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A nonlocal Burgers equation in atmospheric dynamical system and its exact solutions*

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From a two-vortex interaction model in atmospheric and oceanic systems, a nonlocal counterpart with shifted parity and delayed time reversal is derived by a simple AB reduction. To obtain some approximate analytic solutions of this nonlocal system, the multi-scale expansion method is applied to get an AB-Burgers system. Various exact solutions of the AB-Burgers equation, including elliptic periodic waves, kink waves and solitary waves, are obtained and shown graphically. To show the applications of these solutions in describing correlated events, a simple approximate solution for the two-vortex interaction model is given to show two correlated dipole blocking events at two different places. Furthermore, symmetry reduction solutions of the nonlocal AB-Burgers equation are also given by using the standard Lie symmetry method.

Keywords: AB–BA equivalence principle, nonlocal Burgers equation, periodic waves

PACS: 02.30.Jr, 02.30.Ik, 05.45.Yv, 47.35.Fg **DOI:** 10.1088/1674-1056/28/1/010201

1. Introduction

Interest in nonlocal systems has been on the rise since Ablowitz and Musslimani[1] in 2013 introduced a nonlocal Schrödinger (NLS) equation

$$\mathrm{i}q_t(x,t) = q_{xx}(x,t) \pm q(x,t)q^*(-x,t)q(x,t), \tag{1}$$

with $*$ being the complex conjugate and q being a complex valued function of the real variables x and t. The NLS equation (1) is parity–time (PT) symmetric and found to be an integrable infinite-dimensional Hamiltonian equation. Besides, it has a Lax pair, infinitely many conservation laws and can be solved by using the inverse scattering transformation (IST) method. Since then, a lot of nonlocal nonlinear systems have been studied,[2–7] including the nonlocal Ablowitz–Ladik, nonlocal saturable discrete NLS, coupled nonlocal NLS, nonlocal Korteweg–de Vries (KdV), nonlocal modified KdV (mKdV) equations. In Ref. [8], new reverse in space and time or time alone nonlocal nonlinear integrable equations were introduced by symmetry reductions of general AKNS scattering problems, such as nonlocal nonlinear Schrödinger, modified Korteweg–de Vries, sine-Gordon, (1+1) and (2+1) dimensional three-wave interactions, derivative NLS, loop soliton, Davey–Stewartson (DS), partially PT symmetric DS, and partially reverse space–time DS equations.

On the other hand, in the context of Alice–Bob (AB) systems, Lou *et al.* constructed many nonlocal systems with shifted parity, charge conjugation, and delayed time reversal by using AB–BA equivalence principle and $\hat{P}_s$–$\hat{T}_d$–C principle with various exact solutions being explicitly given.[9,10] Among the various methods for solving nonlocal equations, the function expansion method is very powerful in obtaining different types of exact solutions, such as soliton solutions, cnoidal wave solutions, soliton-cnoidal interaction solutions, etc.[3,5,11] To give possible application of these solutions, in Refs. [3], [5], and [9], two correlated dipole blocking events in atmosphere were derived by using the multi-scale expansion method.

The paper is organized as follows. In Section 2, a general nonlocal Burgers equation with shifted parity and delayed time reversal is derived by using the multi-scale expansion method from a two-vortex interaction model, which is a special case of the nonlinear inviscid dissipative and barotropic vorticity equation in a β-plane channel. In Section 3, to give some concrete exact solutions of the nonlocal Burgers equation, the function expansion method is applied to obtain some kinds of periodic wave solutions as well as soliton solutions. In Section 3, a special approximate solution of the nonlocal two-vortex interaction model is obtained and analyzed to give appropriate explanation of two correlated blocking events, which is an important large-scale weather phenomenon in mid-high latitudes in the atmosphere. In Section 4, the nonlocal Burgers equation is studied by using the standard Lie symmetry method and the general forms of symmetry reduction solutions are obtained as well as their corresponding symmetry reduction equations. The last section is devoted to a summary and discussion.

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2. Derivation of a nonlocal Burgers equation

In the study of atmospheric and oceanic dynamical systems, there is a fundamental nonlinear, inviscid, nondissipative, and equivalent barotropic vorticity equation in a β-plane channel

$$(\partial_t + u\partial_x + v\partial_y)(v_x - u_y - \lambda_0^2\psi) + \beta\psi_x = 0, \tag{2}$$

where ψ is the stream function, $u = -\psi_y$ and $v = -\psi_x$ are the two components of the dimensionless velocity, $\beta = (\omega_0/R_0)\cos\phi_0$ is the Rossby parameter with the earth's radius R_0, the angular frequency of the earth's rotation ω_0, and the latitude φ_0, and λ_0 is related to the Coriolis force f_0, the gravitational acceleration g, and the average height of the atmosphere through the relation $\lambda_0 = f_0/\sqrt{gH}\lambda$. When taking $\beta = 0$ in Eq. (2), it reduces to the well-known Euler equation, which has been extensively studied.[12,13] To study the interactions among multiple vortices, by neglecting higher order small interactions among different vertices, Jia *et al.* established a simplified model[14]

$$\omega_i = \psi_{ixx} + \psi_{iyy}, \tag{3}$$

$$\omega_{it} + [\psi_i, \omega_i] + \varepsilon\sum_{j\neq i}^{N}[\psi_i, \omega_j] - C\sum_{j=1}^{N}[\omega_i, \omega_j] + \beta\psi_{ix} = 0, \tag{4}$$

where $[\psi_i, \omega_j], j \neq i$ denotes the i–j-th stream–vorticity (SVI) interactions and $[\omega_i, \omega_j]$ denotes the i–j-th vorticity–vorticity interactions (VVI). C represents the strength of VVI, and ε is a small parameter introduced by considering that the SVI between two faraway (both in space and time) vortices should be small. For the special case of $N = 2$ in Eqs. (3) and (4), they are changed to

$$\omega_i = \psi_{ixx} + \psi_{iyy}, \quad i = 1, 2, \tag{5}$$

$$\omega_{1t} + [\psi_1, \omega_1] + \varepsilon[\psi_1, \omega_2] - C[\omega_1, \omega_2] + \beta\psi_{1x} = 0, \tag{6}$$

$$\omega_{2t} + [\psi_2, \omega_2] + \varepsilon[\psi_2, \omega_1] - C[\omega_2, \omega_1] + \beta\psi_{2x} = 0, \tag{7}$$

which describes the two-vortex interaction. The corresponding nonlocal system can be obtained by the following AB reduction

$$\psi_2 = \hat{P}_s^x\hat{T}_d\psi_1 = \psi_1(-x + x_0, y, -t + t_0), \tag{8}$$

$$\omega_2 = \hat{P}_s^x\hat{T}_d\omega_1 = \omega_1(-x + x_0, y, -t + t_0). \tag{9}$$

To derive a nonlocal Burgers type equation from Eqs. (5)–(9), according to the multi-scale expansion method, the following long wave approximation assumption is assumed:

$$\xi = \varepsilon(x - c_0 t), \quad \tau = \varepsilon^2 t, \tag{10}$$

where ε is a small parameter, and c_0 is an arbitrary constant. The stream function ψ_1 can be expanded as

$$\psi_1 = -c_0 y - \int u_0\,\mathrm{d}y + \psi_{11}(\xi, y, \tau), \tag{11}$$

where the first two terms of the above expansion represents a linear background westerly, $u_0 \equiv u_0(y)$ is an arbitrary function of y, and the last term can be expanded as

$$\psi_{11}(\xi, y, \tau) = \varepsilon\phi_{11} + \varepsilon^2\phi_{12} + \varepsilon^3\phi_{13} + O(\varepsilon^4). \tag{12}$$

Meanwhile, the constants C and β can also be expanded as

$$C = \varepsilon^2\delta, \quad \beta = \varepsilon^2\mu, \tag{13}$$

which means that both the strength of VVI and the effect of the rotation of the earth are much smaller.

Substituting Eqs. (11) with Eqs. (10), (12), (13) into Eqs. (6) and (7) and vanishing coefficients of different powers of ε, several different equations are obtained, among which, the ones corresponding to $O(\varepsilon)$ are

$$u_0\phi_{11yy\xi} - u_{0yy}\phi_{11\xi} = 0, \tag{14}$$

$$u_0\phi_{21yy\xi} - u_{0yy}\phi_{21\xi} = 0, \tag{15}$$

where $\phi_{21} = \hat{P}_s^\xi\hat{T}_d^\tau\phi_{11} = \phi_{11}(-\xi + \xi_0, y, -\tau + \tau_0)$.

Suppose ϕ_{11} and ϕ_{12} in Eqs. (14) and (15) having the following variable separation form:

$$\phi_{11} = G_0(y)A(\xi, \tau) \equiv G_0 A, \tag{16}$$

$$\phi_{21} = G_0 B(\xi, \tau) \equiv G_0 B, \tag{17}$$

with $B = \hat{P}_s^\xi\hat{T}_d^\tau A$. Then substitute Eqs. (16) and (17) into Eqs. (14) and (15) to find

$$G_0 u_{0,yy} - u_0 G_{0,yy} = 0. \tag{18}$$

A general form of solution for G_0 is

$$G_0 = u_0\left(C_1\int\frac{1}{u_0^2}\,\mathrm{d}y + C_2\right), \tag{19}$$

with C_1 and C_2 being arbitrary constants.

Now vanishing the coefficients of $O(\varepsilon^2)$ leads to

$$\begin{aligned} &u_0\phi_{12yy\xi} - u_{0yy}\phi_{12\xi} \\ =\ &\phi_{11y}\phi_{11yy\xi} + (u_{0yy} - \phi_{11yyy})\phi_{11\xi} \\ &+ (u_0 + c_0)\phi_{21yy\xi} - \phi_{11yy\tau}, \end{aligned} \tag{20}$$

$$\begin{aligned} &u_0\phi_{22yy\xi} - u_{0yy}\phi_{22\xi} \\ =\ &\phi_{21y}\phi_{21yy\xi} + (u_{0yy} - \phi_{21yyy})\phi_{21\xi} \\ &+ (u_0 + c_0)\phi_{11yy\xi} - \phi_{21yy\tau}, \end{aligned} \tag{21}$$

where $\phi_{22} = \hat{P}_s^\xi\hat{T}_d^\tau\phi_{12} = \phi_{12}(-\xi + \xi_0, y, -\tau + \tau_0)$.

To solve Eqs. (20) and (21), unlike the traditional y average trick, we assume

$$\phi_{12} = G_1 B + G_2 A + G_3 A^2 + G_4 B^2 + G_5 AB + G_6 A_\xi, \tag{22}$$

$$\phi_{22} = \hat{P}_s^\xi\hat{T}_d^\tau\phi_{12}, \tag{23}$$

where $G_i \equiv G_i(y)$ $(i = 1, \ldots, 6)$ are functions to be determined later. By substituting Eqs. (22) and (23) into Eqs. (20) and (21)

and requiring all the coefficients of different derivatives of A and B proportional to each other up to a constant, we obtain the desired nonlocal Burgers equation

$$A_\tau + (e_2A + e_3B + e_4)A_\xi + (e_3A + 2e_5B - e_6)B_\xi + e_1A_{\xi\xi} = 0, \tag{24}$$

$$B = \hat{P}_s^\xi \hat{T}_d^\tau A, \tag{25}$$

with

$$G_1 = u_0\left(m_{11} + \int \frac{-m_{12} + \int (u_0G_0'' + c_0G_0'' - e_6G_0'')\,\mathrm{d}y}{u_0^2}\,\mathrm{d}y\right), \tag{26}$$

$$G_2 = u_0\left(m_7 + \int \frac{m_8 + \int (u_0''G_0 + e_4G_0'')\,\mathrm{d}y}{u_0^2}\,\mathrm{d}y\right), \tag{27}$$

$$G_3 = u_0\left(m_3 - \frac{1}{2}\int \frac{G_0G_0'' - G_0'^2 - e_2G_0' - m_4}{u_0^2}\,\mathrm{d}y\right), \tag{28}$$

$$G_4 = u_0\left(m_9 + \int \frac{m_{10} + e_5G_0'}{u_0^2}\,\mathrm{d}y\right), \tag{29}$$

$$G_5 = u_0\left(m_5 + \int \frac{m_6 + e_3G_0'}{u_0^2}\,\mathrm{d}y\right), \tag{30}$$

$$G_6 = u_0\left(m_1 + \int \frac{m_2 + e_1G_0'}{u_0^2}\,\mathrm{d}y\right), \tag{31}$$

where m_i $(i = 1, \ldots, 12)$ are arbitrary constants.

3. Exact solutions of the nonlocal Burgers equation and their applications

To obtain exact solutions of the nonlocal Burgers equation (24), a direct and simple way is to apply the function expansion method to it. After some routine calculations, two periodic wave solutions are obtained, the first one is

$$A = -\frac{a_2e_4 + a_2e_6 + b_2}{2a_2(e_3 - 2e_5)} + c\cdot \mathrm{sn}\left[a_2\left(\xi - \frac{1}{2}\xi_0\right) + b_2\left(\tau - \frac{1}{2}\tau_0\right), m\right], \tag{32}$$

with

$$e_1 = 0, \quad e_2 = 2e_3 - 2e_5, \tag{33}$$

and the other one is

$$A = a_1 + c\cdot \mathrm{cn}\left[a_3\left(\xi - \frac{1}{2}\xi_0\right) + b_3\left(\tau - \frac{1}{2}\tau_0\right), m\right], \tag{34}$$

with

$$a_3 = \frac{-b_3}{e_4 - e_6}, \quad e_1 = 0, \quad e_2 = -2e_3 - 2e_5, \tag{35}$$

where a_1, a_2, b_2, b_3 are arbitrary constants, and sn and cn are Jacobi elliptic sine and cosine functions, respectively, with modulus m. Figure 1(a) shows the solution of Eq. (32) with $B = \hat{P}_s^\xi \hat{T}_d^\tau A$, and the parameters are fixed as

$$a_2 = b_2 = e_3 = e_4 = e_6 = c = 1, \quad e_5 = 2, \quad a_1 = \frac{1}{2}, \quad e_2 = -2, \quad \xi_0 = \tau_0 = 0, \quad m = 0.9, \tag{36}$$

at a specific time $\tau = 1$. Figure 1(b) shows the solution of Eq. (34) with $B = \hat{P}_s^\xi \hat{T}_d^\tau A$, and the parameters are fixed as

$$a_1 = a_3 = b_3 = e_3 = e_4 = e_5 = c = 1, \quad e_1 = 0, \quad e_2 = -4, \quad e_6 = 2, \quad \tau_0 = 2, \quad \xi_0 = 0, \quad m = 0.9, \tag{37}$$

at the same specific time $\tau = 1$. From Figs. 1(a) and 1(b), it is obvious that solution (32) is $\hat{P}_s^\xi \hat{T}_d^\tau$ symmetry breaking while solution (34) is $\hat{P}_s^\xi \hat{T}_d^\tau$ symmetry invariant. When the modulus m in Eqs. (32) and (34) approaches to unity, they reduce to kink and bright solitons, respectively, which are shown in Figs. 2(a) and 2(b).

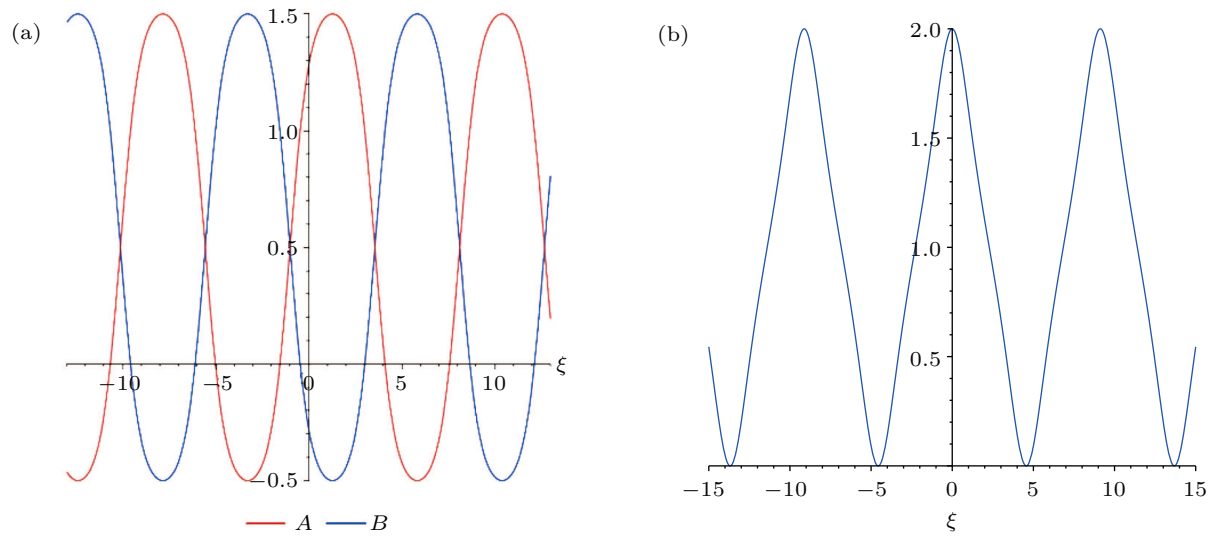


Fig. 1. Two specific solutions of the nonlocal Burgers equation (24) with (25) expressed by (a) Eq. (32) and (b) Eq. (34) at a specific time $\tau = 1$. The parameters are fixed as (a) $a_2 = b_2 = e_3 = e_4 = e_6 = c = 1$, $e_5 = 2$, $a_1 = 1/2$, $e_2 = -2$, $\xi_0 = \tau_0 = 0$, $m = 0.9$; (b) $a_1 = a_3 = b_3 = e_3 = e_4 = e_5 = c = 1$, $e_1 = 0$, $e_2 = -4$, $e_6 = 2$, $\tau_0 = 2$, $\xi_0 = 0$, $m = 0.9$.

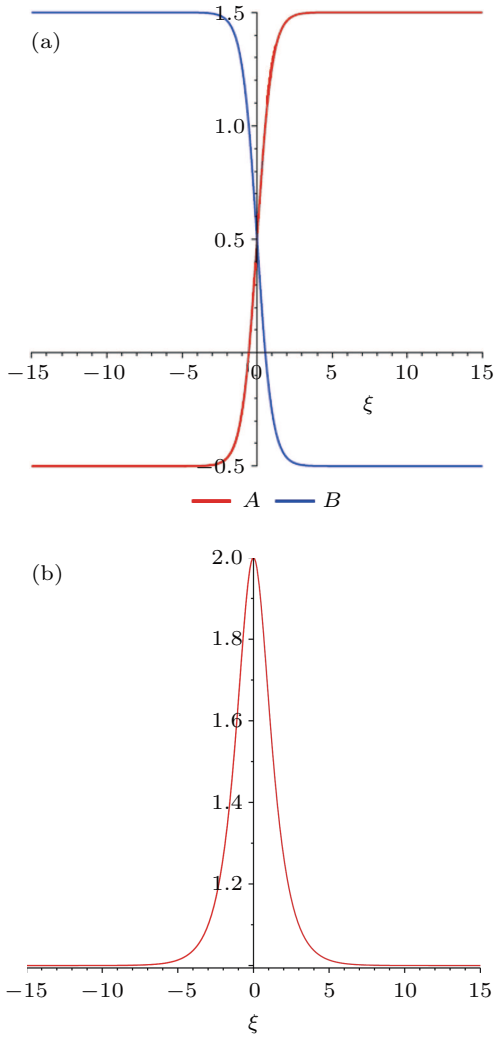


Fig. 2. Two soliton solutions of the nonlocal Burgers equation (24) with (25) expressed by (a) Eq. (32) and (b) Eq. (34) at a specific time $\tau = 1$. The parameters are fixed as (a) $a_2 = b_2 = e_3 = e_4 = e_6 = c = 1$, $e_5 = 2$, $a_1 = 1/2$, $e_2 = -2$, $\xi_0 = \tau_0 = 0$, $m = 1$; (b) $a_1 = a_3 = b_3 = e_3 = e_4 = e_5 = c = 1$, $e_1 = 0$, $e_2 = -4$, $e_6 = 2$, $\tau_0 = 2$, $\xi_0 = 0$, $m = 1$.

Atmospheric blocking is an important large-scale weather phenomenon in mid-high latitudes in the atmosphere that has a profound effect on climates.[15] In the local case, using the multiple-scale method, many various nonlinear systems, such as KdV, mKdV, and NLS equations, have been obtained, meanwhile their exact solutions were used to simulate the dynamical behavior of dipole blocking and monopole blocking events.[16–19] Inspired by these works, we seek physical applications of exact solutions of the nonlocal Burgers equation (24) with (25). To this end, up to $O(\varepsilon)$, we first obtain the approximate solution of Eqs. (6) and (7) from Eqs. (12) and (16)

$$\psi_1 = -c_0 y - \int u_0 \mathrm{d}y + \varepsilon G_0(y) A(\xi, \tau), \ \psi_2 = \hat{P}_s^{\xi} \hat{T}_d^{\tau} \psi_1, \quad (38)$$

with $\xi = \varepsilon(x - c_0 t)$, $\tau = \varepsilon^2 t$, and G_0 is given by Eq. (19). When taking $A(\xi, \tau)$ in Eq. (38) as the solution in Eq. (34) and fixing the arbitrary functions and parameters as

$$u_0 = \sin\frac{y}{4}, \quad (39)$$

$$\begin{aligned} &e_1 = c_0 = C_1 = \tau_0 = \xi_0 = 0, \\ &a_1 = b_2 = c = e_3 = e_5 = e_6 = \tau = 1, \ C_2 = 200, \\ &a_2 = -1, \ e_2 = -4, \ e_4 = 2, \ \varepsilon = 0.1, \end{aligned} \quad (40)$$

two correlated dipole blocking events are obtained, which are shown graphically in Figs. 3(a) and 3(b).

It should be noted that the existence of an arbitrary function u_0 in Eq. (38) provides potential explanations for related atmosphere phenomena, not only the dipole blocking events mentioned here, but also, for example, monopole blocking events. Furthermore, the approximate solution of Eqs. (6) and (7) can also be taken to the $O(\varepsilon^2)$ order, which can be applied to find more complicated physical applications.

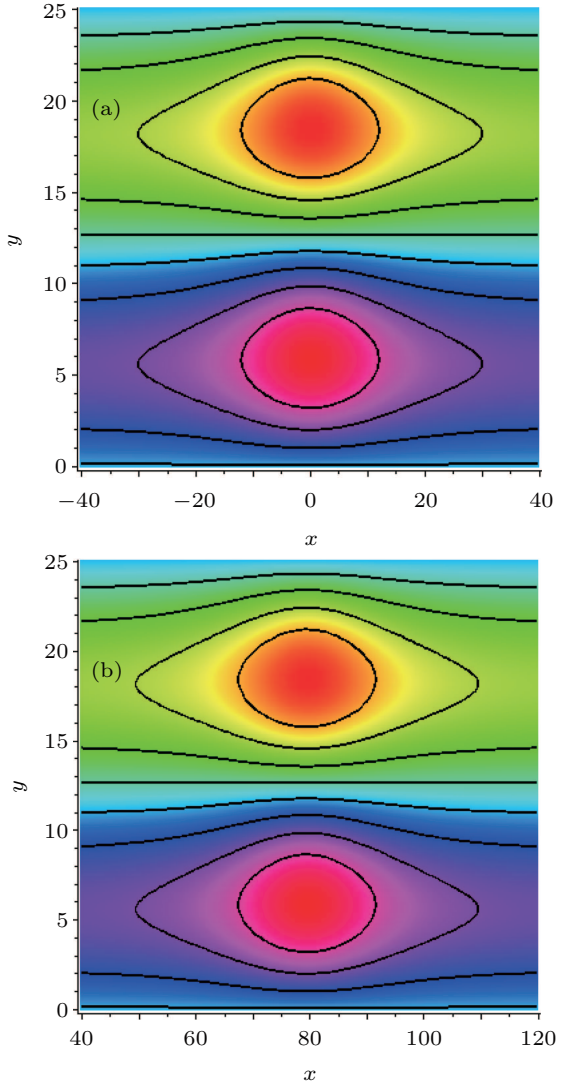


Fig. 3. Two correlated theoretical dipole events given by Eq. (38): (a) ψ_1, (b) ψ_2.

4. Symmetry reduction solutions of the nonlocal Burgers system

It is well known that symmetry analysis plays an important role in simplifying and even completely solving complicated nonlinear problems.[20,21] In this section, using the stan-

dard Lie group method, we study the Lie symmetry of the nonlocal Burgers system (24) with (25) in the form

$$V = X\frac{\partial}{\partial\xi} + T\frac{\partial}{\partial\tau} + \bar{A}\frac{\partial}{\partial A} + \bar{B}\frac{\partial}{\partial B}. \tag{41}$$

In other words, the nolocal Burgers system (24) and its counterpart

$$\begin{aligned} &B_\tau + (e_2B + e_3A + e_4)B_\xi + (e_3B + 2e_5A - e_6)A_\xi \\ &+ e_1B_{\xi\xi} = 0 \end{aligned} \tag{42}$$

with (25) are invariant under the transformation

$$\{\xi, \tau, A, B\} \to \{\xi + \varepsilon X, \tau + \varepsilon T, A + \varepsilon\bar{A}, B + \varepsilon\bar{B}\} \tag{43}$$

with the infinitesimal parameter ε. Equivalently, the symmetry in the form (41) can be written as a function form as

$$\sigma_A = XA_\xi + TA_\tau - \bar{A}, \tag{44a}$$
$$\sigma_B = XB_\xi + TB_\tau - \bar{B}, \tag{44b}$$

where σ_A and σ_B satisfy the linearized equations of Eqs. (24), (42), and (25), i.e.,

$$\begin{aligned} &(e_2\sigma_A + e_3\sigma_B)A_\xi + (e_3\sigma_A + 2e_5\sigma_B)B_\xi + e_1\sigma_{A,\xi\xi} + \sigma_{A,\tau} \\ &+ (e_2A + e_3B + e_4)\sigma_{A,\xi} + (e_3A + 2e_5B - e_6)\sigma_{B,\xi} = 0, \end{aligned} \tag{45a}$$
$$\begin{aligned} &(e_2\sigma_B + e_3\sigma_A)B_\xi + (e_3\sigma_B + 2e_5\sigma_A)A_\xi + e_1\sigma_{B,\xi\xi} + \sigma_{B,\tau} \\ &+ (e_2B + e_3A + e_4)\sigma_{B,\xi} + (e_3B + 2e_5A - e_6)\sigma_{A,\xi} = 0, \end{aligned} \tag{45b}$$
$$\sigma_B = \sigma_A(-\xi + \xi_0, -\tau + \tau_0). \tag{45c}$$

Substituting Eq. (44) with Eqs. (24) and (42) into Eq. (45) and vanishing all the coefficients of the independent partial derivatives of variables A and B, a system of over determined linear equations for $X, T, \bar{A}, \bar{B}$ is obtained. Calculated with a computer, we obtain the desired solitons

$$\begin{aligned} X &= \frac{c_1}{2}\xi + \frac{c_1(e_2e_6 + e_3e_4 + e_3e_6 + 2e_4e_5)}{2(e_3 + 2e_5)}\tau, \\ T &= c_1\tau, \quad \bar{A} = -\frac{c_1}{2}A + \frac{c_1e_6}{2(e_3 + 2e_5)}, \\ \bar{B} &= -\frac{c_1}{2}B + \frac{c_1e_6}{2(e_3 + 2e_5)}, \end{aligned} \tag{46}$$

with c_1 being an arbitrary constant and $\xi_0 = \tau_0 = 0$.

Substituting Eq. (46) into Eq. (44), one obtains

$$\begin{aligned} \sigma_A &= \left[\frac{c_1}{2}\xi + \frac{c_1(e_2e_6 + e_3e_4 + e_3e_6 + 2e_4e_5)}{2(e_3 + 2e_5)}\tau\right]A_\xi \\ &+ c_1\tau A_\tau + \frac{c_1}{2}A - \frac{c_1e_6}{2(e_3 + 2e_5)}, \end{aligned} \tag{47}$$
$$\begin{aligned} \sigma_B &= \left[\frac{c_1}{2}\xi + \frac{c_1(e_2e_6 + e_3e_4 + e_3e_6 + 2e_4e_5)}{2(e_3 + 2e_5)}\tau\right]B_\xi \\ &+ c_1\tau B_\tau + \frac{c_1}{2}B - \frac{c_1e_6}{2(e_3 + 2e_5)}. \end{aligned} \tag{48}$$

The group invariant solutions of the nonlocal Burgers system can be obtained by solving Eqs. (47) and (48) under the symmetry constraints $\sigma_A = \sigma_B = 0$, alternatively, solving the corresponding characteristic equation

$$\begin{aligned} &\frac{\mathrm{d}\xi}{\frac{c_1}{2}\xi + \frac{c_1(e_2e_6 + e_3e_4 + e_3e_6 + 2e_4e_5)}{2(e_3 + 2e_5)}\tau} = \frac{\mathrm{d}\tau}{c_1\tau} \\ &= \frac{\mathrm{d}A}{-\frac{c_1}{2}A + \frac{c_1e_6}{2(e_3 + 2e_5)}} = \frac{\mathrm{d}B}{-\frac{c_1}{2}B + \frac{c_1e_6}{2(e_3 + 2e_5)}}. \end{aligned} \tag{49}$$

By solving Eq. (49), the symmetry reduction solutions of the nonlocal Burgers system (24) with (25) are

$$A = \frac{\tilde{A}}{\sqrt{\tau}} + \frac{e_6}{e_3 + 2e_5}, \tag{50}$$
$$B = \frac{\tilde{B}}{\sqrt{\tau}} + \frac{e_6}{e_3 + 2e_5}, \tag{51}$$

where $\tilde{A}$ and $\tilde{B}$ are invariant functions of

$$\Xi = \frac{\xi}{\sqrt{\tau}} - \frac{\sqrt{\tau}(e_2e_6 + e_3e_4 + e_3e_6 + 2e_4e_5)}{e_3 + 2e_5}.$$

Substituting Eqs. (50) and (51) into the nonlocal Burgers system (24), (42) with (25) yields the symmetry reduction equations

$$\begin{aligned} &(2e_2\tilde{A} + 2e_3\tilde{B} - \Xi)\tilde{A}_\Xi + (2e_3\tilde{A} + 4e_5\tilde{B})\tilde{B}_\Xi \\ &+ 2e_1\tilde{A}_{\Xi\Xi} - \tilde{A} = 0, \end{aligned} \tag{52}$$
$$\begin{aligned} &(2e_2\tilde{B} + 2e_3\tilde{A} - \Xi)\tilde{B}_\Xi + (2e_3\tilde{B} + 4e_5\tilde{A})\tilde{A}_\Xi \\ &+ 2e_1\tilde{B}_{\Xi\Xi} - \tilde{B} = 0, \end{aligned} \tag{53}$$
$$\tilde{B}(\Xi) = -\mathrm{i}\tilde{A}(\mathrm{i}\Xi), \tag{54}$$

with i being the imaginary unit number.

5. Discussion and conclusion

In summary, a nonlocal Burgers equation with shifted parity and delayed time reversal is derived from a two-vortex interaction model in atmospheric and oceanic systems. Some exact solutions of the nonlocal Burgers equations are obtained by using the function expansion method, which include periodic wave solutions and soliton solutions. As an illustration of applications of these solutions, an approximate solution of the nonlocal two-vortex interaction model is obtained and used to simulate two correlated dipole blocking events graphically. To study the nonlocal Burgers system from a symmetry perspective, we also apply the standard Lie symmetry method to giving the general symmetry reduction solutions and symmetry reduction equations.

From the derivation of the nonlocal Burgers equation, it is clear that the multi-scale expansion method is powerful in obtaining the desired nonlocal nonlinear equations. In this sense, it is hoped to derive other types of nonlocal equations from many physically important models and use their approximate solutions to analyze correlated phenomena, which deserve to

be studied in the future. In the other aspect, finding symmetries and conservation laws for any nonlinear differential equations has important meaning, which is a challenging task for nonlocal nonlinear systems because of the nonlocal property. To solve this difficulty, a direct and simple way is to introduce new dependent variables to enlarge the nonlocal equation to a localized coupled nonlinear system. Then the symmetries and conservation laws of the original nonlocal system can be derived from the counterparts of the localized coupled system. Furthermore, symmetry reduction solutions of nonlocal equations can also be given by the similar procedure. All of these ideas will be carried out in our future works.

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