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Conservation laws for Birkhoffian systems of Herglotz type*

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Conservation laws for the Birkhoffian system and the constrained Birkhoffian system of Herglotz type are studied. We propose a new differential variational principle, called the Pfaff–Birkhoff–d'Alembert principle of Herglotz type. Birkhoff's equations for both the Birkhoffian system and the constrained Birkhoffian system of Herglotz type are obtained. According to the relationship between the isochronal variation and the nonisochronal variation, the conditions of the invariance for the Pfaff–Birkhoff–d'Alembert principle of Herglotz type are given. Then, the conserved quantities for the Birkhoffian system and the constrained Birkhoffian system of Herglotz type are deduced. Furthermore, the inverse theorems of the conservation theorems are also established.

Keywords: Birkhoffian system, conversation law, differential variational principle, variational problem of Herglotz type

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1. Introduction

It is well known that conservation laws play an important role in the study of complex dynamical systems. The existence of certain conserved laws is useful for us to find out the local physical properties and dynamical behaviors, even if the differential equations of motion are nonintegrable. By using conservation laws, the differential equations of motion can be reduced. What is more, conservation laws are also important in the stability analysis of dynamical systems. Therefore, the study of conservation laws has always been a hot topic in analytical mechanics.[1] In fact, there are many methods to find conservation laws of mechanical systems. The traditional method is to derive the cyclic integral and generalized energy integral directly from the differential equations of motion. In addition, conservation laws can be established by using the invariance of the action functional, the differential equations of motion, and the dynamical functions under the infinitesimal transformations, i.e., Noether conserved quantity,[2–6] Hojman conserved quantity,[7–10] and Mei conserved quantity.[11–14] Besides, the conservation laws can also be studied by the differential variational principle, such as the d'Alembert principle,[15] Jourdain's principle,[16–19] Gauss's principle,[19,20] and Pfaff–Birkhoff–d'Alembert principle.[20,21] In this paper, the conservation laws are constructed by the differential variational principle of Herglotz type.

The classical integral variational principles, such as the most famous Hamilton principle and Lagrange principle, are all extreme value problems of the action functional for a holonomic conservative system. For a non-conservative system, however, there is generally no action functional that makes its variation equal to zero. For example, the Hamilton principle of a holonomic non-conservative system, i.e.,

$$\int_{t_0}^{t_1} (\delta T + Q_s \delta q_s)\,\mathrm{d}t = 0$$

is not a principle of stationary action.[22] In fact, it is a quasi variational principle.

Different from the classical variational principle above, the Herglotz variational principle[23] gives a variational description of a holonomic non-conservative system. The Herglotz variational principle can describe all physical processes that the classical variational principle can, and can be applied to solve non-conservative problems where the classical variational principle is not applicable. The action functional of the Herglotz variational principle is defined by a differential equation. If its Lagrangian or Hamiltonian does not depend on the action functional, the Herglotz variational principle reduces to the classical integral variational principle.

In recent years, the Herglotz variational principle and its symmetries have been applied in finite and infinite dimensional non-conservative dynamic systems, quantum systems, thermodynamics, optimal control theory, and other fields.[24–33] In Ref. [34], the simple and physically meaningful Lagrangians of Herglotz type were constructed, which describe a wide range of non-conservative classical and quantum systems, for example, vibrating string under viscous forces, non-conservative electromagnetic theory, non-

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conservative Schrödinger equation, non-conservative Klein–Gordon equation, etc.

American physicist Santilli[35] pointed out that Birkhoffian mechanics is the most general possible mechanics, which can be applied to hadron physics, space mechanics, statistical mechanics, biophysics, engineering, etc. Mei[36] proposed that Birkhoffian mechanics is a new stage in the development of analytical mechanics. Therefore, it is of great significance to study Birkhoffian mechanics.[37–40]

The sufficient and necessary conditions that the system of first-order differential equations

$$C_{\mu\nu}\left(t,a^{\nu}\right)\dot{a}^{\nu}+D_{\mu}\left(t,a^{\nu}\right)=0,\ \ (\mu,\nu=1,2,\ldots,2n)$$

has a Birkhoffian representation are that the system satisfies the self-adjoint conditions

$$\begin{cases} C_{\mu\nu}+C_{\nu\mu}=0, \\ \dfrac{\partial C_{\mu\nu}}{\partial a^{\tau}}+\dfrac{\partial C_{\nu\tau}}{\partial a^{\mu}}+\dfrac{\partial C_{\tau\mu}}{\partial a^{\nu}}=0,\ \ (\mu,\nu,\tau=1,2,\ldots,2n). \\ \dfrac{\partial C_{\mu\nu}}{\partial t}=\dfrac{\partial D_{\mu}}{\partial a^{\nu}}-\dfrac{\partial D_{\nu}}{\partial a^{\mu}}, \end{cases}$$

Therefore, it is difficult to convert a general holonomic non-conservative system or a nonholonomic system into a Birkhoffian system, while it is relatively easy to convert the system into a generalized Birkhoffian system.[40] However, similar to the extension of the Hamilton principle to non-conservative systems, it is also difficult to extend the Pfaff–Birkhoff principle

$$\delta\int_{t_0}^{t_1}\left[R_{\mu}\left(t,a^{\nu}\left(t\right)\right)\dot{a}^{\mu}\left(t\right)-B\left(t,a^{\nu}\left(t\right)\right)\right]\mathrm{d}t=0$$

to the generalized Birkhoffian system or to the constrained Birkhoffian system. The generalized Pfaff–Birkhoff principle

$$\int_{t_0}^{t_1}\left[\delta\left(R_{\mu}\dot{a}^{\mu}-B\right)+\Lambda_{\mu}\left(t,a^{\nu}\right)\delta a^{\mu}\right]\mathrm{d}t=0$$

cannot be expressed as a variation of an action functional equal to zero.

On the basis of the Herglotz variational principle, we proposed the variational principle of the Birkhoffian system of the Herglotz type,[41,42] and established a variational description of the generalized Birkhoffian system, which can systematically deal with the Birkhoffian system and the generalized Birkhoffian system. When the Birkhoffian B and Birkhoff's functions R_{μ} do not depend on the action functional, this principle reduces the classical Pfaff–Birkhoff principle.

In this paper, we propose a new differential variational principle, the Pfaff–Birkhoff–d'Alembert principle of Herglotz type, which provides a simple method to give variational descriptions of holonomic non-conservative processes and nonholonomic non-conservative processes in Birkhoffian systems. By use of the new differential variational principle, we will study the conservation laws for the Birkhoffian system and the constrained Birkhoffian system. At the same time, the inverse theorem of the conservation theorem will be given.

2. Pfaff–Birkhoff–d'Alembert principle of Herglotz type

The variational problem for a Birkhoffian system of Herglotz type can be described as follows.[41]

Determine the functions $a^{\mu}\left(t\right)$ satisfying the given boundary conditions

$$a^{\mu}\left(t\right)|_{t=t_0}=a_0^{\mu},\ \ a^{\mu}\left(t\right)|_{t=t_1}=a_1^{\mu},\ \ (\mu=1,2,\ldots,2n),\quad(1)$$

such that integral functional $z\left(t\right)$ reaches its extreme at $t=t_1$, i.e.,

$$z\left(t_1\right)\rightarrow\mathrm{extr},$$

where $z\left(t\right)$ is defined by the differential equation

$$\dot{z}\left(t\right)=R_{\mu}\left(t,a^{\nu}\left(t\right),z\left(t\right)\right)\dot{a}^{\mu}\left(t\right)-B\left(t,a^{\nu}\left(t\right),z\left(t\right)\right),\quad(2)$$

and satisfies the initial condition

$$z\left(t\right)|_{t=t_0}=z_0.\quad(3)$$

Here, $B\left(t,a^{\nu}\left(t\right),z\left(t\right)\right)$ is the Birkhoffian of Herglotz type, and $R_{\mu}\left(t,a^{\nu}\left(t\right),z\left(t\right)\right)$ are Birkhoff's functions of Herglotz type. $a^{\mu}\left(t\right)\left(\mu=1,2,\ldots,2n\right)$ are Birkhoff's variables, and a_0^{μ}, a_1^{μ}, z_0 are constants.

The above extreme problem can be called the Pfaff–Birkhoff principle of Herglotz type.

A new type of differential variational principle can be derived from the Pfaff–Birkhoff principle of Herglotz type. To solve the above extreme problem, we can easily obtain[41]

$$\begin{aligned}&\int_{t_0}^{t_1}\exp\left[-\int_{t_0}^{t}\left(\frac{\partial R_{\mu}}{\partial z}\dot{a}^{\mu}-\frac{\partial B}{\partial z}\right)\mathrm{d}\theta\right]\\&\times\left\{\left(\frac{\partial R_{\nu}}{\partial a^{\mu}}-\frac{\partial R_{\mu}}{\partial a^{\nu}}\right)\dot{a}^{\nu}-\frac{\partial B}{\partial a^{\mu}}-\frac{\partial R_{\mu}}{\partial t}\right.\\&\left.+\left(R_{\mu}\frac{\partial R_{\nu}}{\partial z}-\frac{\partial R_{\mu}}{\partial z}R_{\nu}\right)\dot{a}^{\nu}+\frac{\partial R_{\mu}}{\partial z}B-R_{\mu}\frac{\partial B}{\partial z}\right\}\delta a^{\mu}\,\mathrm{d}t=0,\\&(\mu,\nu=1,2,\ldots,2n).\end{aligned}\quad(4)$$

Consider the arbitrariness of the integral interval $[t_0,t_1]$, so

$$\begin{aligned}&\exp\left[-\int_{t_0}^{t}\left(\frac{\partial R_{\mu}}{\partial z}\dot{a}^{\mu}-\frac{\partial B}{\partial z}\right)\mathrm{d}\theta\right]\left\{\left(\frac{\partial R_{\nu}}{\partial a^{\mu}}-\frac{\partial R_{\mu}}{\partial a^{\nu}}\right)\dot{a}^{\nu}\right.\\&-\frac{\partial B}{\partial a^{\mu}}-\frac{\partial R_{\mu}}{\partial t}+\left(R_{\mu}\frac{\partial R_{\nu}}{\partial z}-\frac{\partial R_{\mu}}{\partial z}R_{\nu}\right)\dot{a}^{\nu}\\&\left.+\frac{\partial R_{\mu}}{\partial z}B-R_{\mu}\frac{\partial B}{\partial z}\right\}\delta a^{\mu}=0,\ \ (\mu,\nu=1,2,\ldots,2n).\end{aligned}\quad(5)$$

The principle (5) is called the Pfaff–Birkhoff–d'Alembert principle of Herglotz type, which is a new differential variational principle.

3. Birkhoff's equations of Herglotz type

Making use of the Pfaff–Birkhoff–d'Alembert principle of Herglotz type, we can deduce Birkhoff's equations of Herglotz type.

If $\delta a^{\mu}\ (\mu=1,2,\ldots,2n)$ in principle (5) are independent of each other, we have

$$\exp\left[-\int_{t_0}^{t}\left(\frac{\partial R_{\mu}}{\partial z}\dot{a}^{\mu}-\frac{\partial B}{\partial z}\right)\mathrm{d}\theta\right]\left[\left(\frac{\partial R_{\nu}}{\partial a^{\mu}}-\frac{\partial R_{\mu}}{\partial a^{\nu}}\right)\dot{a}^{\nu}-\frac{\partial B}{\partial a^{\mu}}-\frac{\partial R_{\mu}}{\partial t}+\left(R_{\mu}\frac{\partial R_{\nu}}{\partial z}-\frac{\partial R_{\mu}}{\partial z}R_{\nu}\right)\dot{a}^{\nu}+\frac{\partial R_{\mu}}{\partial z}B-R_{\mu}\frac{\partial B}{\partial z}\right]=0,\quad(\mu=1,2,\ldots,2n). \tag{6}$$

The above equations are Birkhoff's equations for the Birkhoffian system of Herglotz type.

If $2n$ Birkhoff's variables a^{μ} are not independent of each other, they are subject to $2m\ (m<n)$ constraints

$$f_{\beta}(t,a^{\mu})=0,\quad(\beta=1,2,\ldots,2m;\quad m<n). \tag{7}$$

Taking the calculation of the variation to formula (7), we can obtain the restriction of constraints (7) on the virtual displacements

$$\frac{\partial f_{\beta}}{\partial a^{\mu}}\delta a^{\mu}=0. \tag{8}$$

From principle (5) and restriction (8), and by using the usual Lagrange multiplier method, we obtain

$$\exp\left[-\int_{t_0}^{t}\left(\frac{\partial R_{\mu}}{\partial z}\dot{a}^{\mu}-\frac{\partial B}{\partial z}\right)\mathrm{d}\theta\right]\left[\left(\frac{\partial R_{\nu}}{\partial a^{\mu}}-\frac{\partial R_{\mu}}{\partial a^{\nu}}\right)\dot{a}^{\nu}-\frac{\partial B}{\partial a^{\mu}}-\frac{\partial R_{\mu}}{\partial t}+\left(R_{\mu}\frac{\partial R_{\nu}}{\partial z}-\frac{\partial R_{\mu}}{\partial z}R_{\nu}\right)\dot{a}^{\nu}+\frac{\partial R_{\mu}}{\partial z}B-R_{\mu}\frac{\partial B}{\partial z}+\lambda_{\beta}\frac{\partial f_{\beta}}{\partial a^{\mu}}\right]=0,\ (\mu=1,2,\ldots,2n), \tag{9}$$

where λ_{β} are the constraint multipliers. The above equations are Birkhoff's equations for the constrained Birkhoffian system of Herglotz type.

Remark 1 If neither the Birkhoffian B nor the Birkhoffian functions R_{μ} contain z, i.e., $B=B(t,a^{\nu}(t))$, $R_{\mu}=R_{\mu}(t,a^{\nu}(t))$, then equation (2) can be written as

$$z=\int_{t_0}^{t_1}\left[R_{\mu}(t,a^{\nu}(t))\dot{a}^{\mu}(t)-B(t,a^{\nu}(t))\right]\mathrm{d}t. \tag{10}$$

Formula (10) is the classical Pfaff action. Then, equation (6) reduces to

$$\left(\frac{\partial R_{\nu}}{\partial a^{\mu}}-\frac{\partial R_{\mu}}{\partial a^{\nu}}\right)\dot{a}^{\nu}-\frac{\partial B}{\partial a^{\mu}}-\frac{\partial R_{\mu}}{\partial t}=0,\quad(\mu,\nu=1,2,\ldots,2n), \tag{11}$$

and equation (9) can be expressed as

$$\left(\frac{\partial R_{\nu}}{\partial a^{\mu}}-\frac{\partial R_{\mu}}{\partial a^{\nu}}\right)\dot{a}^{\nu}-\frac{\partial B}{\partial a^{\mu}}-\frac{\partial R_{\mu}}{\partial t}+\lambda_{\beta}\frac{\partial f_{\beta}}{\partial a^{\mu}}=0,\quad(\mu,\nu=1,2,\ldots,2n). \tag{12}$$

Equations (11) and (12) are Birkhoff's equations of a classical Birkhoffian system and a constrained Birkhoffian system, respectively.

4. Isochronal variation and nonisochronal variation

The isochronal variation and nonisochronal of the variables a^{μ} respectively are

$$\delta a^{\mu}=\bar{a}^{\mu}(t)-a^{\mu}(t), \tag{13}$$

$$\Delta a^{\mu}=\bar{a}^{\mu}(t+\Delta t)-a^{\mu}(t). \tag{14}$$

Since the variation Δt is small enough, making expansion $\bar{a}^{\mu}(t+\Delta t)$ and keeping the first order items, we have

$$\bar{a}^{\mu}(t+\Delta t)=\bar{a}^{\mu}(t)+\dot{a}^{\mu}\Delta t. \tag{15}$$

From formulae (13)–(15), we easily obtain

$$\Delta a^{\mu}=\delta a^{\mu}+\dot{a}^{\mu}\Delta t,\quad(\mu=1,2,\ldots,2n). \tag{16}$$

Introducing the infinitesimal generators of space F_{μ} and the infinitesimal generators of time f, we have

$$\Delta a^{\mu}=\varepsilon F_{\mu}(t,a^{\nu}),\ \Delta t=\varepsilon f(t,a^{\nu}),\ (\mu=1,2,\ldots,2n). \tag{17}$$

Thus, we can obtain

$$\delta a^{\mu}=\varepsilon\left[F_{\mu}(t,a^{\nu})-\dot{a}^{\mu}f(t,a^{\nu})\right],\quad(\mu=1,2,\ldots,2n), \tag{18}$$

where ε is an infinitesimal parameter.

5. The conditions of the invariance for the Pfaff–Birkhoff–d'Alembert principle of Herglotz type

Substituting Eq. (18) into Eq. (5), we obtain

$$\varepsilon\left\{\exp\left[-\int_{t_0}^{t}\left(\frac{\partial R_{\mu}}{\partial z}\dot{a}^{\mu}-\frac{\partial B}{\partial z}\right)\mathrm{d}\theta\right]\left[\left(\frac{\partial R_{\nu}}{\partial a^{\mu}}-\frac{\partial R_{\mu}}{\partial a^{\nu}}\right)\dot{a}^{\nu}-\frac{\partial B}{\partial a^{\mu}}-\frac{\partial R_{\mu}}{\partial t}+\left(R_{\mu}\frac{\partial R_{\nu}}{\partial z}-\frac{\partial R_{\mu}}{\partial z}R_{\nu}\right)\dot{a}^{\nu}+\frac{\partial R_{\mu}}{\partial z}B-R_{\mu}\frac{\partial B}{\partial z}\right]\left(F_{\mu}-\dot{a}^{\mu}f\right)\right\}=0. \tag{19}$$

Considering

$$\left(\frac{\partial R_{\nu}}{\partial a^{\mu}}-\frac{\partial R_{\mu}}{\partial a^{\nu}}\right)\dot{a}^{\nu}\dot{a}^{\mu}=0, \tag{20}$$

$$\left(R_{\mu}\frac{\partial R_{\nu}}{\partial z}-\frac{\partial R_{\mu}}{\partial z}R_{\nu}\right)\dot{a}^{\nu}\dot{a}^{\mu}=0, \tag{21}$$

$$\dot{B}=\frac{\partial B}{\partial t}+\frac{\partial B}{\partial a^{\nu}}\dot{a}^{\nu}+\frac{\partial B}{\partial z}(R_{\nu}\dot{a}^{\nu}-B), \tag{22}$$

$$\dot{R}_{\mu}=\frac{\partial R_{\mu}}{\partial t}+\frac{\partial R_{\mu}}{\partial a^{\nu}}\dot{a}^{\nu}+\frac{\partial R_{\mu}}{\partial z}(R_{\nu}\dot{a}^{\nu}-B), \tag{23}$$

we can rewrite formula (19) as

$$\varepsilon\Bigg\{\exp\left[-\int_{t_0}^{t}\left(\frac{\partial R_\mu}{\partial z}\dot{a}^\mu-\frac{\partial B}{\partial z}\right)\mathrm{d}\theta\right]\times\left[\left(\frac{\partial R_\mu}{\partial t}f+\frac{\partial R_\mu}{\partial a^\nu}F_\nu\right)\dot{a}^\mu+R_\mu\dot{F}_\mu-B\dot{f}-\frac{\partial B}{\partial a^\mu}F_\mu-\frac{\partial B}{\partial t}f\right]-\frac{\mathrm{d}}{\mathrm{d}t}\left[\exp\left(-\int_{t_0}^{t}\left(\frac{\partial R_\mu}{\partial z}\dot{a}^\mu-\frac{\partial B}{\partial z}\right)\mathrm{d}\theta\right)\times\left(R_\mu F_\mu-Bf\right)\right]\Bigg\}=0. \tag{24}$$

On the left hand side of Eq. (24) plus and minus the function

$$\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\left[G\exp\left(-\int_{t_0}^{t}\left(\frac{\partial R_\mu}{\partial z}\dot{a}^\mu-\frac{\partial B}{\partial z}\right)\mathrm{d}\theta\right)\right],$$

we obtain

$$\varepsilon\Bigg\{\exp\left(-\int_{t_0}^{t}\left(\frac{\partial R_\mu}{\partial z}\dot{a}^\mu-\frac{\partial B}{\partial z}\right)\mathrm{d}\theta\right)\times\left[\left(\frac{\partial R_\mu}{\partial t}f+\frac{\partial R_\mu}{\partial a^\nu}F_\nu\right)\dot{a}^\mu+R_\mu\dot{F}_\mu-B\dot{f}-\frac{\partial B}{\partial a^\mu}F_\mu-\frac{\partial B}{\partial t}f-\left(\frac{\partial R_\mu}{\partial z}\dot{a}^\mu-\frac{\partial B}{\partial z}\right)G+\dot{G}\right]-\frac{\mathrm{d}}{\mathrm{d}t}\left[\exp\left(-\int_{t_0}^{t}\left(\frac{\partial R_\mu}{\partial z}\dot{a}^\mu-\frac{\partial B}{\partial z}\right)\mathrm{d}\theta\right)\times\left(R_\mu F_\mu-Bf+G\right)\right]\Bigg\}=0, \tag{25}$$

where $G=G(t,a^\mu,z)$ is a gauge function. Equation (22) is the transformation of the condition of the invariance for the Pfaff–Birkhoff–d'Alembert principle of Herglotz type, or the deformation of the Pfaff–Birkhoff–d'Alembert principle of Herglotz type under infinitesimal transformations.

6. Conserved quantity for a Birkhoffian system of Herglotz type

According to Eq. (25), if there exists a gauge function G, which can make the infinitesimal generators of space F_μ and the infinitesimal generators of time f satisfy the following conditions:

$$\left(\frac{\partial R_\mu}{\partial t}f+\frac{\partial R_\mu}{\partial a^\nu}F_\nu\right)\dot{a}^\mu+R_\mu\dot{F}_\mu-B\dot{f}-\frac{\partial B}{\partial a^\mu}F_\mu-\frac{\partial B}{\partial t}f-\left(\frac{\partial R_\mu}{\partial z}\dot{a}^\mu-\frac{\partial B}{\partial z}\right)G+\dot{G}=0, \tag{26}$$

then the system has a conserved quantity, which can be written in the following form

$$I=\exp\left(-\int_{t_0}^{t}\left(\frac{\partial R_\mu}{\partial z}\dot{a}^\mu-\frac{\partial B}{\partial z}\right)\mathrm{d}\theta\right)\left(R_\mu F_\mu-Bf+G\right)=\text{const.} \tag{27}$$

So we can draw the following conclusion.

Theorem 1 For the Birkhoffian system of Herglotz type (6), if there exists a gauge function G, and the infinitesimal generators F_μ, f satisfy the conditions (26), then the system has a conserved quantity, which can be written in the form of formula (27).

Particularly, when $G\equiv 0$, Theorem 1 can be expressed as follows.

Theorem 2 For the Birkhoffian system of Herglotz type (6), if the infinitesimal generators F_μ, f satisfy the following conditions

$$\left(\frac{\partial R_\mu}{\partial t}f+\frac{\partial R_\mu}{\partial a^\nu}F_\nu\right)\dot{a}^\mu+R_\mu\dot{F}_\mu-B\dot{f}-\frac{\partial B}{\partial a^\mu}F_\mu-\frac{\partial B}{\partial t}f=0, \tag{28}$$

then the system has a conserved quantity

$$I=\exp\left(-\int_{t_0}^{t}\left(\frac{\partial R_\mu}{\partial z}\dot{a}^\mu-\frac{\partial B}{\partial z}\right)\mathrm{d}\theta\right)\left(R_\mu F_\mu-Bf\right)=\text{const.} \tag{29}$$

Theorem 2 has been given by Ref. [31].

7. Conserved quantity for a constrained Birkhoffian system of Herglotz type

Substituting Eq. (18) into Eq. (8), and taking note of the arbitrariness of the parameter ε, we have

$$\frac{\partial f_\beta}{\partial a^\mu}\left(F_\mu-\dot{a}^\mu f\right)=0,\quad(\beta=1,2,\ldots,2m), \tag{30}$$

which are the restriction of the constraints (7) on the infinitesimal generators F_μ and f.

By Eqs. (7) and (9), we can obtain the constraint multipliers λ_β, which are the functions of t, a^μ, and z. Then, substituting them into Eq. (9), we have

$$\exp\left[-\int_{t_0}^{t}\left(\frac{\partial R_\mu}{\partial z}\dot{a}^\mu-\frac{\partial B}{\partial z}\right)\mathrm{d}\theta\right]\left[\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial t}+\left(R_\mu\frac{\partial R_\nu}{\partial z}-\frac{\partial R_\mu}{\partial z}R_\nu\right)\dot{a}^\nu+\frac{\partial R_\mu}{\partial z}B-R_\mu\frac{\partial B}{\partial z}+\Lambda_\mu\right]=0,\quad(\mu=1,2,\ldots,2n), \tag{31}$$

where

$$\Lambda_\mu=\Lambda_\mu(t,a^\mu,z)=\lambda_\beta\frac{\partial f_\beta}{\partial a^\mu}. \tag{32}$$

Equations (31) are called Birkhoff's equations of the free Birkhoffian system corresponding to the constrained Birkhoffian system (7), (9) of Herglotz type, for short, Birkhoff's equations of the corresponding free Birkhoffian system. If the initial conditions satisfy the constraint of Eq. (7), then the solution of Eq. (31) gives the motion of the constrained Birkhoffian system.

Similar to Theorem 1, we have the following theorem.

Theorem 3 For the corresponding free Birkhoffian system (31), if there exists a gauge function $G(t,a^\mu,z)$, and the infinitesimal generators F_μ, f satisfy the following conditions:

$$\left(\frac{\partial R_\mu}{\partial t}f+\frac{\partial R_\mu}{\partial a^\nu}F_\nu\right)\dot{a}^\mu+R_\mu\dot{F}_\mu-B\dot{f}-\frac{\partial B}{\partial a^\mu}F_\mu-\frac{\partial B}{\partial t}f+\Lambda_\mu\left(F_\mu-\dot{a}^\mu f\right)-\left(\frac{\partial R_\mu}{\partial z}\dot{a}^\mu-\frac{\partial B}{\partial z}\right)G+\dot{G}=0, \tag{33}$$

then the system has a conserved quantity, which can be written in the form of formula (27).

Substituting the restriction Eq. (30) into formula (33), we can obtain formula (26). Therefore, we derive the following theorem.

Theorem 4 For the constrained Birkhoffian system (7), (9) of Herglotz type, if there exists a gauge function $G(t,a^\mu,z)$, and the infinitesimal generators F_μ, f satisfy the conditions (26) and the generators satisfy the restriction Eq. (30), then the system has a conserved quantity written in the form of formula (27).

Remark 2 If neither the Birkhoffian B nor the Birkhoffian functions R_μ contain z, the conserved quantity (27) reduces to the conserved quantity for the classical Birkhoffian system

$$I=R_\mu F_\mu-Bf+G=\text{const.} \tag{34}$$

8. Inverse theorems of conservation theorems

For the constrained Birkhoffian system (7), (9) of Herglotz type, assume the system has a conserved quantity

$$I=I(t,a^\mu,z)=\text{const.} \tag{35}$$

Differentiating both sides of formula (33) with respect to time t, we obtain

$$\frac{\mathrm{d}I}{\mathrm{d}t}=\frac{\partial I}{\partial t}+\frac{\partial I}{\partial a^\mu}\dot{a}^\mu+\frac{\partial I}{\partial z}\left(R_\mu\dot{a}^\mu-B\right)=0. \tag{36}$$

Multiplying the two sides of Eq. (31) by $F_\mu-\dot{a}^\mu f$, we have

$$\exp\left[-\int_{t_0}^{t}\left(\frac{\partial R_\mu}{\partial z}\dot{a}^\mu-\frac{\partial B}{\partial z}\right)\mathrm{d}\theta\right]\left[\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial t}+\left(R_\mu\frac{\partial R_\nu}{\partial z}-\frac{\partial R_\mu}{\partial z}R_\nu\right)\dot{a}^\nu+\frac{\partial R_\mu}{\partial z}B-R_\mu\frac{\partial B}{\partial z}+\Lambda_\mu\right]\left(F_\mu-\dot{a}^\mu f\right)=0. \tag{37}$$

Add Eqs. (36) and (37), and let the coefficients of $\dot{a}^\mu$ be zero. Obviously, the items containing $\dot{a}^\mu\dot{a}^\nu$ are equal to zero, so we have

$$\frac{\partial I}{\partial a^\mu}+\frac{\partial I}{\partial z}R_\mu+\exp\left(-\int_{t_0}^{t}\left(\frac{\partial R_\mu}{\partial z}\dot{a}^\mu-\frac{\partial B}{\partial z}\right)\mathrm{d}\theta\right)\times\left[\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)F_\nu+\left(R_\nu\frac{\partial R_\mu}{\partial z}-\frac{\partial R_\nu}{\partial z}R_\mu\right)F_\nu+\left(\frac{\partial B}{\partial a^\mu}+\frac{\partial R_\mu}{\partial t}-\frac{\partial R_\mu}{\partial z}B+R_\mu\frac{\partial B}{\partial z}-\Lambda_\mu\right)f\right]=0,\quad(\mu=1,2,\cdots,2n). \tag{38}$$

Then, let the integral (35) be equal to the conserved quantity (27), i.e.,

$$\exp\left(-\int_{t_0}^{t}\left(\frac{\partial R_\mu}{\partial z}\dot{a}^\mu-\frac{\partial B}{\partial z}\right)\mathrm{d}\theta\right)\times\left(R_\mu F_\mu-Bf+G\right)=I. \tag{39}$$

Thus, we draw the following conclusion.

Theorem 5 For the constrained Birkhoffian system (7), (9) of Herglotz type, if a conserved quantity of the system is known, then the infinitesimal generators F_μ, f and the gauge function G can be found by formulae (38), (39) and the restriction Eq. (30).

If the constraints do not exist, from Theorem 5, we obtain the inverse theorem of the conservation theorem for the Birkhoffian system of Herglotz type as follows.

Theorem 6 For the Birkhoffian system of Herglotz type (6), if a conserved quantity (35) of the system is known, then the infinitesimal generators F_μ, f and the gauge function G can be found by formula (39) and the following formula

$$\frac{\partial I}{\partial a^\mu}+\frac{\partial I}{\partial z}R_\mu+\exp\left(-\int_{t_0}^{t}\left(\frac{\partial R_\mu}{\partial z}\dot{a}^\mu-\frac{\partial B}{\partial z}\right)\mathrm{d}\theta\right)\times\left[\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)F_\nu+\left(R_\nu\frac{\partial R_\mu}{\partial z}-\frac{\partial R_\nu}{\partial z}R_\mu\right)F_\nu+\left(\frac{\partial B}{\partial a^\mu}+\frac{\partial R_\mu}{\partial t}-\frac{\partial R_\mu}{\partial z}B+R_\mu\frac{\partial B}{\partial z}-\Lambda_\mu\right)f\right]=0. \tag{40}$$

Specially, when $G\equiv 0$, Theorem 6 gives the result of Ref. [41].

9. Example

Consider a fourth-order Birkhoffian system of Herglotz type, whose Birkhoffian and Birkhoff's functions are

$$B=\frac{1}{2}\left[\left(a^1\right)^2+\left(a^2\right)^2+\left(a^3\right)^2+\left(a^4\right)^2\right]+\gamma z,\quad R_1=a^3,\ R_2=a^4,\ R_3=0,\ R_4=0. \tag{41}$$

Supposing the constraints of this system are

$$f_1=a^1+a^2-c_1=0, \tag{42}$$

$$f_2=a^3+a^4-c_2=0, \tag{43}$$

we try to study its conserved quantity.

By Eq. (9), we have

$$-\dot{a}^3-a^1-a^3\gamma+\lambda_1=0,\ -\dot{a}^4-a^2-a^4\gamma+\lambda_1=0,\quad \dot{a}^1-a^3+\lambda_2=0,\ \dot{a}^2-a^4+\lambda_2=0. \tag{44}$$

From Eq. (44) and the constraints (42), (43), we obtain

$$\lambda_1=\frac{1}{2}\left(a^1+a^2\right)+\frac{\gamma}{2}\left(a^3+a^4\right),\ \lambda_2=\frac{1}{2}\left(a^3+a^4\right). \tag{45}$$

The conditions (26) give

$$\begin{aligned}&F_3\dot{a}^1+F_4\dot{a}^2+a^3\dot{F}_1+a^4\dot{F}_2\\&-\frac{1}{2}\left[\left(a^1\right)^2+\left(a^2\right)^2+\left(a^3\right)^2+\left(a^4\right)^2\right]f\\&-\gamma z\dot{f}-a^1F_1-a^2F_2-a^3F_3-a^4F_4+\gamma G+\dot{G}=0.\end{aligned}\tag{46}$$

According to the restriction Eq. (30), we have

$$F_1-\dot{a}^1f+F_2-\dot{a}^2f=0,\tag{47}$$

$$F_3-\dot{a}^3f+F_4-\dot{a}^4f=0.\tag{48}$$

The solution of Eq. (46) is

$$\begin{aligned}&f=0,\ F_1=-F_2=\frac{\gamma}{2}\left(a^1-a^2\right)+\left(a^3-a^4\right),\\&F_3=-F_4=-\left(a^1-a^2\right)-\frac{\gamma}{2}\left(a^3-a^4\right),\\&G=\frac{1}{2}\left(a^1-a^2\right)^2-\frac{1}{2}\left(a^3-a^4\right)^2.\end{aligned}\tag{49}$$

Obviously, the generators F_μ and f given by formula (49) satisfy Eqs. (47) and (48). Therefore, we obtain the conserved quantity by Theorem 1 as

$$\begin{aligned}I=&\frac{1}{2}\exp\left[\gamma(t-t_0)\right]\left[\left(a^1-a^2\right)^2\right.\\&\left.+\gamma\left(a^1-a^2\right)\left(a^3-a^4\right)+\left(a^3-a^4\right)^2\right]\\=&\ \text{const.}\end{aligned}\tag{50}$$

Next, by Theorem 5 and a given conserved quantity, we can obtain the corresponding infinitesimal generators. Assume there is a quantity (50) of this system, from formula (38), we obtain

$$\begin{aligned}&\exp\left[\gamma(t-t_0)\right]\left[\left(a^1-a^2\right)+\frac{\gamma}{2}\left(a^3-a^4\right)\right]+\exp\left[\gamma(t-t_0)\right]\\&\times\left\{F_3+\left[a^1+a^3\gamma-\frac{\gamma}{2}\left(a^3+a^4\right)-\frac{1}{2}\left(a^1+a^2\right)\right]f\right\}=0,\\&\exp\left[\gamma(t-t_0)\right]\left[-\left(a^1-a^2\right)-\frac{\gamma}{2}\left(a^3-a^4\right)\right]+\exp\left[\gamma(t-t_0)\right]\\&\times\left\{F_4+\left[a^2+a^4\gamma-\frac{\gamma}{2}\left(a^3+a^4\right)-\frac{1}{2}\left(a^1+a^2\right)\right]f\right\}=0,\\&\exp\left[\gamma(t-t_0)\right]\left[\frac{\gamma}{2}\left(a^1-a^2\right)+\left(a^3-a^4\right)\right]+\exp\left[\gamma(t-t_0)\right]\\&\times\left\{-F_1+\left[a^3-\frac{1}{2}\left(a^3+a^4\right)\right]f\right\}=0,\\&\exp\left[\gamma(t-t_0)\right]\left[-\frac{\gamma}{2}\left(a^1-a^2\right)-\left(a^3-a^4\right)\right]+\exp\left[\gamma(t-t_0)\right]\\&\times\left\{-F_2+\left[a^4-\frac{1}{2}\left(a^3+a^4\right)\right]f\right\}=0.\end{aligned}\tag{51}$$

Formula (39) gives

$$\begin{aligned}&\exp\left[\gamma(t-t_0)\right]\left\{a^3F_1+a^4F_2-\frac{1}{2}\left[\left(a^1\right)^2+\left(a^2\right)^2\right.\right.\\&\left.\left.+\left(a^3\right)^2+\left(a^4\right)^2\right]f-\gamma zf+G\right\}=I.\end{aligned}\tag{52}$$

Equations (51) and (52) have the solutions

$$\begin{aligned}f&=0,\ F_1=\frac{\gamma}{2}\left(a^1-a^2\right)+\left(a^3-a^4\right)=-F_2,\\F_3&=-\left(a^1-a^2\right)-\frac{\gamma}{2}\left(a^3-a^4\right)=-F_4,\\G&=\frac{1}{2}\left(a^1-a^2\right)^2-\frac{1}{2}\left(a^3-a^4\right)^2;\end{aligned}\tag{53}$$

$$\begin{aligned}f&=-2,\ F_1=\frac{\gamma}{2}\left(a^1-a^2\right)=-F_2,\\F_3&=\frac{\gamma}{2}\left(a^3-a^4\right)=-F_4,\\G&=-2\gamma z-\frac{1}{2}\left(a^1+a^2\right)^2-\frac{1}{2}\left(a^3+a^4\right)^2;\end{aligned}\tag{54}$$

$$\begin{aligned}f&=\frac{\left(a^1-a^2\right)^2-\left(a^3-a^4\right)^2}{\left(a^3-a^4\right)^2-B},\\F_1&=-F_2=\frac{\gamma}{2}\left(a^1-a^2\right)+\left(a^3-a^4\right)+\frac{1}{2}\left(a^3-a^4\right)f,\\F_3&=-F_4=-\left(a^1-a^2\right)-\frac{\gamma}{2}\left(a^3-a^4\right)\\&\quad-\left[\frac{1}{2}\left(a^1-a^2\right)+\frac{\gamma}{2}\left(a^3-a^4\right)\right]f,\\G&=0.\end{aligned}\tag{55}$$

Taking note of constraints (42) and (43), generators (53), (54), and (55) all satisfy the restriction of Eqs. (47) and (48). Therefore, they are the infinitesimal generators and the gauge function for the constrained Birkhoffian system of Herglotz type.

10. Conclusion

The Pfaff–Birkhoff–d'Alembert principle of Herglotz type, a new differential variational principle, is proposed. Birkhoff's equations for the Birkhoffian system (6) and the constrained Birkhoffian system (9) of Herglotz type are obtained, respectively. Not only are the conservation theorem and its inverse theorem for the Birkhoffian system of Herglotz type (6) established, but also the ones for the constrained Birkhoffian system of Herglotz type (9) are given. When the gauge function $G=0$, the conservation theorems and inverse theorems become the conclusions of Ref. [41]. On the other hand, according to the relationship among Lagrangian systems, Hamiltonian systems, and Birkhoffian systems, the conservation theorems of this paper can reduce to the ones for the Lagrangian system or the Hamiltonian system of Herglotz type when Birkhoffian B and Birkhoff's functions R_μ are taken to some special values. Consequently, the results of this paper are more general and can even solve the problems of nonholonomic systems.

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