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Fractional squeezing-Hankel transform based on the induced entangled state representations*

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Based on the fact that the quantum mechanical version of Hankel transform kernel (the Bessel function) is just the transform between $|q,r\rangle$ and $(s,r'|$, two induced entangled state representations are given, and working with them we derive fractional squeezing-Hankel transform (FrSHT) caused by the operator $\mathrm{e}^{-\mathrm{i}\alpha\left(a_1^{\dagger}a_2^{\dagger}+a_1a_2\right)}\mathrm{e}^{-\mathrm{i}\pi a_2^{\dagger}a_2}$, which is an entangled fractional squeezing transform operator. The additive rule of the FrSHT can be explicitly proved.

Keywords: induced entangled state representation, entangled fractional squeezing transform, fractional squeezing-Hankel transform

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1. Introduction

The fractional Fourier transform (FrFT), an extension of the Fourier transform, was first developed by Kober[1] and then rediscovered by Namias,[2,3] who used the FrFT in the context of quantum mechanics as a way to solve certain problems involving quantum harmonic oscillators. However, they were Mendlovic, Ozaktas, and Lohmann who firstly introduced the FrFT to optics in 1993,[4–6] and the optical implementation of FrFT was recognized based on the fact that a piece of graded-index fiber (GRIN) of proper length can perform a Fourier transform optically. Nowadays, the FrFT has been developed and utilized by many researchers, and been used in almost all applications where Fourier transforms were used.[7,8] The one-dimensional α-angle FrFT of a function g is defined as[9]

$$\begin{aligned} &F_\alpha\left[g\left(p\right)\right] \\ &= \frac{\mathrm{e}^{\mathrm{i}(\alpha/2-\pi/4)}}{\sqrt{2\pi\sin\alpha}}\int_{-\infty}^{\infty}\exp\left[\frac{\mathrm{i}\left(p^2+x^2\right)}{2\tan\alpha}-\frac{\mathrm{i}px}{\sin\alpha}\right]g\left(x\right)\mathrm{d}x \\ &= \int_{-\infty}^{\infty}h\left(p,x\right)g\left(x\right)\mathrm{d}x. \end{aligned} \tag{1}$$

FrFT satisfies the additive rule (or the compositional rule) $F_\alpha\circ F_\beta\left[g\right]=F_{\alpha+\beta}\left[g\right]$, i.e., two successive FrFTs with parameters α and β are equivalent to a single FrFT with parameter $\alpha+\beta$. In addition, in Ref. [10], by converting the triangular functions in the integration kernel $h\left(p,x\right)$ of the FrFT to the hyperbolic functions, we have found the quantum mechanical fractional squeezing transform (FrST) which satisfies additivity and we derived the unitary operator responsible for the FrST which can be seen in Ref. [10].

In Refs. [11]–[14], based on the two mutually conjugate entangled state representations $|\eta\rangle$ and $|\xi\rangle$, Fan *et al.* have constructed the complex fractional Fourier transform (CFrFT) of the form

$$F_\alpha\left[f\left(\eta\right)\right]=\int_{-\infty}^{\infty}\frac{\mathrm{d}^2\xi}{\pi}K_\alpha\left(\eta,\xi\right)f\left(\xi\right), \tag{2}$$

where the integration kernel $K_\alpha\left(\eta,\xi\right)$ is

$$\begin{aligned} K_\alpha\left(\eta,\xi\right) = &\frac{\mathrm{e}^{\mathrm{i}\left(\alpha-\frac{\pi}{2}\right)}}{2\sin\alpha} \\ &\times\exp\left[\frac{\mathrm{i}\left(|\eta|^2+|\xi|^2\right)}{2\tan\alpha}-\frac{\mathrm{i}\left(\xi\eta^*+\xi^*\eta\right)}{2\sin\alpha}\right]. \end{aligned} \tag{3}$$

The additive rule satisfied by CFrFT was also proved. Later, Dragoman has shown that the kernel of CFrFT can be classically produced with rotated astigmatic optical systems that mimic the quantum entanglement.[15] More applications of the CFrFT can be found in Ref. [14]. Based on the work,[10,14] Chen *et al.* have proposed an entangled fractional squeezing transform (EFrST) and described its fractional property by using two mutually conjugate entangled state representations $|\eta\rangle$ and $|\xi\rangle$, which will be introduced in Section 2.

On the other hand, in mathematical physics the Hankel transform is very useful in dealing with many linear partial differential equations with appropriate initial-boundary-value problems which have obvious physical background, such as the axisymmetric diffusion equation,[16] the vibration of a large plate,[17] axisymmetric Cauchy–Poisson water wave equation,[18] and acoustic radiation equation.[19] The Hankel transform of a function $f(r)$ is defined as[20]

$$\mathcal{H}_n\left[f(r)\right]=\int_0^{\infty}r\mathrm{J}_n(kr)f(r)\mathrm{d}r=\tilde{f}_n(k), \tag{4}$$

where $\mathrm{J}_n(kr)$ is the Bessel function of order n. The inverse

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Hankel tranform is

$$\mathcal{H}_n^{-1}\left[\tilde{f}_n(k)\right]=\int_0^\infty kJ_n(kr)\tilde{f}_n(k)\mathrm{d}k, \tag{5}$$

provided that the integral exists.

An interesting question thus naturally arises: how to introduce the fractional squeezing-Hankel transform (FrSHT)? In this work we shall set forth the FrSHT by making full use of quantum mechanical entangled state representations theory, i.e., generating FrSHT in $|q,r\rangle$ and $|s,r'\rangle$ representation, which are respectively induced from two mutually conjugate entangled state representations $|\eta\rangle$ and $|\xi\rangle$ (see Eqs. (11) and (12) below). In Section 2, employing the integration technique within an ordered product[21] of operators with which we can directly perform integration over Dirac's ket–bra operators,[22] we briefly introduce the EFrST and illustrate the unitary operator responsible for the EFrST. In Sections 3–4, we introduce two entangled states $|q,r\rangle$ from $|\eta\rangle$, $|s,r'\rangle$ from $|\xi\rangle$, which respectively constitute the so-called induced entangled state representations. It is remarkable that the overlap $(s,r'|\,q,r\rangle=\frac{1}{2}\delta_{s,q}J_s(rr')$ is just the Bessel function. In Section 5, we calculate the matrix element $(s,r|\,\mathfrak{S}_\alpha\,|q,r'\rangle$ which is the integration kernel of FrSHT. The additive rule of FrSHT is also illustrated in Section 6.

2. The entangled fractional squeezing transform

Through converting the triangular functions in the integration kernel $K_\alpha(\eta,\xi)$ in Eq. (3) into hyperbolic functions, i.e., $\tan\alpha\to\tanh\alpha$, $\sin\alpha\to\sinh\alpha$, we will get the integration kernel of an EFrST[10]

$$\mathfrak{S}_\alpha(\eta,\xi)=\frac{e^{i(\alpha-\frac{\pi}{2})}}{2\sinh\alpha}\exp\left[\frac{i\left(|\eta|^2+|\xi|^2\right)}{2\tanh\alpha}-i\frac{\xi\eta^*+\xi^*\eta}{2\sinh\alpha}\right]. \tag{6}$$

From the point of view of quantum mechanics, $\mathfrak{S}_\alpha(\eta,\xi)$ of the α-angle EFrST is proved to be the composite transformation including both the basis changing and unitary transformation generated by the unitary operator $\mathfrak{S}_\alpha$

$$\mathfrak{S}_\alpha(\eta,\xi)=\langle\eta|\,\mathfrak{S}_\alpha\,|\xi\rangle. \tag{7}$$

$\mathfrak{S}_\alpha$ is called the EFrST operator, which has the form[10]

$$\mathfrak{S}_\alpha=e^{-i\alpha\left(a_1^\dagger a_2^\dagger+a_1a_2\right)}e^{-i\pi a_2^\dagger a_2}=e^{-i\pi a_2^\dagger a_2}e^{i\alpha\left(a_1^\dagger a_2^\dagger+a_1a_2\right)}, \tag{8}$$

where $e^{-i\alpha\left(a_1^\dagger a_2^\dagger+a_1a_2\right)}$ is a two-mode squeezing operator. Then we can get the relationship

$$\begin{aligned}\mathfrak{S}_\alpha^\dagger&=e^{-i\alpha\left(a_1^\dagger a_2^\dagger+a_1a_2\right)}e^{i\pi a_2^\dagger a_2}=e^{-i\alpha\left(a_1^\dagger a_2^\dagger+a_1a_2\right)}e^{-i\pi a_2^\dagger a_2}\\&=\mathfrak{S}_\alpha.\end{aligned} \tag{9}$$

$e^{-i\pi a_2^\dagger a_2}$ is indispensable to the EFrST which is called the core operator to EFrST, which we emphasize as being essential to the EFrST, since by "fractional" we mean the additive rule (or the compositional rule) $\mathfrak{S}_\alpha\circ\mathfrak{S}_\beta[f](\eta)=\mathfrak{S}_{\alpha+\beta}[f](\eta)$, i.e.,

$$\begin{aligned}&\mathfrak{S}_\alpha\circ\mathfrak{S}_\beta[f](\eta)\\&=\int\frac{d^2\xi}{\pi}\int\frac{d^2\xi'}{\pi}\mathfrak{S}_\alpha(\eta,\xi')[\mathfrak{S}_\beta(\eta',\xi)_{\eta'=\xi'}]f(\xi)\\&=\int\frac{d^2\xi}{\pi}\int\frac{d^2\xi'}{\pi}\langle\eta|\,e^{-i\pi a_2^\dagger a_2}e^{i\alpha\left(a_1^\dagger a_2^\dagger+a_1a_2\right)}\left|\xi'\right\rangle_{\eta'=\xi'}\langle\eta'|\\&\quad\times e^{-i\pi a_2^\dagger a_2}e^{i\beta\left(a_1^\dagger a_2^\dagger+a_1a_2\right)}|\xi\rangle\langle\xi|\,f\rangle\\&=\int\frac{d^2\xi}{\pi}\int\frac{d^2\xi'}{\pi}\langle\eta|\,e^{-i\pi a_2^\dagger a_2}e^{i\alpha\left(a_1^\dagger a_2^\dagger+a_1a_2\right)}\left|\xi'\right\rangle\langle\xi'|\,e^{-i\pi a_2^\dagger a_2}\\&\quad\times e^{i\beta\left(a_1^\dagger a_2^\dagger+a_1a_2\right)}|\xi\rangle\langle\xi|\,f\rangle\\&=\langle\eta|\,e^{-i\pi a_2^\dagger a_2}\exp[i(\alpha+\beta)(a_1^\dagger a_2^\dagger+a_1a_2)]\,|f\rangle\\&=\mathfrak{S}_{\alpha+\beta}[f](\eta).\end{aligned} \tag{10}$$

In Eq. (6), $|\eta\rangle$ and $|\xi\rangle$ are the two mutually conjugate entangled state representations,[23] that is,

$$\begin{aligned}|\eta\rangle&=\exp\left(-\frac{1}{2}|\eta|^2+\eta a_1^\dagger-\eta^*a_2^\dagger+a_2^\dagger a_1^\dagger\right)|00\rangle,\\\eta&=\eta_1+i\eta_2,\end{aligned} \tag{11}$$

and

$$\begin{aligned}|\xi\rangle&=\exp\left(-\frac{1}{2}|\xi|^2+a_1^\dagger\xi+\xi^*a_2^\dagger-a_2^\dagger a_1^\dagger\right)|00\rangle,\\\xi&=\xi_1+i\xi_2,\end{aligned} \tag{12}$$

which satisfy the completeness relations

$$\int\frac{d^2\eta}{\pi}|\eta\rangle\langle\eta|=1,\quad\int\frac{d^2\xi}{\pi}|\xi\rangle\langle\xi|=1, \tag{13}$$

and their overlap is also in Fourier transform type $\langle\eta|\,\xi\rangle=\frac{1}{2}\exp[(\eta^*\xi-\eta\xi^*)/2]=\frac{1}{2}\exp[i(\eta_1\xi_2-\eta_2\xi_1)]$.

In Section 3, we derive the induced entangled representation $|q,r\rangle$ from $|\eta\rangle$, $|s,r'\rangle$ from $|\xi\rangle$, respectively, and show that the mutual transform between $|q,r\rangle$ and $|s,r'\rangle$ is just the Hankel transform. If we do so, the Hankel transform can be studied under the quantum mechanical background and the Hankel transform of quantum state vectors can be naturally introduced. In the following, we propose the fractional squeezing-Hankel transform in the two induced entangled state representations.

3. Two induced entangled representations $|q,r\rangle$ from $|\eta\rangle$, $|s,r'\rangle$ from $|\xi\rangle$ and their overlap

By taking $\eta=re^{i\theta}$, $\xi=r'e^{i\varphi}$ from $|\eta\rangle$, we can define[24]

$$|q,r\rangle\equiv\frac{1}{2\pi}\int_0^{2\pi}\left|\eta=re^{i\theta}\right\rangle e^{-iq\theta}\mathrm{d}\theta. \tag{14}$$

It can be proved that $|q,r\rangle$ is the common eigenvector of the two-mode number-difference operator $a_1^\dagger a_1-a_2^\dagger a_2$ and $(a_1-a_2^\dagger)(a_1^\dagger-a_2)$, since $\left[a_1^\dagger a_1-a_2^\dagger a_2,(a_1-a_2^\dagger)(a_1^\dagger-a_2)\right]=0$,

$$\left(a_1^\dagger a_1-a_2^\dagger a_2\right)|q,r\rangle=q\,|q,r\rangle, \tag{15}$$

$$(a_1-a_2^\dagger)(a_1^\dagger-a_2)\,|q,r\rangle = r^2\,|q,r\rangle\,, \tag{16}$$

and $|q,r\rangle$ also satisfies completeness relation and makes up a complete-orthonormal representation, that is,

$$\langle q,r|\,q',r'\rangle = \delta_{q,q'}\frac{1}{2r}\delta\left(r-r'\right), \tag{17}$$

$$\sum_{q=-\infty}^{\infty}\int_0^\infty \mathrm{d}r^2\,|q,r\rangle\,\langle q,r| = 1. \tag{18}$$

Similarly, we can also introduce $|s,r')$ from $|\xi\rangle$ as (the symbol $|\ ,\ )$ is introduced for distinguishing it from $|\ ,\ \rangle$)

$$|s,r') = \frac{1}{2\pi}\int_0^{2\pi}\left|\xi = r'\mathrm{e}^{\mathrm{i}\varphi}\right\rangle \mathrm{e}^{-\mathrm{i}s\varphi}\mathrm{d}\varphi, \tag{19}$$

where $|s,r')$ is the common eigenvector of the two-mode number-difference operator $a_1^\dagger a_1 - a_2^\dagger a_2$ and $(a_1^\dagger + a_2)(a_1 + a_2^\dagger)$, which satisfy $\left[a_1^\dagger a_1 - a_2^\dagger a_2, (a_1^\dagger + a_2)(a_1 + a_2^\dagger)\right] = 0$. Simultaneously, they obey

$$\begin{aligned}&\left(a_1^\dagger a_1 - a_2^\dagger a_2\right)|s,r') = s\,|s,r')\,,\\ &(a_1^\dagger + a_2)(a_1 + a_2^\dagger)\,|s,r') = r'^2\,|s,r')\,.\end{aligned} \tag{20}$$

$|s,r')$ is complete-orthonormal representation that

$$\begin{aligned}&\sum_{s=-\infty}^{\infty}\int_0^\infty \mathrm{d}r'^2\,|s,r')\,(s,r'| = 1,\\ &(s,r'|\,s,r'') = \delta_{s,s'}\frac{1}{2r'}\delta\left(r'-r''\right).\end{aligned} \tag{21}$$

From the point of view of Schmidt's decomposition, we can know that both $|q,r\rangle$ and $|s,r')$ are called induced entangled states. Noting both the integrals in Eqs. (14) and (19) can be performed throughly, the explicit forms of $|s,r')$ and $|q,r\rangle$ can be obtained, but in this work we do not need them.

According to Eqs. (14) and (19), we can calculate

$$\begin{aligned}&(s,r'|\,q,r\rangle\\ &= \frac{1}{4\pi^2}\int_0^{2\pi}\mathrm{d}\varphi\mathrm{e}^{\mathrm{i}s\varphi}\left\langle\xi = r'\mathrm{e}^{\mathrm{i}\varphi}\right|\int_0^{2\pi}\mathrm{d}\theta\left|\eta = r\mathrm{e}^{\mathrm{i}\theta}\right\rangle\mathrm{e}^{-\mathrm{i}q\theta}\\ &= \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}q\theta}\mathrm{e}^{\mathrm{i}s\varphi}\exp\left[\mathrm{i}rr'\sin\left(\theta-\varphi\right)\right]\mathrm{d}\theta\mathrm{d}\varphi\\ &= \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi}\mathrm{e}^{\mathrm{i}s\varphi}\mathrm{e}^{-\mathrm{i}q\theta}\sum_{m=-\infty}^{\infty}\mathrm{J}_m\left(rr'\right)\exp\left[\mathrm{i}m\left(\theta-\varphi\right)\right]\\ &= \frac{1}{2}\delta_{s,q}\mathrm{J}_s\left(rr'\right),\end{aligned} \tag{22}$$

and we have used the following formula of $\mathrm{J}_s(x)$:

$$\mathrm{e}^{\mathrm{i}x\sin t} = \sum_{s=-\infty}^{\infty}\mathrm{J}_s(x)\,\mathrm{e}^{\mathrm{i}st}, \tag{23}$$

where J_s is the Bessel function

$$\mathrm{J}_s(x) = \sum_{k=0}^{\infty}\frac{(-1)^s}{k!\,(s+k)!}\left(\frac{x}{2}\right)^{s+2k}. \tag{24}$$

By taking

$$\langle q,r|\,g\rangle = g\left(q,r\right),\ \ (s,r'|\,g\rangle = \mathcal{G}\left(s,r'\right), \tag{25}$$

and employing Eqs. (18) and (22), we see

$$\begin{aligned}\mathcal{G}\left(s,r'\right) &= (s,r'|\,g\rangle = \sum_{q=-\infty}^{\infty}\int_0^\infty \mathrm{d}r^2\,(s,r'|\,q,r\rangle\,\langle q,r|\,g\rangle\\ &= \int_0^\infty r\mathrm{d}r\mathrm{J}_s\left(rr'\right)g\left(s,r\right) \equiv \mathcal{H}\left[g\left(s,r\right)\right],\end{aligned} \tag{26}$$

which is just the Hankel transform of $g(q,r)$, well-defined mathematically. We can see that the quantum mechanical version of Hankel transform kernel is just the entangled states transform between $|q,r\rangle$ and $(s,r'|$. The reciprocal transform of Eq. (26) is

$$\begin{aligned}g\left(q,r\right) &= \langle q,r|\,g\rangle = \sum_{s=-\infty}^{\infty}\int_0^\infty \mathrm{d}r'^2\,\langle q,r|\,s,r')\,(s,r'|\,g\rangle\\ &= \int_0^\infty r'\mathrm{d}r'\mathrm{J}_s\left(rr'\right)\mathcal{G}\left(q,r'\right) = \mathcal{H}^{-1}\left[\mathcal{G}\left(q,r'\right)\right].\end{aligned} \tag{27}$$

Thus, we can say that the Hankel transform has its representation in quantum mechanical, that is, $(s,r'|\,q,r\rangle$ and $|s,r')$ is the s-order Hankel transform of $|s,r\rangle$.

In the following section, based on the entangled fractional squeezing transform operator and the form of Hankel transform, we can calculate the fractional squeezing-Hankel transform.

4. The fractional squeezing-Hankel transform in the induced entangled state representation

Using the entangled fractional squeezing transform (6), we now consider the integration

$$\begin{aligned}\mathfrak{S}_{\alpha;q,s}\left(r,r'\right) &\equiv \frac{1}{4\pi^2}\int_0^{2\pi}\mathrm{d}\varphi\int_0^{2\pi}\mathrm{d}\theta\mathrm{e}^{\mathrm{i}q\theta}\mathrm{e}^{-\mathrm{i}s\varphi}\,\langle\eta|\,\mathfrak{S}_\alpha\,|\xi\rangle\\ &= \frac{1}{4\pi^2}\int_0^{2\pi}\mathrm{d}\varphi\int_0^{2\pi}\mathrm{d}\theta\mathrm{e}^{-\mathrm{i}s\varphi}\mathrm{e}^{\mathrm{i}q\theta}\frac{1}{2\mathrm{i}\sinh\alpha}\\ &\quad\times\exp\left[\frac{\mathrm{i}\left(|\eta|^2+|\xi|^2\right)}{2\tanh\alpha} - \mathrm{i}\frac{\xi\eta^*+\xi^*\eta}{2\sinh\alpha}\right].\end{aligned} \tag{28}$$

In reference to Eqs. (14) and (19), we see that a transformation matrix element is essential in the induced entangled state representation

$$\begin{aligned}&\mathfrak{S}_{\alpha;q,s}\left(r,r'\right)\\ &= \langle q,r|\,\mathfrak{S}_\alpha\,|s,r')\\ &= \frac{1}{4\pi^2}\int_0^{2\pi}\mathrm{d}\varphi\int_0^{2\pi}\mathrm{d}\theta\mathrm{e}^{\mathrm{i}q\theta}\left\langle\eta = r\mathrm{e}^{\mathrm{i}\theta}\right|\mathfrak{S}_\alpha\left|\xi = r'\mathrm{e}^{\mathrm{i}\varphi}\right\rangle\mathrm{e}^{-\mathrm{i}s\varphi},\end{aligned} \tag{29}$$

where

$$\begin{aligned}&\left\langle\eta = r\mathrm{e}^{\mathrm{i}\theta}\right|\mathfrak{S}_\alpha\left|\xi = r'\mathrm{e}^{\mathrm{i}\varphi}\right\rangle\\ &= \frac{1}{2\mathrm{i}\sinh\alpha}\exp\left[\frac{\mathrm{i}\left(r^2+r'^2\right)}{2\tanh\alpha} - \mathrm{i}\frac{rr'\left(\mathrm{e}^{-\mathrm{i}(\theta-\varphi)}+\mathrm{e}^{\mathrm{i}(\theta-\varphi)}\right)}{2\sinh\alpha}\right]\end{aligned}$$

$$= \frac{1}{2\mathrm{i}\sinh\alpha}\exp\left[\frac{\mathrm{i}\left(r^2+r'^2\right)}{2\tanh\alpha}-\mathrm{i}\frac{rr'}{\sinh\alpha}\cos\left(\theta-\varphi\right)\right]$$
$$= \frac{1}{2\mathrm{i}\sinh\alpha}\exp\left[\frac{\mathrm{i}\left(r^2+r'^2\right)}{2\tanh\alpha}-\mathrm{i}\frac{rr'}{\sinh\alpha}\sin\left(\frac{\pi}{2}-\theta+\varphi\right)\right]. \tag{30}$$

Using Eq. (23), we can get

$$\exp\left[-\mathrm{i}\frac{rr'}{\sinh\alpha}\sin\left(\frac{\pi}{2}-\theta+\varphi\right)\right]$$
$$= \sum_{m=-\infty}^{\infty}\mathrm{J}_m\left(-\frac{rr'}{\sinh\alpha}\right)\mathrm{e}^{\mathrm{i}m\left(\frac{\pi}{2}-\theta+\varphi\right)}, \tag{31}$$

and we have

$$\begin{aligned}
&\frac{1}{4\pi^2}\int_0^{2\pi}\mathrm{d}\varphi\int_0^{2\pi}\mathrm{d}\theta\mathrm{e}^{\mathrm{i}q\theta}\mathrm{e}^{-\mathrm{i}s\varphi}\\
&\times\sum_{m=-\infty}^{\infty}\mathrm{J}_m\left(-\frac{rr'}{\sinh\alpha}\right)\mathrm{e}^{\mathrm{i}m\left(\frac{\pi}{2}-\theta+\varphi\right)}\\
&=\frac{1}{4\pi^2}\int_0^{2\pi}\mathrm{d}\varphi\int_0^{2\pi}\mathrm{d}\theta\mathrm{e}^{\mathrm{i}q\theta}\mathrm{e}^{-\mathrm{i}s\varphi}\\
&\times\sum_{m=-\infty}^{\infty}\mathrm{J}_m\left(-\frac{rr'}{\sinh\alpha}\right)\mathrm{e}^{-\mathrm{i}m\theta}\mathrm{e}^{\mathrm{i}m\varphi}\mathrm{e}^{\mathrm{i}m\frac{\pi}{2}}\\
&=\frac{1}{4\pi^2}\sum_{m=-\infty}^{\infty}\mathrm{J}_m\left(-\frac{rr'}{\sinh\alpha}\right)\mathrm{i}^m\\
&\times\int_0^{2\pi}\mathrm{e}^{(\mathrm{i}m-\mathrm{i}s)\varphi}\mathrm{d}\varphi\int_0^{2\pi}\mathrm{e}^{(\mathrm{i}q-\mathrm{i}m)\theta}\mathrm{d}\theta\\
&=\frac{1}{4\pi^2}\sum_{m=-\infty}^{\infty}\mathrm{J}_m\left(-\frac{rr'}{\sinh\alpha}\right)\mathrm{i}^m4\pi^2\delta s,m\delta q,m\\
&=\mathrm{J}_q\left(-\frac{rr'}{\sinh\alpha}\right)\delta_{s,q}\mathrm{i}^q.
\end{aligned} \tag{32}$$

Substituting Eq. (32) into Eq. (29) generates

$$\begin{aligned}
&\langle q,r|\,\mathfrak{S}_\alpha\left|s,r'\right)\\
&=\frac{1}{2\mathrm{i}\sinh\alpha}\exp\left[\frac{\mathrm{i}\left(r^2+r'^2\right)}{2\tanh\alpha}\right]\mathrm{J}_q\left(-\frac{rr'}{\sinh\alpha}\right)\delta_{s,q}\mathrm{i}^q,
\end{aligned} \tag{33}$$

and its complex conjugate is

$$\begin{aligned}
&\langle q,r|\,\mathfrak{S}_\alpha\left|s,r'\right)^*\\
&=\left(s,r'\right|\mathfrak{S}_\alpha^\dagger\left|q,r\right\rangle\\
&=\left(s,r'\right|\mathfrak{S}_\alpha\left|q,r\right\rangle=\frac{1}{-2\mathrm{i}\sinh\alpha}\exp\left[\frac{-\mathrm{i}\left(r^2+r'^2\right)}{2\tanh\alpha}\right]\mathrm{J}_q\\
&\times\left(-\frac{rr'}{\sinh\alpha}\right)\delta_{s,q}\left(-\mathrm{i}\right)^q\\
&=\frac{1}{2\sinh\alpha}\exp\left[\frac{-\mathrm{i}\left(r^2+r'^2\right)}{2\tanh\alpha}\right]\mathrm{J}_q\\
&\times\left(-\frac{rr'}{\sinh\alpha}\right)\delta_{s,q}\left(-\mathrm{i}\right)^{q-1},
\end{aligned} \tag{34}$$

so

$$\begin{aligned}
&\left(s,r'\right|\mathfrak{S}_\alpha\left|q,r\right\rangle\\
&=\frac{1}{2\sinh\alpha}\exp\left[\frac{-\mathrm{i}\left(r^2+r'^2\right)}{2\tanh\alpha}\right]\mathrm{J}_q\\
&\times\left(-\frac{rr'}{\sinh\alpha}\right)\delta_{s,q}\left(-\mathrm{i}\right)^{q-1}\\
&=\left(s,r\right|\mathfrak{S}_\alpha\left|q,r'\right\rangle\equiv\mathfrak{S}_\alpha;s,q(r,r').
\end{aligned} \tag{35}$$

Letting

$$\langle q,r|\,f\rangle=f\left(q,r\right),\ \left|F\right\rangle=\mathfrak{S}_\alpha\left|f\right\rangle, \tag{36}$$

and then using the completeness relations (18) and (35), we have

$$\begin{aligned}
&\left(s,r\right|\left.F\right\rangle\\
&=\left(s,r\right|\mathfrak{S}_\alpha\left|f\right\rangle\\
&=\sum_{q=-\infty}^{\infty}\int_0^{\infty}\mathrm{d}r'^2\left(s,r\right|\mathfrak{S}_\alpha\left|q,r'\right\rangle\left\langle q,r'\right|\left.f\right\rangle\\
&=\sum_{q=-\infty}^{\infty}\int_0^{\infty}\mathrm{d}r'^2\frac{1}{2\sinh\alpha}\exp\left[\frac{-\mathrm{i}\left(r^2+r'^2\right)}{2\tanh\alpha}\right]\mathrm{J}_q\\
&\times\left(-\frac{rr'}{\sinh\alpha}\right)\delta_{s,q}\left(-\mathrm{i}\right)^{q-1}f(q,r')\\
&=\int_0^{\infty}\mathrm{d}r'^2\frac{\left(-\mathrm{i}\right)^{s-1}}{2\sinh\alpha}\exp\left[\frac{-\mathrm{i}\left(r^2+r'^2\right)}{2\tanh\alpha}\right]\mathrm{J}_s\\
&\times\left(-\frac{rr'}{\sinh\alpha}\right)f\left(s,r'\right).
\end{aligned} \tag{37}$$

Comparing with the Hankel transform in Eq. (4), we see that equation (37) represents fractional squeezing-Hankel transform of $f\left(s,r'\right)$.

5. The additive rule of FrSHT

Now we need to demonstrate the additive rule of FrSHT. Using Eqs. (35) and (37), we can express FrSHT as

$$\begin{aligned}
&\sum_{q'=-\infty}^{\infty}\int_0^{\infty}\mathrm{d}r''^2\mathfrak{S}_{\beta;q,q'}(r',r'')f(q',r'')\\
&=\sum_{q'=-\infty}^{\infty}\int_0^{\infty}\mathrm{d}r''^2\left(q,r'\right|\mathfrak{S}_\beta\left|q',r''\right\rangle\left\langle q',r''\right|\left.f\right\rangle.
\end{aligned} \tag{38}$$

It then follows the two successive FrSHTs

$$\begin{aligned}
&\dot{G}_\alpha\circ\dot{G}_\beta\left[f\right](s,r)\\
&=\sum_{q=-\infty}^{\infty}\int_0^{\infty}\mathrm{d}r'^2\mathfrak{S}_{\alpha;s,q}\left(r,r'\right)\\
&\times\sum_{q'=-\infty}^{\infty}\int_0^{\infty}\mathrm{d}r''^2\mathfrak{S}_{\beta;q,q'}\left(r',r''\right)f\left(q',r''\right)\\
&=\sum_{q=-\infty}^{\infty}\int_0^{\infty}\mathrm{d}r'^2\delta_{s,q}\frac{\left(-\mathrm{i}\right)^{q-1}}{2\sinh\alpha}\\
&\times\exp\left[-\frac{\mathrm{i}\left(r^2+r'^2\right)}{2\tanh\alpha}\right]\mathrm{J}_q\left(-\frac{rr'}{\sinh\alpha}\right)\\
&\times\sum_{q'=-\infty}^{\infty}\int_0^{\infty}\mathrm{d}r''^2\delta_{q,q'}\frac{\left(-\mathrm{i}\right)^{q'-1}}{2\sinh\beta}
\end{aligned}$$

$$\times \exp\left[-\frac{\mathrm{i}\left(r'^2+r''^2\right)}{2\tanh\beta}\right] \mathrm{J}_{q'}\left(-\frac{r'r''}{\sinh\beta}\right) f\left(q',r''\right)$$
$$= \frac{(-1)^{s-1}}{4\sinh\alpha\sinh\beta} \exp\left[-\frac{\mathrm{i}r^2}{2\tanh\alpha}\right]$$
$$\times \int_0^\infty \mathrm{d}r''^2 \exp\left[-\frac{\mathrm{i}r''^2}{2\tanh\beta}\right] f\left(s,r''\right)$$
$$\times \int_0^\infty \mathrm{d}r'^2 \exp\left[-\frac{\mathrm{i}r'^2}{2}\frac{\tanh\alpha+\tanh\beta}{\tanh\alpha\tanh\beta}\right] \mathrm{J}_s$$
$$\times \left(-\frac{rr'}{\sinh\alpha}\right) \mathrm{J}_s\left(-\frac{r'r''}{\sinh\beta}\right), \tag{39}$$

where we have used the integration formula

$$\int_0^\infty \mathrm{d}r^2 \mathrm{e}^{-\lambda r^2} \mathrm{J}_s\left(vr\right) \mathrm{J}_s\left(ur\right)$$
$$= \frac{1}{\lambda} \exp\left[-\frac{1}{4\lambda}\left(v^2+u^2\right)+\mathrm{i}s\frac{\pi}{2}\right] \mathrm{J}_s\left(-\frac{\mathrm{i}vu}{2\lambda}\right). \tag{40}$$

Further, using

$$-\frac{1}{\tanh\alpha}+\frac{\tanh\beta\tanh\alpha}{(\tanh\alpha+\tanh\beta)}\frac{1}{\sinh^2\alpha}$$
$$= -\frac{1}{\tanh\left(\alpha+\beta\right)} \tag{41}$$

and

$$\frac{1}{\sinh\alpha\cosh\beta+\sinh\beta\cosh\alpha} = \frac{1}{\sinh\left(\alpha+\beta\right)}$$
$$= \frac{1}{\sinh\alpha\sinh\beta}\frac{\tanh\alpha\tanh\beta}{(\tanh\alpha+\tanh\beta)}, \tag{42}$$

equation (39) becomes

$$\dot{G}_\alpha \circ \dot{G}_\beta\left[f\right](s,r)$$
$$= \frac{(-1)^{s-1}}{4\sinh\alpha\sinh\beta}\frac{2\tanh\alpha\tanh\beta}{i(\tanh\alpha+\tanh\beta)} \exp\left(-\frac{\mathrm{i}r^2}{2\tanh\alpha}\right) \int_0^\infty \mathrm{d}r''^2 \exp\left(-\frac{\mathrm{i}r''^2}{2\tanh\beta}\right) f(s,r'')$$
$$\times \exp\left[-\frac{1}{2\mathrm{i}(\tanh\alpha+\tanh\beta)}\left(\frac{r^2}{\sinh^2\alpha}+\frac{r''^2}{\sinh^2\beta}\right)+\mathrm{i}s\frac{\pi}{2}\right] \mathrm{J}_s\left(-\frac{\tanh\alpha\tanh\beta}{\tanh\alpha+\tanh\beta}\frac{rr''}{\sinh\alpha\sinh\beta}\right)$$
$$= \frac{(-1)^{s-1}}{2\mathrm{i}\sinh(\alpha+\beta)} \exp\left(-\frac{\mathrm{i}r^2}{2\tanh\alpha}\right) \int_0^\infty \mathrm{d}r''^2 \exp\left(-\frac{\mathrm{i}r''^2}{2\tanh\beta}\right) f(s,r'')\mathrm{e}^{\mathrm{i}s\frac{\pi}{2}} \mathrm{J}_s\left(-\frac{rr''}{\sinh(\alpha+\beta)}\right)$$
$$\times \exp\left[-\frac{r^2}{2i}\left(-\frac{1}{\tanh(\alpha+\beta)}+\frac{1}{\tanh\alpha}\right)-\frac{r''^2}{2\mathrm{i}}\left(-\frac{1}{\tanh(\alpha+\beta)}+\frac{1}{\tanh\beta}\right)\right]$$
$$= \frac{(-\mathrm{i})^{s-1}}{2\sinh(\alpha+\beta)} \int_0^\infty \mathrm{d}r''^2 \exp\left[-\frac{\mathrm{i}(r^2+r''^2)}{2\tanh(\alpha+\beta)}\right] \mathrm{J}_s\left(-\frac{rr''}{\sinh(\alpha+\beta)}\right) f(s,r'') = \dot{G}_{\alpha+\beta}\left[f\right](s,r), \tag{43}$$

which is the additive rule for the fractional squeezing-Hankel transform.

6. Conclusion

Stemming from the entangled fractional squeezing transform in the entangled state representation, we have shown that the integral kernel of a fractional squeezing-Hankel transform is equivalent to the matrix element of the operator $\mathfrak{S}_\alpha = \mathrm{e}^{-\mathrm{i}\alpha\left(a_1^\dagger a_2^\dagger+a_1a_2\right)}\mathrm{e}^{-\mathrm{i}\pi a_2^\dagger a_2}$ in the induced entangled state representations $\left(s,r'\right|$ and $\left|q,r\right\rangle$, and $\left(s,r'\right|\left.q,r\right\rangle = \frac{1}{2}\delta_{s,q}\mathrm{J}_s\left(rr'\right)$ is essential to our derivation. Additionally, the additive rule of the fractional squeezing-Hankel transform is also presented. The FrSHT obtained above may be used to solve the Laplace equation to calculate the capacitance between two plates with a medium in mathematical physics. Besides, we consider that the zero order FrSHT and its integration properties may be useful to solve the axisymmetric stress system of three-dimensional band cracks.

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