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Quantum uncertainty relations of quantum coherence and dynamics under amplitude damping channel*

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In this paper, we discuss quantum uncertainty relations of quantum coherence through a different method from Ref. [52]. Some lower bounds with parameters and their minimal bounds are obtained. Moreover, we find that for two pairs of measurement bases with the same maximum overlap, quantum uncertainty relations and lower bounds with parameters are different, but the minimal bounds are the same. In addition, we discuss the dynamics of quantum uncertainty relations of quantum coherence and their lower bounds under the amplitude damping channel (ADC). We find that the ADC will change the uncertainty relations and their lower bounds, and their tendencies depend on the initial state.

Keywords: quantum uncertainty relation, quantum coherence, amplitude damping channel, single qubit state

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1. Introduction

In quantum theory, the uncertainty relation is a fundamental consequence of the superposition principle and the incompatible nature of observables. The uncertainty relation is a powerful tool in many quantum information tasks, such as quantum key distribution,[1,2] quantum random number generation,[3,4] entanglement witness,[5] and Einstein–Podolsky–Rosen (EPR) steering.[6,7] The original formulation of this concept was given by Heisenberg in 1927,[8] and then further generalized by Robertson[9] for two arbitrary observables in 1929, in which a lower bound of total variance of two observables was given. Such an uncertainty relation based on variance generally relies on the definition of observables. Deutsch[10] and Maassen *et al.*[11] proposed an uncertainty relation based on the Shannon entropy of measurement outcomes. Moreover, the majorization entropic uncertainty relation[12] and the strong majorization entropic uncertainty relation[13] were considered based on the Rényi entropy and Tsallis entropy of the measurement outcomes by using the majorization technique. Kurzyk *et al.* discussed entropic uncertainty relations in terms of the Tsallis entropies for states with a fixed amount of entanglement.[14] Several other discussions on the uncertainty relation have been published.[15–19]

Quantum coherence is one of the most important physical resources in quantum mechanics, which can be used in quantum optics,[20] quantum information and quantum computation,[21] thermodynamics,[22,23] and low temperature thermodynamics.[24–26] Many considerable efforts have recently been made to quantify the coherence of quantum states from the perspective of resource theory.[27–50] The authors of Ref. [27] proposed a rigorous framework to quantify coherence. Consequently, various coherence measures have been defined based on this framework, such as l_1-norm of coherence,[27] relative entropy of coherence (REOC),[27] quantum coherence via skew information (SIC),[28] and the geometric coherence (GC).[29]

As quantum coherence is a basis-dependent notion, we may ask if coherence respects some uncertainty relations for two or more incompatible bases. A concept of quantum uncertainty relation of quantum coherence was proposed in Refs. [51]–[53], and the sum of quantum coherence based on two measurement bases was called the quantum uncertainty relation of quantum coherence. Generally, the uncertainty relation of a measurement contains classical (predictable) and quantum (unpredictable) parts that originate from classical noise and quantum effect, respectively.[54] The quantum uncertainty relation of quantum coherence was considered as a quantum part (or genuine quantum uncertainty relation).[52] For the qubit case, quantum uncertainty relations based on two measurement bases were considered by using REOC, l_1 norm of coherence, and coherence of formation.[52] They gave the lower bounds of the sum of the quantum coherence for a qubit state based on two measurement bases. These bounds are related to the maximum eigenvalue of the state and the maximum overlap of the two measurement bases. However, one will want to know the following question. (i) For a fixed single qubit state and two pairs of measurement bases which have the same maximum overlap, will they give the same lower bound? In the first place of this work, we try to answer this question. We first consider quantum uncertainty relations of the quan-

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tum coherence through a different method from Ref. [52]. We will obtain some lower bounds with parameters, and their minimal bounds. Moreover, we can find that for two pairs of measurement bases which have the same maximum overlap, the quantum uncertainty relations and lower bounds with parameters are different, but the minimal bounds are the same.

Any quantum system inevitably suffers from environmental effect. Therefore, it is important to investigate how the noisy environment influences the entropic uncertainty. Dynamics of the entropic uncertainty relations under noisy channels were explored.[55–58] Naturally, we believe that it is also important to investigate another question, i.e., (ii) how does the noisy environment influences the quantum uncertainty relation of the quantum coherence? In the next place of the paper, we will consider dynamics of quantum uncertainty relations of the quantum coherence and their lower bounds under the amplitude damping channel (ADC). We find that ADC changes the uncertainty relations and their lower bounds, and their tendencies depend on the initial state. In this paper, we only consider the influence under the amplitude damping channel, and other Markovian channels can be considered by the similar method.

This paper is organized as follows. We briefly recall some notions we are going to use in our analysis in Section 2. In Section 3 we discuss quantum uncertainty relations of the quantum coherence in single state. In Section 4 we discuss dynamics of quantum uncertainty relations of the quantum coherence under ADC, and we summarize our results in Section 5.

2. Preliminaries

In this section, we review some notions related to quantifying the quantum coherence. Considering a finite-dimensional Hilbert space H with $d = \dim(H)$. Let $\{|i\rangle, i = 1,2,\ldots,d\}$ be a particular basis of H. A state is called an incoherent state if and only if its density operator is diagonal in this basis, and the set of all the incoherent states is usually denoted as Δ. Baumgratz *et al.*[27] have proposed that the quantum coherence can be measured by a function C that maps a state ρ to a non-negative real value, moreover, C must satisfy the following properties: $C(\rho) \geq 0$ and $C(\rho) = 0$ if and only if $\rho \in \Delta$; $C(\rho) \geq C(\Phi(\rho))$, where Φ is any incoherent completely positive and trace preserving map; $C(\rho) \geq \sum_i p_i C(\rho_i)$, where $p_i = \mathrm{tr}(K_i \rho K_i^\dagger)$, $\rho_i = \frac{K_i \rho K_i^\dagger}{\mathrm{tr}(K_i \rho K_i^\dagger)}$, for all K_i with $\sum_i K_i K_i^\dagger = I$ and $K_i \Delta K_i^\dagger \subseteq \Delta$; $\sum_i p_i C(\rho_i) \geq C(\sum_i p_i \rho_i)$ for any ensemble $\{p_i, \rho_i\}$.

In accordance with the criterion, several coherence measures have been studied. It has been shown that l_1 norm of coherence,[27] relative entropy of coherence,[27] the quantum coherence via skew information,[28] and geometric coherence[29] satisfy the above four conditions.

The relative entropy of coherence[27] is defined as

$$C_{\mathrm{r}}(\rho) = \min_{\delta \in I} S(\rho||\delta) = S(\rho_{\mathrm{diag}}) - S(\rho), \tag{1}$$

where $S(\rho||\delta) = \mathrm{tr}(\rho \log \rho - \rho \log \delta)$ is the quantum relative entropy, $S(\rho) = \mathrm{tr}(\rho \log \rho)$ is the von Neumann entropy, and $\rho_{\mathrm{diag}} = \sum_i |i\rangle\langle i|\rho|i\rangle\langle i|$. In this paper, log has a base of 2, and ln is the natural logarithm.

l_1 norm of coherence[27] is defined as

$$C_{l_1}(\rho) = \sum_{i \neq j} |\rho_{ij}|, \tag{2}$$

where ρ_{ij} are entries of ρ. The coherence measure defined by the l_1 norm is based on the minimal distance of ρ to the set of incoherent states Δ, $C_D(\rho) = \min_{\delta \in \Delta} D(\rho, \delta)$ with D being the l_1 norm, and $0 \leq C_{l_1}(\rho) \leq d - 1$.

The quantum coherence via skew information[28] of ρ is defined as

$$C_{\mathrm{SI}}(\rho) = \sum_{k=1}^{d} I(\rho, |k\rangle\langle k|) = 1 - \sum_{k=1}^{d} \langle k|\sqrt{\rho}|k\rangle, \tag{3}$$

where $I(\rho, |k\rangle\langle k|) = -\frac{1}{2}\mathrm{tr}\{[\sqrt{\rho}, |k\rangle\langle k|]\}^2$ represents the skew information subject to the projector $|k\rangle\langle k|$.

The geometric coherence is defined by Streltsov *et al.*[29] as follows:

$$C_{\mathrm{g}}(\rho) = 1 - \max_{\delta \in \Delta} F(\rho, \delta), \tag{4}$$

where $F(\rho, \delta) = \| \sqrt{\rho}\sqrt{\delta} \|^2$. For pure states, GC takes the form

$$C_{\mathrm{g}}(\rho) = \frac{1}{2}[1 - \sqrt{1 - 4|\rho_{01}|^2}], \tag{5}$$

where $\rho_{01} = \langle x_0|\rho|x_1\rangle$ is the off-diagonal element of ρ in the incoherent basis.

3. Quantum uncertainty relations of quantum coherence

In this section, we consider quantum uncertainty relations of the above four coherence measures for single-qubit states. Considering a single qubit state ρ with spectral decomposition

$$\rho = p|\varphi\rangle\langle\varphi| + (1-p)|\varphi_\perp\rangle\langle\varphi_\perp|, \tag{6}$$

where $p \in [\frac{1}{2}, 1]$ represents maximum eigenvalue of ρ, pure state $|\varphi\rangle = \mathrm{e}^{-\mathrm{i}\beta}\sin\alpha|0\rangle + \mathrm{e}^{\mathrm{i}\beta}\cos\alpha|1\rangle$ represents the eigenvector corresponding to the maximum eigenvalue p, $\alpha, \beta \in [0, \pi]$. In fact, any single qubit pure state can be represented in the above form if a global phase is neglected. We know any two orthogonal bases are related by a unitary transformation $U \in U(d)$. We choose two orthogonal bases X and Y which are related to computational base by the unitary transformations $M(\theta)$ and $M(\theta + \varepsilon)$, i.e., $|x_i\rangle = M(\theta)|i\rangle$, $|y_j\rangle = M(\theta + \varepsilon)|j\rangle$, where $i, j = 0, 1$ and

$$M(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in SO(2) \tag{7}$$

is a real rotation matrix. Hereafter, we assume $\theta, \varepsilon \in [0, \frac{\pi}{2}]$, and denote $c = \max\limits_{i,j=0,1} |\langle x_i|y_j\rangle|^2$. By a routine calculation, we have

$$c = \begin{cases} \cos^2\varepsilon, & \varepsilon \le \frac{\pi}{4}, \\ \sin^2\varepsilon, & \varepsilon > \frac{\pi}{4}. \end{cases}$$

For convenience, we first show the following lemma.

Lemma 1 For any pure state $|\varphi\rangle = \mathrm{e}^{-\mathrm{i}\beta}\sin\alpha|0\rangle + \mathrm{e}^{\mathrm{i}\beta}\cos\alpha|1\rangle$, $p \in [\frac{1}{2}, 1]$, we denote $a = |\langle x_0|\varphi\rangle|^2$, then there exists $\omega \in [0, 2\pi]$ such that $a = \sin^2(\theta + \omega)$.

Proof We have

$$\begin{aligned} a &= |\langle x_0|\varphi\rangle|^2 = |\,\mathrm{e}^{-\mathrm{i}\beta}\sin\alpha\cos\theta + \mathrm{e}^{\mathrm{i}\beta}\cos\alpha\sin\theta\,|^2 \\ &= \cos^2\beta(\sin\alpha\cos\theta + \cos\alpha\sin\theta)^2 \\ &\quad + \sin^2\beta(\cos\alpha\sin\theta - \sin\alpha\cos\theta)^2 \\ &= \cos^2\beta\sin^2(\theta+\alpha) + \sin^2\beta\sin^2(\theta-\alpha). \end{aligned} \tag{8}$$

Let $f(x) = \sin^2(\theta + x)$, it is obvious that $f(x)$ is a continuous function. Let $f_{\max} = \max\{(\sin^2(\theta-\alpha), \sin^2(\theta+\alpha)\}$, $f_{\min} = \min\{(\sin^2(\theta-\alpha), \sin^2(\theta+\alpha)\}$, we have $f_{\min} \le a \le f_{\max}$. By boundary value theorem, there exists $\omega \in [-\alpha, \alpha]$ such that $a = \sin^2(\theta + \omega)$.

Given a single qubit state ρ, we can find a corresponding value ω. Therefore, for convenience, we replace $\theta + \omega$ with θ in the following. Hence, we have $|\langle x_0|\varphi\rangle|^2 = \sin^2(\theta)$, furthermore, $|\langle x_0|\varphi_\perp\rangle|^2 = |\langle x_1|\varphi\rangle|^2 = \cos^2(\theta)$, and $|\langle x_1|\varphi_\perp\rangle|^2 = \sin^2(\theta)$. Similarly, we have $|\langle y_0|\varphi\rangle|^2 = |\langle y_1|\varphi_\perp\rangle|^2 = \sin^2(\theta + \varepsilon)$, and $|\langle y_0|\varphi_\perp\rangle|^2 = |\langle y_1|\varphi\rangle|^2 = \cos^2(\theta + \varepsilon)$.

After the projective measurements X and Y, the states of the system are given, respectively, by

$$\begin{aligned} \rho_x &= p_x|x_0\rangle\langle x_0| + (1-p_x)|x_1\rangle\langle x_1|, \\ \rho_y &= p_y|y_0\rangle\langle y_0| + (1-p_y)|y_1\rangle\langle y_1|, \end{aligned} \tag{9}$$

where $p_x = |\langle x_0|\rho|x_0\rangle = p\sin^2(\theta) + (1-p)\cos^2(\theta)$, $p_y = |\langle y_0|\rho|y_0\rangle = p\sin^2(\theta+\varepsilon) + (1-p)\cos^2(\theta+\varepsilon)$.

Based on the above preliminaries, we consider quantum uncertainty relations of the above quantum coherence measures. Instead of optimizing the uncertainty relations over all possible states ρ, we will optimize them over θ. We first present quantum uncertainty relations of REOC as follows.

Theorem 1 Given a single qubit state with the form of Eq. (6), and the above two orthogonal bases X and Y, a quantum uncertainty relation of REOC can be obtained as follows:

$$\begin{aligned} &C_{\mathrm{r}}^X(\rho) + C_{\mathrm{r}}^Y(\rho) \\ &\ge [1 - H(p)][\sin^2(2\theta) + \sin^2(2\theta + 2\varepsilon)] = B_{\mathrm{r}_1}(\theta) \\ &\ge 2[1 - H(p)](1-c) = B_{\mathrm{r}_1}. \end{aligned} \tag{10}$$

Proof By the proposition 1 in Ref. [14], when $\alpha \to 1$, we can obtain the following inequality:

$$H(px + (1-p)(1-x)) \ge 4[1 - H(p)]x(1-x) + H(p). \tag{11}$$

By using the above inequality, we have

$$\begin{aligned} &C_{\mathrm{r}}^X(\rho) + C_{\mathrm{r}}^Y(\rho) \\ &= H(p_x) + H(p_y) - 2H(p) \\ &\ge 4[1 - H(p)][\sin^2(\theta)\cos^2(\theta) + \sin^2(\theta+\varepsilon)\cos^2(\theta+\varepsilon)] \\ &= [1 - H(p)][\sin^2(2\theta) + \sin^2(2\theta + 2\varepsilon)]. \end{aligned} \tag{12}$$

In the following, we will prove that $f(\theta) = \sin^2(2\theta) + \sin^2(2\theta + 2\varepsilon)$ have the minimum value $2(1-c)$.

$$\begin{aligned} f(\theta) &= \sin^2(2\theta) + \sin^2(2\theta + 2\varepsilon) \\ &= [\sin(2\theta) + \sin(2\theta + 2\varepsilon)]^2 - 2\sin(2\theta)\sin(2\theta + 2\varepsilon) \\ &= 4\sin^2(2\theta + \varepsilon)\cos^2\omega + \cos(4\theta + 2\omega) - \cos 2\omega \\ &= 4\sin^2(2\theta + \varepsilon)\cos^2\omega + 1 - 2\sin^2(2\theta + \varepsilon) - \cos 2\omega \\ &= 2\cos 2\omega\sin^2(2\theta + \varepsilon) + 1 - \cos 2\omega. \end{aligned} \tag{13}$$

The third equality comes from product to sum formula. When $\varepsilon \le \frac{\pi}{4}$, $f(\theta)$ achieves a unique minimum value $2(1-c)$, for $\theta = \frac{\pi}{2} - \frac{\varepsilon}{2}$; when $\varepsilon \ge \frac{\pi}{4}$, $f(\theta)$ achieves a unique minimum value $2(1-c)$, for $\theta = \frac{\pi}{4} - \frac{\varepsilon}{2}$.

Theorem 1 gives a lower bound $B_{\mathrm{r}_1}(\theta)$ with parameters θ, the value θ depends on orthogonal base X and eigenvector $|\varphi\rangle$. We obtain the minimum lower bound B_{r_1} by optimizing the value θ. In Ref. [52], Yuan *et al.* have given a quantum uncertainty relation of REOC

$$\begin{aligned} C_{\mathrm{r}}^X(\rho) + C_{\mathrm{r}}^Y(\rho) &\ge H\left(\frac{\sqrt{2p-1}(2\sqrt{c}-1)+1}{2}\right) - H(\rho) \\ &= B_{\mathrm{r}_2}. \end{aligned} \tag{14}$$

We compare B_{r_1} with B_{r_2} for the case of $p = 0.65$, 0.8, and 0.95 in Figs. 1(a), 1(b), and 1(c), respectively, and we find that B_{r_1} is more optimal than B_{r_2} when ε is close to $\frac{\pi}{4}$, but the opposite result can be reached when ε is away from $\frac{\pi}{4}$. $B_{\mathrm{r}_1}(\theta)$, B_{r_1}, B_{r_2}, and $C_{\mathrm{r}}^X(\rho) + C_{\mathrm{r}}^Y(\rho)$ are compared for special $p = 0.75$ and $\varepsilon = \frac{\pi}{4}$ in Fig. 1(d).

According to Eq. (10) and Fig. 1(d), we can answer the question (i) in Section 1 aiming to REOC. For two pairs of measurement bases which have the same overlap, i.e., the same values ε, we can obtain different uncertainties of the quantum coherence and different lower bounds $B_{\mathrm{r}_1}(\theta)$, but the same lower B_{r_1} and B_{r_2} can be obtained.

Now, we consider a quantum uncertainty relation of SIC based on bases X and Y, and give a lower bound of $C_{\mathrm{SI}}^X(\rho) + C_{\mathrm{SI}}^Y(\rho)$. In Fig. 2(a), we compare $C_{\mathrm{SI}}^X(\rho) + C_{\mathrm{SI}}^Y(\rho)$ and B_{SI} for $p = 0.75$ and $\varepsilon = \frac{\pi}{5}$.

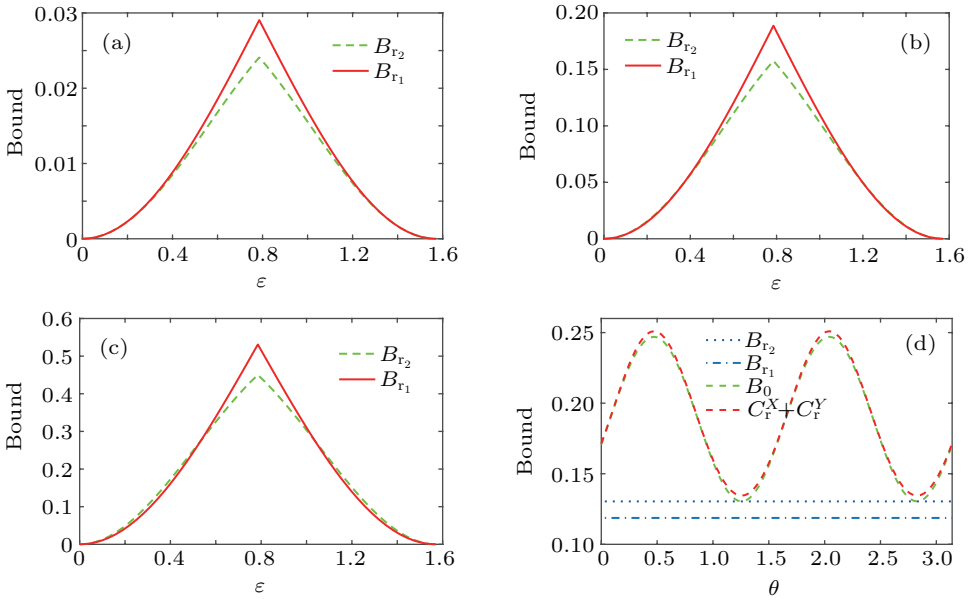


Fig. 1. (color online) The comparison between B_{r_1} and B_{r_2} for the cases of (a) $p=0.65$, (b) $p=0.8$, and (c) $p=0.95$; (d) $B_{r_1}(\theta)$, B_{r_1}, B_{r_2}, and $C_r^X(\rho)+C_r^Y(\rho)$ are compared for the case of $\varepsilon=\frac{\pi}{4}$ and $p=0.75$.

Theorem 2 Given a single qubit state with the form of Eq. (6), and the above two orthogonal bases X and Y, a quantum uncertainty relation of SIC can be obtained as follows:

$$\begin{aligned} & C_{\mathrm{SI}}^X(\rho)+C_{\mathrm{SI}}^Y(\rho) \\ = & \frac{1-2\sqrt{p(1-p)}}{2}[\sin^2(2\theta)+\sin^2(2\theta+2\varepsilon)] \\ \geq & [1-2\sqrt{p(1-p)}](1-c)=B_{\mathrm{SI}}. \end{aligned} \tag{15}$$

If $\varepsilon \leq \frac{\pi}{4}$, inequality becomes quality when $\theta=\frac{\pi}{2}-\frac{\varepsilon}{2}$; if $\varepsilon \geq \frac{\pi}{4}$, inequality becomes quality when $\theta=\frac{\pi}{4}-\frac{\varepsilon}{2}$.

Proof Since

$$|\langle x_0|\sqrt{\rho}|x_0\rangle|^2=\sqrt{p}\sin^2(2\theta)+\sqrt{1-p}\cos^2(2\theta)$$

and

$$|\langle x_1|\sqrt{\rho}|x_1\rangle|^2=\sqrt{p}\cos^2(2\theta)+\sqrt{1-p}\sin^2(2\theta),$$

then

$$\begin{aligned} C_{\mathrm{SI}}^X(\rho) &= 1-|\langle x_0|\sqrt{\rho}|x_0\rangle|^2-|\langle x_1|\sqrt{\rho}|x_1\rangle|^2 \\ &= 2(1-2\sqrt{p(1-p)})\sin^2(\theta)\cos^2(\theta) \\ &= \frac{1-2\sqrt{p(1-p)}}{2}\sin^2(2\theta). \end{aligned} \tag{16}$$

Replacing θ with $\theta+\varepsilon$, we can obtain $C_{\mathrm{SI}}^Y(\rho)=\frac{1-2\sqrt{p(1-p)}}{2}\sin^2(2\theta+2\varepsilon)$. Therefore, we have

$$\begin{aligned} & C_{\mathrm{SI}}^X(\rho)+C_{\mathrm{SI}}^Y(\rho) \\ = & \frac{1-2\sqrt{p(1-p)}}{2}[\sin^2(2\theta)+\sin^2(2\theta+2\varepsilon)] \\ \geq & [1-2\sqrt{p(1-p)}](1-c). \end{aligned} \tag{17}$$

The inequality comes from the proof of Theorem 1. If $\varepsilon \leq \frac{\pi}{4}$, inequality becomes quality when $\theta=\frac{\pi}{2}-\frac{\varepsilon}{2}$; if $\varepsilon \geq \frac{\pi}{4}$, $f(\theta)$ achieves a unique minimum value $2(1-c)$, when $\theta=\frac{\pi}{4}-\frac{\varepsilon}{2}$.

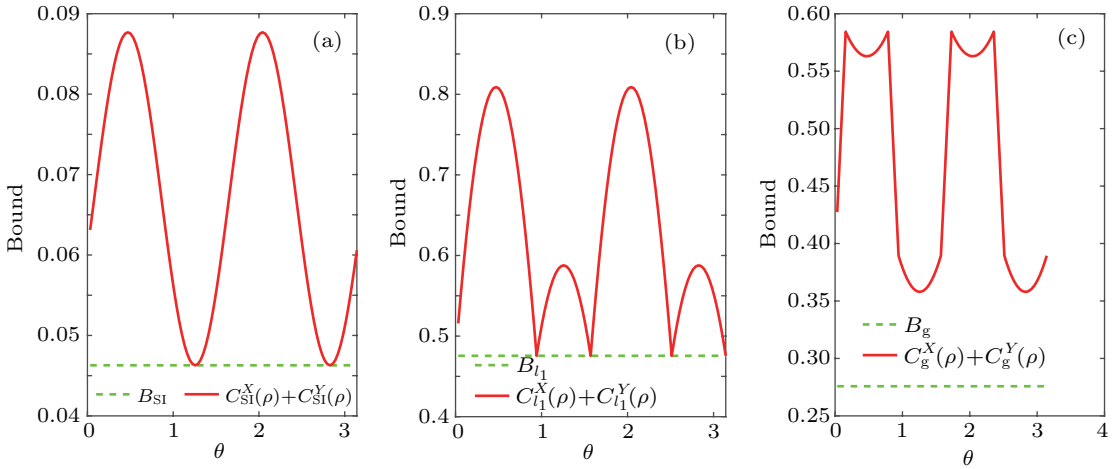


Fig. 2. (color online) For $\varepsilon=\frac{\pi}{5}$ and $p=0.75$, (a) $C_{\mathrm{SI}}^X(\rho)+C_{\mathrm{SI}}^Y(\rho)$ and B_{SI}, (b) $C_{l_1}^X(\rho)+C_{l_1}^Y(\rho)$ and B_{l_1}, and (c) $C_g^X(\rho)+C_g^Y(\rho)$ and B_g are compared.

According to Eq. (15) and Fig. 2(a), we can answer the question (i) in Section 1 aiming to SIC. For two pairs of measurement bases which have the same overlap, i.e., the same values ε, we can obtain different red uncertainty relations of SIC, but the same lower $B_{\rm SI}$ will be obtained.

In Ref. [52], Yuan *et al.* have given a quantum uncertainty relation of l_1-norm of coherence based on two fixed bases, and a lower bound of $C^X_{l_1}(\rho)+C^Y_{l_1}(\rho)$ has been obtained. This bound can be saturated in some special cases. In the following, we will show this bound from our perspective. Our perspective will provide a simple method to prove the lower bound. In Fig. 2(b), we compare $C^X_{l_1}(\rho)+C^Y_{l_1}(\rho)$ with B_{l_1}.

Theorem 3 Given a single qubit state with the form of Eq. (6), and the above two orthogonal bases X and Y, a quantum uncertainty relation of l_1 norm of coherence can be obtained as follows:

$$\begin{aligned} & C^X_{l_1}(\rho)+C^Y_{l_1}(\rho) \\ =\ & |(2p-1)|[|\sin(2\theta)|+|\sin(2\theta+2\varepsilon)|] \\ \geq\ & |(2p-1)||\sin(2\varepsilon)| = 2|(2p-1)|\sqrt{c(1-c)} = B_{l_1}. \end{aligned} \tag{18}$$

Inequality can be saturated when $\theta=0$, or $\theta=\pi-\varepsilon$.

Proof By a direct calculation, we obtain $C^X_{l_1}(\rho) = 2|\langle x_0|\rho|x_1\rangle| = 2|(2p-1)||\sin(2\theta)|$ and $C^Y_{l_1}(\rho) = 2|(2p-1)||\sin(2\theta+2\varepsilon)|$. Hence, we have

$$\begin{aligned} & C^X_{l_1}(\rho)+C^Y_{l_1}(\rho) \\ =\ & 2|(2p-1)|[|\sin(2\theta)|+|\sin(2\theta+2\varepsilon)|] \\ \geq\ & |(2p-1)||\sin(2\varepsilon)| = |(2p-1)|2\sqrt{c(1-c)}. \end{aligned} \tag{19}$$

In the following, we will show $f(\alpha)=|\sin(\alpha)|+|\sin(\alpha+\beta)|\geq|\sin(\beta)|$, where $\beta\in[0,\frac{\pi}{2}]$. It is obvious that f is a period function, and the minimal positive period is $T=\pi$. Considering $\alpha\in[0,\pi]$. When $\alpha\in[0,\pi-\beta]$, in this case, $\alpha+\beta\in[\beta,\pi]$, hence $f(\alpha)=\sin(\alpha)+\sin(\alpha+\beta)$, $f''(\alpha)=-\sin(\alpha)-\sin(\alpha+\beta)\leq 0$, then f is a concave function. Therefore, f has a minimal value of $f_{\min}=\sin(\beta)$ for $\theta=0$ or $\theta=\pi-\beta$. When $\theta\in[\pi-\beta,\pi]$, $f(\theta)=\sin(\theta)-\sin(\theta+\beta)$. In the same way, we can find $f_{\min}=\sin(\beta)$ for $\theta=\pi$. Replacing α and β with 2θ and 2ε, respectively, we can prove our result. Moreover, inequality can be saturated when $\theta=0$, or $\theta=\pi-\varepsilon$.

According to Eq. (18) and Fig. 2(b), we can answer the question (i) in Section 1 aiming to the l_1 norm of coherence. For two pairs of measurement bases which have the same overlap, i.e., the same values ε, we can obtain different uncertainties of l_1 norm of coherence, but the same lower bound B_{l_1} can be obtained. In addition to REOC and l_1 norm of coherence, the uncertainty relation of the coherence of formation has been discussed in Ref. [52] for single qubit state. By using the way similar to Ref. [52], we can also answer the question (i) in Section 1 aiming to the coherence of formation. We may find an optimal lower bound than the previous one if we find a suitable inequality about entropy. This is a problem to be solved.

Next, we consider a quantum uncertainty relation of GC based on bases X and Y, and give a lower bound of $C^X_{\rm g}(\rho)+C^Y_{\rm g}(\rho)$. In Fig. 2(c), we compare $C^X_{\rm g}(\rho)+C^Y_{\rm g}(\rho)$ and $B_{\rm g}$ for $p=0.75$ and $\varepsilon=\frac{\pi}{5}$.

Theorem 4 Given a single qubit state with the form of Eq. (6), and the above two orthogonal bases X and Y, a quantum uncertainty relation of GC can be obtained as follows:

$$C^X_{\rm g}(\rho)+C^Y_{\rm g}(\rho)\geq 1-\sqrt{1-2(2p-1)\sqrt{c(1-c)}} = B_{\rm g}. \tag{20}$$

Proof According to Eq. (4), we obtain

$$\begin{aligned} C^X_{\rm g}(\rho) &= \frac{1}{2}[1-\sqrt{1-(4p-2)|\sin(2\theta)|}], \\ C^Y_{\rm g}(\rho) &= \frac{1}{2}[1-\sqrt{1-(4p-2)|\sin(2\theta+2\varepsilon)|}]. \end{aligned} \tag{21}$$

Therefore, we have

$$\begin{aligned} & C^X_{\rm g}(\rho)+C^Y_{\rm g}(\rho) \\ \geq\ & 1-\frac{1}{2}[\sqrt{1-(4p-2)|\sin(2\theta)|} \\ & +\sqrt{1-(4p-2)|\sin(2\theta+2\varepsilon)|}] \\ \geq\ & 1-\sqrt{1-2(2p-1)\sqrt{c(1-c)}} = B_{\rm g}. \end{aligned} \tag{22}$$

The second inequality comes from the inequality $\frac{\sqrt{x}+\sqrt{y}}{2}\leq\sqrt{\frac{x+y}{2}}$.

According to Eq. (20) and Fig. 2(c), we can answer the question (i) in Section 1 aiming to GC. For two pairs of measurement bases which have the same overlap, i.e., the same values ε, we can obtain different uncertainty relations of GC, but the same lower bound $B_{\rm g}$ can be obtained.

4. Dynamics of quantum uncertainty relations of quantum coherence under amplitude damping channel

In this section, to answer the question (ii) in Section 1, we discuss dynamics of quantum uncertainty relations of quantum coherence under amplitude damping channel for single-qubit states. Our task is to study dynamics of left-hand sides (LHSs) and right-hand sides (RHSs) of Eqs. (10), (15), (18), and (20) under ADC. In fact, we can consider other Markovian channels by a similar method. ADC can be characterized by the Kraus' operators $K^{\rm AD}_0=|0\rangle\langle 0|+\sqrt{q}|1\rangle\langle 1|$, $K^{\rm AD}_1=\sqrt{1-q}|0\rangle\langle 1|$, where parameters $q\in[0,1]$. We consider single-qubit mixed states with the form

$$\rho(t,z)=\begin{bmatrix} \frac{1+z}{2} & \frac{t}{2} \\ \frac{t}{2} & \frac{1-z}{2} \end{bmatrix}, \tag{23}$$

with $t^2+z^2\leq 1$. Using a simple calculation, the maximum eigenvalue of ρ is $p=\frac{1+\sqrt{C}}{2}$, where $C=t^2+z^2$. Using ADC into the state with the form of Eq. (23), we obtain

$$\varepsilon(\rho)=\begin{bmatrix}\frac{1+z}{2}+(1-q)\frac{1-z}{2} & \sqrt{q}\frac{t}{2}\\ \sqrt{q}\frac{t}{2} & q\frac{1-z}{2}\end{bmatrix}. \tag{24}$$

Through a direct calculation, the maximum eigenvalue of $\varepsilon(\rho)$ is $p'=\frac{1+\sqrt{D}}{2}$, where $D=1+qt^2-q[1-z^2+(1-q)(1-z)^2]$. We substitute the state $\varepsilon(\rho)$ into Eqs. (10), (15), (18), and (20), and we denote four new bounds by B'_{r_1}, B'_{SI}, B'_{l_1}, and B'_g.

We first compare B_{r_1}, B_{SI}, B_{l_1}, and B_g with B'_{r_1}, B'_{SI}, B'_{l_1}, and B'_g. Due to Eqs. (10), (15), (18), and (20), we find B_{r_1}, B_{SI}, B_{l_1}, and B_g are increasing functions with respect to p for a fixed value ε. Now, we consider $D-C=(q-1)[t^2+z^2-1+q(1-z)^2]$. In the case of $t^2\leq 2z-2z^2$, we have $D-C\geq 0$ for any $q\in[0,1]$. Hence, $p'>p$ holds. Therefore, $B'_{r_1}\geq B_{r_1}$, $B'_{SI}\geq B_{SI}$, $B'_{l_1}\geq B_{l_1}$, and $B'_g\geq B_g$, and inequalities become equalities when $q=1$. In the case of $\sqrt{1-z^2}\geq t^2>2z-2z^2$, if $q\leq\frac{1-t^2-z^2}{(1-z)^2}$, then $D\geq C$, i.e., $p'\geq p$, hence, $B'_{r_1}\geq B_{r_1}$, $B'_{SI}\geq B_{SI}$, $B'_{l_1}\geq B_{l_1}$, and $B'_{l_1}\geq B_{l_1}$. If $q\geq\frac{1-t^2-z^2}{(1-z)^2}$, then $D\leq C$, i.e, $p'\leq p$, hence, $B'_{r_1}\leq B_{r_1}$, $B'_{SI}\leq B_{SI}$, $B'_{l_1}\leq B_{l_1}$, and $B'_g\leq B_g$.

Next, we discuss the variation of B'_{r_1}, B'_{SI}, B'_{l_1}, and B'_g with the coefficient q. In the case of $t^2\leq 2z-2z^2$, $\frac{\partial D}{\partial q}=t^2+z^2-1-(1-2q)(1-z)^2\leq 0$ holds for any $q\in[0,1]$. B'_{r_1}, B'_{SI}, B'_{l_1}, and B'_g will be reduced with q increasing. In the case of $\sqrt{1-z^2}\geq t^2>2z-2z^2$, if $q\geq\frac{1}{2}+\frac{1-(t^2+z^2)}{2(1-z)^2}$, then $\frac{\partial D}{\partial q}\geq 0$, that is to say that p' will be increased with q increasing, hence, B'_{r_1}, B'_{SI}, B'_{l_1}, and B'_g will be increased with q increasing. If $0\leq q<\frac{1}{2}+\frac{1-(t^2+z^2)}{2(1-z)^2}$, then $\frac{\partial D}{\partial q}<0$, that is to say that p' will be decreased with q increasing, hence, B'_{r_1}, B'_{SI}, B'_{l_1}, and B'_g will be decreased with q increasing.

Example 1 Given a fixed value $\varepsilon=\frac{\pi}{4}$, and two initial states of ρ_1 and ρ_2 with parameters $(z_1,t_1)=(\frac{1}{2},\frac{\sqrt{3}}{3})$ and $(z_2,t_2)=(\frac{1}{2},\frac{\sqrt{10}}{4})$. We can easily find $t_1^2<2z_1-2z_1^2$, but $t_2^2>2z_2-2z_2^2$. In Fig. 3, we study the variation of B'_{r_1}, B'_{SI}, B'_{l_1}, and B'_g with the coefficient q. For the initial state ρ_1, these three bounds will first decrease, and then increase as q increases. For the initial state ρ_2, these three bounds will always decrease.

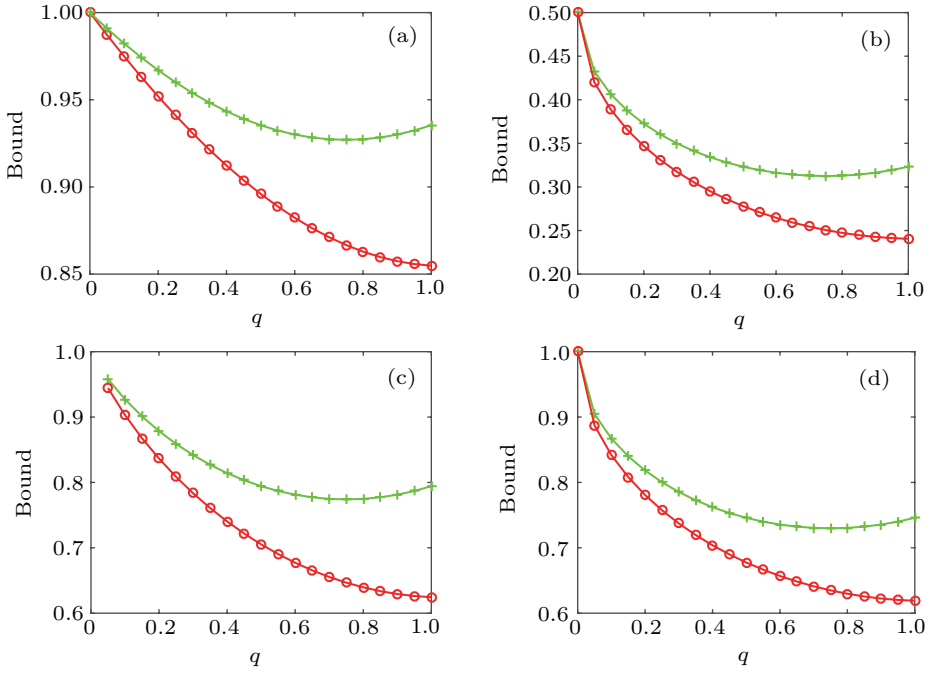


Fig. 3. (color online) $\varepsilon=\frac{\pi}{4}$. (a), (b), (c), and (d) represent the variation of B'_{r_1}, B'_{SI}, B'_{l_1}, and B'_g with the coefficient q, respectively. Red "+" is initial state ρ_1, and green "○" is initial state ρ_2.

In the following, we investigate dynamics of LHSs of Eqs. (10), (15), (18), and (20) under ADC. It is difficult to consider all cases. For simplicity, we only consider some special cases. We treat the LHSs of Eqs. (10), (15), (18), and (20) as some functions with respect to q, and treat other parameters t, z, ε, and θ as constants. We first consider l_1-norm of coherence for single state with the form of Eq. (24).

$$\begin{aligned} f &= C^X_{l_1}(\varepsilon(\rho))+C^Y_{l_1}(\varepsilon(\rho))\\ &= |\sqrt{q}t\cos(2\theta)+[q(1-z)-1]\sin(2\theta)|\\ &\quad +|\sqrt{q}t\cos(2\theta+2\varepsilon)+[q(1-z)-1)]\sin(2\theta+2\varepsilon)|. \end{aligned} \tag{25}$$

We consider some special cases as follows:

(I) In the case of $\cos(2\theta)\geq 0$, $\sin(2\theta)\leq 0$, $\cos(2\theta+2\varepsilon)\geq 0$, and $\sin(2\theta+2\varepsilon)\leq 0$. That is to say $\theta\in[\frac{3}{4}\pi,\pi]\cap[\frac{3}{4}\pi-\varepsilon,\pi-\varepsilon]$. Here, we assume $\varepsilon\in[0,\frac{\pi}{4}]$, the set is empty set if $\varepsilon\in[\frac{\pi}{4},\frac{\pi}{2}]$. In this case,

$$f=\sqrt{q}t\cos(2\theta)+[q(1-z)-1]\sin(2\theta)$$

$$
\begin{aligned}
&+\sqrt{qt}\cos(2\theta+2\varepsilon)+[q(1-z)-1]\sin(2\theta+2\varepsilon),\\
\frac{\partial f}{q} &= \frac{t}{2\sqrt{q}}[\cos(2\theta)+\cos(2\theta+2\varepsilon)]\\
&+(1-z)[\sin(2\theta)+\sin(2\theta+2\varepsilon)].
\end{aligned}
$$

We denote $M=\frac{t[\cos(2\theta)+\cos(2\theta+2\varepsilon)]}{2(z-1)[\sin(2\theta)+\sin(2\theta+2\varepsilon)]}$. M is clearly non-negative. When $M\geq 1$, $\frac{\partial f}{q}\geq 0$ holds for all $q\in[0,1]$, hence f increases as q increases. When $M\in[0,1]$, $\frac{\partial f}{q}\geq 0$ if $q\in[0,M^2]$, and $\frac{\partial f}{q}\leq 0$ if $q\in[M^2,1]$. Therefore, f first increases and then decreases as q increases.

(II) In the case of $\cos(2\theta)<0$, $\sin(2\theta)>0$, $\cos(2\theta+2\varepsilon)<0$, and $\sin(2\theta+2\varepsilon)>0$. That is to say $\theta\in[\frac{1}{4}\pi,\frac{\pi}{2}]\cap[\frac{1}{4}\pi-\varepsilon,\frac{\pi}{2}-\varepsilon]$. Here, we assume $\varepsilon\in[0,\frac{\pi}{4}]$, the set is empty set if $\varepsilon\in[\frac{\pi}{4},\frac{\pi}{2}]$. In this case,

$$
\begin{aligned}
f &= -\sqrt{qt}\cos(2\theta)+[1-q(1-z)]\sin(2\theta)\\
&-\sqrt{qt}\cos(2\theta+2\varepsilon)+[1-q(1-z))]\sin(2\theta+2\varepsilon),\\
\frac{\partial f}{q} &= -\frac{t}{2\sqrt{q}}[\cos(2\theta)+\cos(2\theta+2\varepsilon)]\\
&-(1-z)[\sin(2\theta)+\sin(2\theta+2\varepsilon)]. \qquad (26)
\end{aligned}
$$

Similar to case (I), when $M\geq 1$, $\frac{\partial f}{q}\geq 0$ holds for all $q\in[0,1]$, hence f increases as q increases. When $M\in[0,1]$, $\frac{\partial f}{q}\geq 0$ if $q\in[0,M^2]$, and $\frac{\partial f}{q}\leq 0$ if $q\in[M^2,1]$. Therefore, f first increases and then decreases as q increases.

(III) In the case of $\cos(2\theta)<0$, $\sin(2\theta)>0$, $\cos(2\theta+2\varepsilon)>0$, and $\sin(2\theta+2\varepsilon)<0$. That is to say $\theta\in[\frac{\pi}{4},\frac{\pi}{2}]\cap[\frac{3\pi}{4}-\varepsilon,\pi-\varepsilon]$. Here, $\varepsilon\in[\frac{\pi}{4},\frac{\pi}{2}]$, the set is a empty set if $\varepsilon\in[0,\frac{\pi}{4}]$. In this case,

$$
\begin{aligned}
f &= -\sqrt{qt}\cos(2\theta)-[q(1-z)-1]\sin(2\theta)\\
&+\sqrt{qt}\cos(2\theta+2\varepsilon)+[q(1-z)-1)]\sin(2\theta+2\varepsilon),\\
\frac{\partial f}{q} &= \frac{t}{2\sqrt{q}}[\cos(2\theta+2\varepsilon)-\cos(2\theta)]\\
&+(1-z)[\sin(2\theta+2\varepsilon)-\sin(2\theta)]. \qquad (27)
\end{aligned}
$$

We denote $M_1=\frac{t[\cos(2\theta+2\varepsilon)-\cos(2\theta)]}{2(z-1)[\sin(2\theta+2\varepsilon)-\sin(2\theta)]}$, M_1 is clearly non-negative. When $M_1\geq 1$, $\frac{\partial f}{q}\geq 0$ holds for all $q\in[0,1]$, hence f increases as q increases. When $M_1\in[0,1]$, $\frac{\partial f}{q}\geq 0$ if $q\in[0,M_1^2]$, and $\frac{\partial f}{q}\leq 0$ if $q\in[M_1^2,1]$. Therefore, f first increases and then decreases as q increases.

Example 2 We give the following examples to express the difference of the above cases. In Figs. 4(a)–4(f), we give the variations of Eq. (25) for the situations of (A)–(F) under ADC.

(A) Given a state ρ_1 with parameters $(z_1,t_1)=(\frac{\sqrt{99}}{10},\frac{1}{10})$, and a two-tuples $(\theta_1,\varepsilon_1)=(\frac{3\pi}{4},\frac{\pi}{6})$. In this case $M=5.759>1$.

(B) Given a state ρ_2 with parameters $(z_2,t_2)=(\frac{1}{2},\frac{\sqrt{10}}{4})$, and a two-tuples $(\theta_1,\varepsilon_1)=(\frac{3\pi}{4},\frac{\pi}{6})$. In this case, $M=0.4564<1$.

(C) Given a state ρ_3 with parameters $(z_3,t_3)=(\frac{\sqrt{99}}{10},\frac{1}{10})$, and a two-tuples $(\theta_2,\varepsilon_2)=(\frac{\pi}{4},\frac{\pi}{6})$, In this case $M=5.759>1$.

(D) Given a state ρ_4 with parameters $(z_4,t_4)=(\frac{10}{13},\frac{5}{13})$, and a two-tuples $(\theta_2,\varepsilon_2)=(\frac{\pi}{4},\frac{\pi}{6})$. In this case, $M=0.4811<1$.

(E) Given a state ρ_5 with parameters $(z_5,t_5)=(\frac{1}{2},\frac{1}{6})$, and a two-tuples $(\theta_3,\varepsilon_3)=(\frac{\pi}{2},\frac{\pi}{3})$. In this case, $M_1=0.2887<1$.

(F) Given a state ρ_6 with parameters $(z_6,t_6)=(\frac{1}{2},\frac{\sqrt{10}}{4})$, and a two-tuples $(\theta_3,\varepsilon_3)=(\frac{\pi}{2},\frac{\pi}{3})$. In this case, $M_1=1.3693>1$.

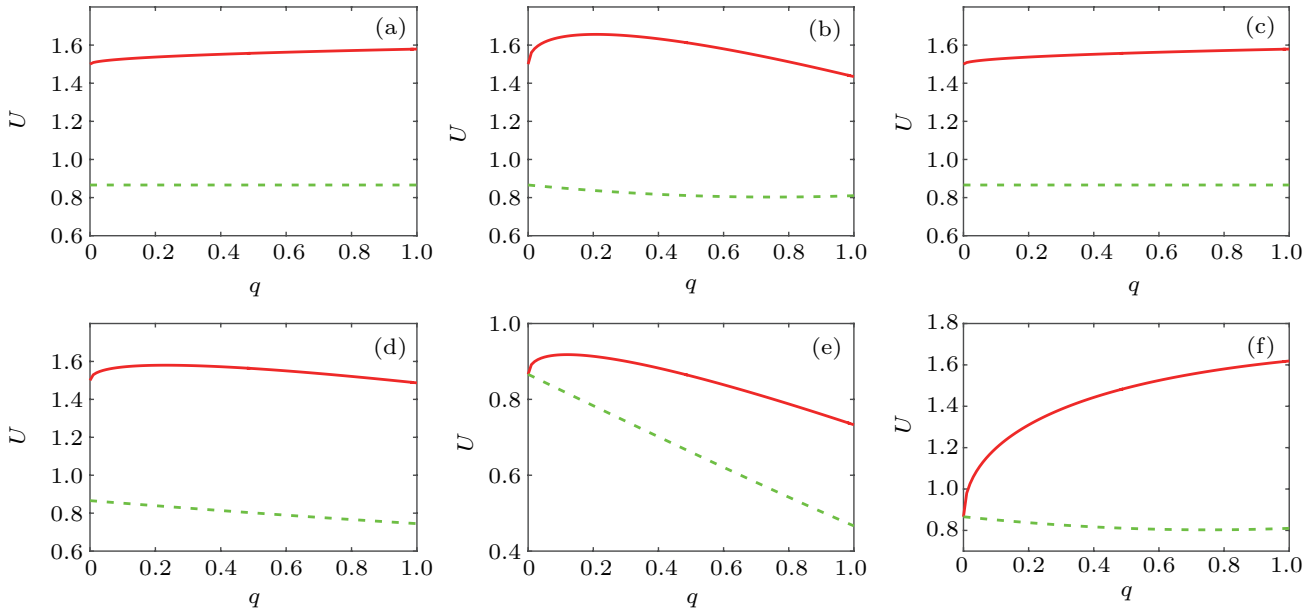


Fig. 4. (color online) The variations of function f and the lower bounds under ADC. The red solid line represents the value of f, and the green dotted line represents the lower bounds. U represents the value of function f or its lower bound. (a)–(f) correspond to (A)–(F) in Example 2, respectively.

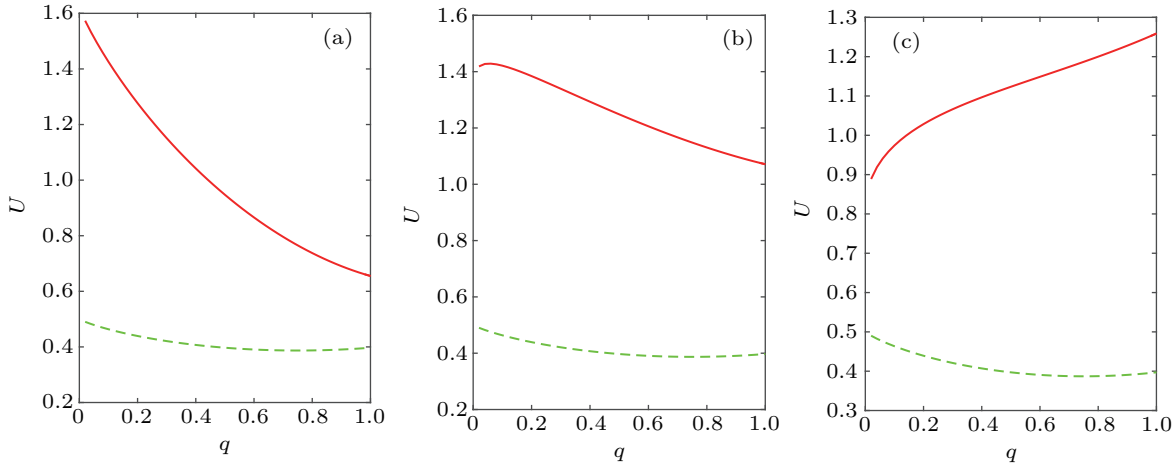


Fig. 5. (color online) The variations of the function g and the lower bounds under ADC. $\varepsilon = \frac{\pi}{6}$ and $p = 0.75$. The red line represents the value of function g, and the green dotted line represents the lower bounds. U represents the value of function g or its lower bound.

Now, we consider dynamic of the LHS of Eq. (10) for the state with the form of Eq. (24) under ADC. We denote $g = C_{\mathrm{r}}^{X}(\varepsilon(\rho)) + C_{\mathrm{r}}^{Y}(\varepsilon(\rho)) = H(p'_x) + H(p'_y) - 2H(p')$, $B_{\mathrm{r}} = 2[1 - H(p')](1-c)$, where $p' = \frac{1+\sqrt{D}}{2}$, and $D = 1 + qt^2 - q[1 - z^2 + (1-q)(1-z)^2]$. $p'_x = \cos^2\theta - q\frac{1-z}{2}\cos 2\theta + \frac{\sqrt{q}t}{2}\sin 2\theta$, $p'_y = \cos^2(\theta+\varepsilon) - q\frac{1-z}{2}\cos 2(\theta+\varepsilon) + \frac{\sqrt{q}t}{2}\sin 2(\theta+\varepsilon)$. We draw g and B_{r} as a function of q. In Fig. 5, we discuss the dynamics of the function g and B_{r} for some initial states with respect to q.

Example 3 Given an initial state ρ with parameters $(t,z) = (\sqrt{\frac{5}{8}}, \frac{1}{2})$, and $\varepsilon = \frac{\pi}{6}$. In Fig. 5(a), for the case of $\theta_1 = \frac{\pi}{6}$, ADC decreases the value of function g as q increases; in Fig. 5(b), for the case of $\theta_2 = \frac{\pi}{4}$, ADC first increases and then decreases the value of function g as q increases; in Fig. 5(c), for the case of $\theta_3 = \frac{\pi}{3}$, ADC increases the value of function g as q increases. In the above three cases, ADC first decreases and then increases B_{r} as q increases.

It is a difficult task to calculate LHSs of Eqs. (15) and (20) for state with the form of Eq. (24). However, we can discuss their dynamics by fixing the parameters ε and θ. In this case, we find LHSs of Eqs. (15) and (20) have the same tendency with RHSs of Eqs. (15) and (20) under ADC.

5. Conclusion

In this paper, we discussed quantum uncertainty relations of quantum coherence through a different method from Ref. [52]. We obtained some lower bounds with parameters, and their minimal bounds. Moreover, we found that for two pairs of measurement bases which have the same maximum overlap, the different quantum uncertainty relations and lower bounds with parameters maybe obtained, but the same minimal bounds can be obtained. This result answered the question (i) posed in Section 1. In the next place, we considered dynamics of quantum uncertainty relations of quantum coherence and their lower bounds under the amplitude damping channel (ADC). We found that ADC changed the uncertainty relations and their lower bounds, and their tendencies depended on the initial state. These discussions answered a part of the question (ii) posed in Section 1.

There are many further issues that need to be solved in the future. We can discuss the uncertainty relations for other coherence measures. We only considered dynamics of of quantum uncertainty relations of quantum coherence under ADC. Dynamics of quantum uncertainty relations of quantum coherence under any incoherent operator[27] (strictly incoherent operation and maximally incoherent operation[45]) is an interesting subject. Quantum uncertainty relations based on many fixed bases are also an interesting subject for future work.

References

[1] Koashi M 2009 *New J. Phys.* **11** 045018
[2] Berta M, Christandl M, Colbeck R, Renes J M and Renner R 2010 *Nat. Phys.* **6** 659
[3] Vallone G, Marangon D G, Tomasin M and Villoresi P 2014 *Phys. Rev. A* **90** 052327
[4] Cao Z, Zhou H, Yuan X and Ma X 2016 *Phys. Rev. X* **6** 011020
[5] Berta M, Coles P J and Wehner 2014 *Phys. Rev. A* **90** 062127
[6] Walborn S P, Salles A, Gomes R M, Toscano F and Ribeiro P H S 2011 *Phys. Rev. Lett.* **106** 130402
[7] Schneeloch J, Broadbent C J, Walborn S P, Cavalcanti E G and Howell J C 2013 *Phys. Rev. A* **87** 062103
[8] Heisenberg W 1927 *Physik Z* **43** 172
[9] Robertson H P 1929 *Phys. Rev.* **34** 163
[10] Deutsch 1983 *Phys. Rev. Lett.* **50** 631
[11] Maassen H and Uffink J B M 1988 *Phys. Rev. Lett.* **60** 1103
[12] Puchała Z, Rudnicki Ł and Życzkowski K 2013 *J. Phys. A: Math. Theor.* **46** 272002
[13] Rudnicki Ł, Puchała Z and Życzkowski K 2014 *Phys. Rev. A* **89** 052115
[14] Kurzyk D, Pawela Ł and Puchała Z 2018 *Quantum Inf. Process.* **17** 193
[15] Ma Z H, Chen Z H and Fei S M 2017 *Sci. China-Phys. Mech. Astron.* **60** 010321
[16] Xiao Y, Guo C, Meng F, Jing N and Yung M H 2017 arXiv:1706.05650 [quant-ph]
[17] Puchała Z, Rudnicki Ł, Krawiec A and Życzkowski K 2018 *J. Phys. A: Math. Theor.* **51** 175306
[18] Chen B, Cao N P, Fei S M and Long G L 2016 *Quantum Inf. Proc.* **15** 3909
[19] Chen B, Fei S M, and Long G L 2016 *Quantum Inf. Proc.* **15** 2639

[20] Scully M O and Zubairy M S 1997 *Quantum Optics* (Cambridge: Cambridge University Press)

[21] Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)

[22] Rodríguez-Rosario C A, Frauenheim T and Aspuru-Guzik A 2013 arXiv:1308.1245 [quant-ph]

[23] Åberg J 2014 *Phys. Rev. Lett.* **113** 150402

[24] Horodecki M and Oppenheim J 2013 *Nat. Commun.* **4** 2059

[25] Lostaglio M, Korzekwa K, Jennings D and Rudolph T 2015 *Phys. Rev. X* **5** 021001

[26] Narasimhachar V and Gour G 2015 *Nat. Commun.* **6** 7689

[27] Baumgratz T, Cramer M and Plenio M B 2014 *Phys. Rev. Lett.* **113** 140401

[28] Yu C S 2017 *Phys. Rev. A* **95** 042337

[29] Streltsov A, Singh U, Dhar H S, Bera M N and Adesso G 2015 *Phys. Rev. Lett.* **115** 020403

[30] Zhou Y, Zhao Q, Yuan X and Ma X 2017 *New J. Phys.* **19** 123033

[31] Shao L H, Xi Z, Fan H and Li Y 2015 *Phys. Rev. A* **91** 042120

[32] Napoli C, Bromley T R, Cianciaruso M, Piani M, Johnston N and Adesso G 2016 *Phys. Rev. Lett.* **116** 150502

[33] Zhang Y R, Shao L H, Li Y and Fan H 2016 *Phys. Rev. A* **93** 012334

[34] Yu X D, Zhang D J, Xu G F and Tong D M 2016 *Phys. Rev. A* **94** 060302

[35] Yao Y, Xiao X, Ge L and Sun C P 2015 *Phys. Rev. A* **92** 022112

[36] Du S, Bai Z and Guo Y 2015 *Phys. Rev. A* **91** 052120

[37] Xi Z, Li Y and Fan H 2015 *Sci. Rep.* **5** 10922

[38] Winter A and Yang D 2016 *Phys. Rev. Lett.* **116** 120404

[39] Bromley T R, Cianciaruso M and Adesso G 2015 *Phys. Rev. Lett.* **114** 210401

[40] Peng Y, Jiang Y and Fan H 2016 *Phys. Rev. A* **93** 032326

[41] Brandão G S L F and Gour G 2015 *Phys. Rev. Lett.* **115** 070503

[42] Liu Z W, Hu X and Lloyd S 2017 *Phys. Rev. Lett.* **118** 060502

[43] Hu M L, Shen S Q and Fan H 2017 *Phys. Rev. A* **96** 052309

[44] Hu M L, Hu X, Peng Y, Zhang Y R and Fan H 2017 arXiv:1703.01852 [quant-ph]

[45] Streltsov A, Adesso G and Plenio M B 2017 *Rev. Mod. Phys.* **89** 041003)

[46] Yang L W and Xia Y J 2016 *Chin. Phys. B* **25** 110303

[47] Gao D Y, Gao Q and Xia Y J 2017 *Chin. Phys. B* **26** 110303

[48] Liu Y, Zou H M and Fang M F 2018 *Chin. Phys. B* **27** 010304

[49] Yang L W and Xia Y J 2017 *Chin. Phys. B* **26** 080302

[50] Yang L W, Han W and Xia Y J 2018 *Chin. Phys. B* **27** 040302

[51] Singh U, Pati A K and Bera M N 2016 *Mathematics* **4** 47

[52] Yuan X, Bai G, Peng T and Ma X 2017 *Phys. Rev. A* **96** 032313

[53] Zhang F G and Li Y M 2018 *Sci. China-Phys. Mech. Astron.* **61** 080312

[54] Yuan X, Zhou H, Cao Z and Ma X 2015 *Phys. Rev. A* **92** 022124

[55] Xu Z Y, Yang W L and Feng M 2012 *Phys. Rev. A* **86** 012113

[56] Yao, C, Chen Z, Ma Z, Severini S and Serafini A 2014 *Sci. China-Phys. Mech. Astron.* **57** 1703

[57] Guo Y N, Tian Q L and Zeng K 2017 *Quantum Inf. Proc.* **16** 310

[58] Ming F, Wang D, Huang A J, Sun W Y and Ye L 2018 *Quantum Inf. Proc.* **17** 9