+1)} \end{pmatrix}, \tag{3}$$

here the initial data $a_n^{(0)} = 1/2$, $b_n^{(0)} = 0$, $c_n^{(0)} = 0$, and the recursion relations ($m \geq 1$)

$$\begin{aligned} &(a_{n+1}^{(m+1)} - a_n^{(m+1)}) + q_n b_{n+1}^{(m+1)} - \left(p_n + \frac{p_n}{q_n}\right)c_n^{(m+1)} = 0,\\ &\left(p_n + \frac{p_n}{q_n}\right)(a_{n+1}^{(m)} + a_n^{(m)}) + (1+p_n+p_nq_n)b_{n+1}^{(m)}\\ &\quad - b_n^{(m+1)} = 0,\\ &c_{n+1}^{(m)} - q_n(a_{n+1}^{(m)} + a_n^{(m)}) - (1+p_n+p_nq_n)c_n^{(m)} = 0. \end{aligned} \tag{4}$$

The typical lattice system in hierarchy (3) is

$$\begin{aligned} p_{n,t_1} &= \frac{q_np_{n+1}(1+q_{n+1})(1+p_n)}{q_{n+1}(1+q_n)} - \frac{p_nq_{n-1}}{q_n(1+q_n)}\\ &\quad - \frac{(1+q_n)p_n^2q_{n-1}}{q_n},\\ q_{n,t_1} &= p_{n+1}q_n^2 + \frac{p_{n+1}}{q_{n+1}}q_n^2 - q_{n-1}. \end{aligned} \tag{5}$$

In Ref. [2], we presented the discrete spectral problem

$$E\phi_n = \begin{pmatrix} \lambda & \lambda\tilde{p}_n \\ \tilde{q}_n & 1+\tilde{p}_n\tilde{q}_n \end{pmatrix}\phi_n, \tag{6}$$

where $\tilde{p}_n$, $\tilde{q}_n$ are the potentials, $\tilde{p}_n = \tilde{p}(n,t)$, and $\tilde{q}_n = \tilde{q}(n,t)$. We set

$$\tilde{\Gamma}_n^{(m)} = \sum_{i=0}^{m} \begin{pmatrix} A_n^{(i)}\lambda^{m-i} & B_n^{(i)}\lambda^{m-i} \\ C_n^{(i)}\lambda^{m-i+1} & -A_n^{(i)}\lambda^{m-i} \end{pmatrix}. \tag{7}$$

By means of the discrete zero curvature representation

$$U_{n_{t_m}} = \tilde{\Gamma}_{n+1}^{(m)}U_n - U_n\tilde{\Gamma}_n^{(m)},$$

we obtain

$$\begin{aligned} \tilde{p}_{n_{t_m}} &= \tilde{p}_nA_n^{(m)} + \tilde{p}_n\tilde{q}_nB_{n+1}^{(m)} + B_{n+1}^{(m)} + \tilde{p}_n = B_n^{(m+1)}, \quad m \geq 0,\\ \tilde{q}_{t_m} &= -\tilde{q}_nA_{n+1}^{(m)} - \tilde{q}_nA_n^{(m)} - \tilde{p}_n\tilde{q}_nC_n^{(m)} - C_n^{(m)}\\ &= -C_{n+1}^{(m+1)}, \quad m \geq 0, \end{aligned} \tag{8}$$

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with

$$\begin{aligned}
&A_n^{(0)} = 1/2, \quad B_n^{(0)} = 0, \quad C_n^{(0)} = 0,\\
&A_{n+1}^{(m+1)} + \tilde{q}_n B_{n+1}^{(m+1)} - A_n^{(m+1)} - \tilde{p}_n C_n^{(m+1)} = 0, \quad m \geq 0,\\
&-B_n^{(m+1)} + \tilde{p}_n A_n^{(m)} + \tilde{p}_n \tilde{q}_n B_{n+1}^{(m)} + B_{n+1}^{(m)} + \tilde{p}_n A_{n+1}^{(m)} = 0,\\
&\quad m \geq 0,\\
&C_{n+1}^{(m+1)} - \tilde{q}_n A_{n+1}^{(m)} - \tilde{q}_n A_n^{(m)} - \tilde{p}_n \tilde{q}_n C_n^{(m)} - C_n^{(m)} = 0,\\
&\quad m \geq 0.
\end{aligned} \tag{9}$$

When $m = 1$, the typical lattice system in hierarchy (8) is

$$\begin{cases}
\tilde{p}_{n_{t_1}} = \tilde{p}_n - \tilde{p}_n^2 q_{n-1},\\
\tilde{q}_{t_1} = \tilde{q}_n^2 p_{n+1} - \tilde{q}_{n-1}.
\end{cases} \tag{10}$$

Remark 1 To show clarity, p, q in Ref. [2] have been replaced by $\tilde{p}_n, \tilde{q}_n$, respectively. For $i \in \mathbf{Z}$, $A_m^{(i)}$, $B_m^{(i)}$, $C_m^{(i)}$ in Ref. [2] have been replaced by $A_{n+i}^{(m)}$, $B_{n+i}^{(m)}$, $C_{n+i}^{(m)}$, respectively.

Proposition 1 Under the transformation between two sets of potentials

$$\tilde{p}_n = p_n + \frac{p_n}{q_n}, \quad \tilde{q}_n = q_n, \tag{11}$$

we can obtain

$$\begin{cases}
A_n^{(m)} = a_n^{(m)}, & m \geq 0,\\
B_n^{(m)} = b_n^{(m)}, & m \geq 0,\\
C_n^{(m)} = c_n^{(m)}, & m \geq 0,
\end{cases} \tag{12}$$

and the integrable hierarchy (3) is changed into the integrable hierarchy (8).

Proof We use the mathematical induction. When $m = 1$, it is obvious that equation (12) holds. For natural number N, we assume

$$A_n^{(N)} = a_n^{(N)}, \quad B_n^{(N)} = b_n^{(N)}, \quad C_n^{(N)} = c_n^{(N)}.$$

Then we have

$$\begin{aligned}
A_n^{(N+1)} &= (1-E)^{-1}(\tilde{q}_n B_{n+1}^{(N+1)} - \tilde{p}C_n^{(N+1)})\\
&= (1-E)^{-1}(\tilde{q}_n E(\tilde{p}_n A_n^{(N)} + \tilde{p}_n \tilde{q}_n B_{n+1}^{(N)}\\
&\quad + B_{n+1}^{(N)} + \tilde{p}_n A_{n+1}^{(N)})) - (\tilde{p}_n)E^{-1}(\tilde{q}_n A_{n+1}^{(N)}\\
&\quad + \tilde{q}_n A_n^{(N)} + \tilde{p}_n \tilde{q}_n C_n^{(N)} + C_n^{(N)})\\
&= (1-E)^{-1}\Bigg\{\Bigg(q_n E\Bigg[\left(p_n + \frac{p_n}{q_n}\right)a_n^{(N)}\\
&\quad + \left(p_n + \frac{p_n}{q_n}\right)q_n b_{n+1}^{(N)} + b_{n+1}^{(N)} + \left(p_n + \frac{p_n}{q_n}\right)a_{n+1}^{(N)}\Bigg]\Bigg)\Bigg\}\\
&\quad - \left(p_n + \frac{p_n}{q_n}\right)E^{-1}\left(q_n a_{n+1}^{(N)} + q_n a_n^{(N)}\right.\\
&\quad \left. + \left(p_n + \frac{p_n}{q_n}\right)q_n c_n^{(N)} + c_n^{(N)}\right)\\
&= (1-E)^{-1}(a_n^{(N+1)} - a_{n+1}^{(N+1)}) = a_n^{(N+1)},\\
B_n^{(N+1)} &= \tilde{p}_n A_n^{(N)} + \tilde{p}_n \tilde{q}_n B_{n+1}^{(N)} + B_{n+1}^{(N)} + \tilde{p}_n A_{n+1}^{(N)}\\
&= \left(p_n + \frac{p_n}{q_n}\right)a_n^{(N)} + \left(p_n + \frac{p_n}{q_n}\right)q_n b_{n+1}^{(N)} + b_{n+1}^{(N)}\\
&\quad + \left(p_n + \frac{p_n}{q_n}\right)a_{n+1}^{(N)} = b_n^{(N+1)},\\
C_n^{(N+1)} &= E^{-1}C_{n+1}^{(N+1)}\\
&= E^{-1}(\tilde{q}_n A_{n+1}^{(N)} + \tilde{q}_n A_n^{(N)} + \tilde{p}_n \tilde{q}_n C_n^{(N)} + C_n^{(N)})\\
&= E^{-1}\left(q_n a_{n+1}^{(N)} + q_n a_n^{(m)} + \left(p_n + \frac{p_n}{q_n}\right)q_n c_n^{(N)} + c_n^{N}\right)\\
&= E^{-1}c_{n+1}^{(N+1)} = c_n^{(N+1)}.
\end{aligned}$$

According to Eqs. (11) and (12), the integrable lattice hierarchy (3) is changed into the integrable lattice hierarchy (8). Specifically, according to the transformation (11), system (5) is changed into integrable lattice system (10). The proof is completed.

Furthermore, in the theory of continuous integrable systems, various explicit solutions of integrable equations have been studied, for instance, abundant lump solutions and interaction solutions.[3–6] Various explicit solutions of the lattice equation (10) can also be discussed. These results will appear in later papers.

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