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Dynamical behaviors of traveling wave solutions to a Fujimoto–Watanabe equation*

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We study dynamical behaviors of traveling wave solutions to a Fujimoto–Watanabe equation using the method of dynamical systems. We obtain all possible bifurcations of phase portraits of the system in different regions of the three-dimensional parameter space. Then we show the required conditions to guarantee the existence of traveling wave solutions including solitary wave solutions, periodic wave solutions, kink-like (antikink-like) wave solutions, and compactons. Moreover, we present exact expressions and simulations of these traveling wave solutions. The dynamical behaviors of these new traveling wave solutions will greatly enrich the previews results and further help us understand the physical structures and analyze the propagation of nonlinear waves.

Keywords: dynamical behaviors, traveling wave solutions, Fujimoto–Watanabe equation, bifurcations

PACS: 02.30.Hq, 02.30.Oz, 05.45.Yv, 05.45.–a **DOI:** 10.1088/1674-1056/27/9/090201

1. Introduction

When Fujimoto and Watanabe[1] classified the third-order polynomial evolution equations of uniform rank with non-constant separants, which admit generalized symmetries (non-trivial Lie–Bäcklund algebras), they derived eight third-order differential equations and two fifth-order differential equations. Sakovich[2] showed that these differential equations can be connected with the famous KdV equation, Qiao equation,[3] and so on. Shi and Wen[4] studied the bifurcation and dynamics of traveling wave solutions to one of the Fujimoto–Watanabe equations. It is well known that traveling wave solution is an important type of solution to nonlinear wave equations. Many methods have been employed to find exact traveling wave solutions of nonlinear wave equations, such as, Jacobi elliptic function method,[5] F-expansion method,[6] (G'/G)-expansion method,[7] Bäcklund transformation method,[8] the simplified Hirota's method,[9] and so on. Our goal in this paper is to seek traveling wave solutions to the following Fujimoto–Watanabe equation:

$$u_t = u^3 u_{xxx} + \frac{3}{2}u^2 u_x u_{xx} + \alpha u^2 u_x, \tag{1}$$

where α is a parameter, and analyze their dynamical behaviors from the perspective of the theory of dynamical systems.[10–24]

More precisely, we look for the traveling wave solutions to Eq. (1) through the transformation $u(x,t) = \varphi(\xi)$ with $\xi = x - ct$, where c is the wave speed. Substituting it into Eq. (1) yields

$$-c\varphi' = \varphi^3\varphi''' + \frac{3}{2}\varphi^2\varphi'\varphi'' + \alpha\varphi^2\varphi', \tag{2}$$

where the prime stands for the derivative with respect to ξ.

Dividing both sides of Eq. (2) by φ^2 and integrating it once, we obtain

$$g + \frac{c}{\varphi} = \varphi\varphi'' + \frac{1}{4}(\varphi')^2 + \alpha\varphi, \tag{3}$$

where g is the integral constant.

Letting $y = \varphi'$, we obtain a three-parameter planar system

$$\begin{cases} \dfrac{\mathrm{d}\varphi}{\mathrm{d}\xi} = y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}\xi} = \dfrac{-y^2/4 - \alpha\varphi + g + c/\varphi}{\varphi}, \end{cases} \tag{4}$$

with first integral

$$H_1(\varphi, y) = \frac{\sqrt{\varphi}}{2}y^2 + \frac{2\alpha}{3\sqrt{\varphi}}\left(\varphi^2 - \frac{3g}{\alpha}\varphi + \frac{3c}{\alpha}\right), \quad \text{for } \varphi > 0, \tag{5}$$

$$H_2(\varphi, y) = \frac{\sqrt{-\varphi}}{2}y^2 - \frac{2\alpha}{3\sqrt{-\varphi}}\left(\varphi^2 - \frac{3g}{\alpha}\varphi + \frac{3c}{\alpha}\right), \quad \text{for } \varphi < 0. \tag{6}$$

The purpose of this paper is to investigate the dynamical behaviors of traveling wave solutions to system (4) in the three-dimensional parameter space (α, c, g) and to show traveling wave solutions to Eq. (1).

2. Bifurcation conditions and phase portraits

Note that system (4) is not well-defined on the line $l : \varphi = 0$. Therefore, we consider the following associated system of system (4):

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$$\begin{cases} \dfrac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \varphi^2 y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}\tau} = -\dfrac{1}{4}\varphi y^2 - \alpha\left(\varphi^2 - \dfrac{g}{\alpha}\varphi - \dfrac{c}{\alpha}\right), \end{cases} \tag{7}$$

where $\mathrm{d}\xi = \varphi^2 \mathrm{d}\tau$.

System (7) has the same level curves as system (4). Therefore, we can analyze the phase portraits of system (4) from those of system (7).

We suppose that $g > 0$. Let $\varphi_\pm = \frac{1}{2}\left(\frac{g}{\alpha} \pm \sqrt{\frac{g^2+4\alpha c}{\alpha^2}}\right)$, $\varphi_* = \frac{g}{\alpha}$, and $\varphi^* = \frac{g}{2\alpha}$. Now we summarize the singular points of system (7) and their relative positions when $\alpha > 0$ in the following lemma. We just omit the other case when $\alpha < 0$, since it can be analyzed similarly.

Lemma 1 When $\alpha > 0$, the singular points of system (7) are as follows.

(a) When $c < \frac{-g^2}{4\alpha}$, system (7) has no singular point.

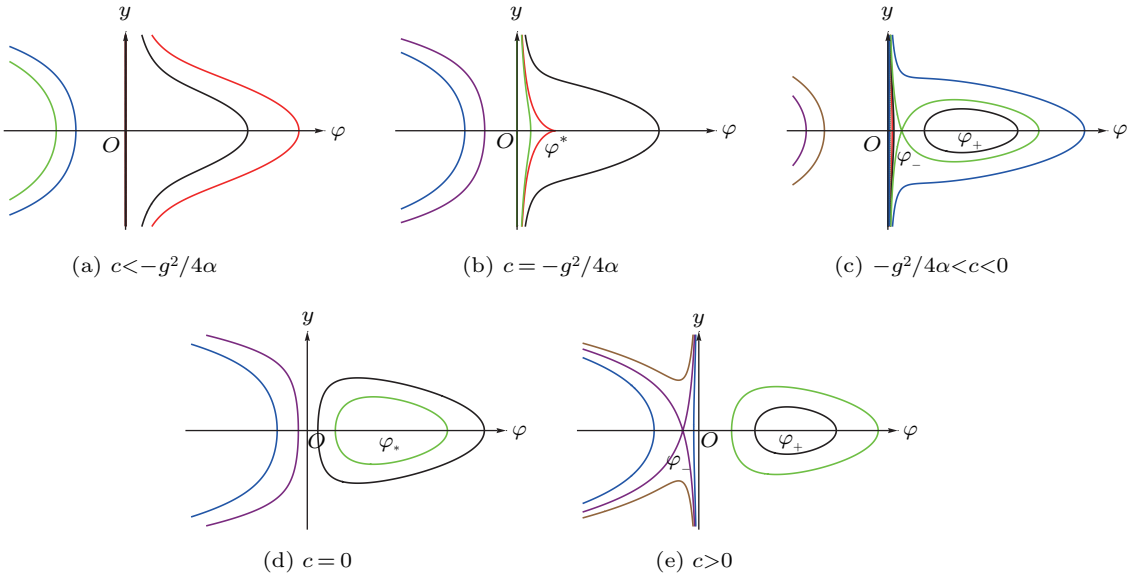
(b) When $c = \frac{-g^2}{4\alpha}$, system (7) has only one singular point $(\varphi^*, 0)$ with $\varphi^* > 0$.

(c) When $\frac{-g^2}{4\alpha} < c < 0$, system (7) has two singular points $(\varphi_-, 0)$ and $(\varphi_+, 0)$ with $0 < \varphi_- < \varphi_+$.

(d) When $c = 0$, system (7) has two singular points $(0, 0)$ and $(\varphi_*, 0)$ with $\varphi_* > 0$.

(e) When $c > 0$, system (7) has two singular points $(\varphi_-, 0)$ and $(\varphi_+, 0)$ with $\varphi_- < 0 < \varphi_+$.

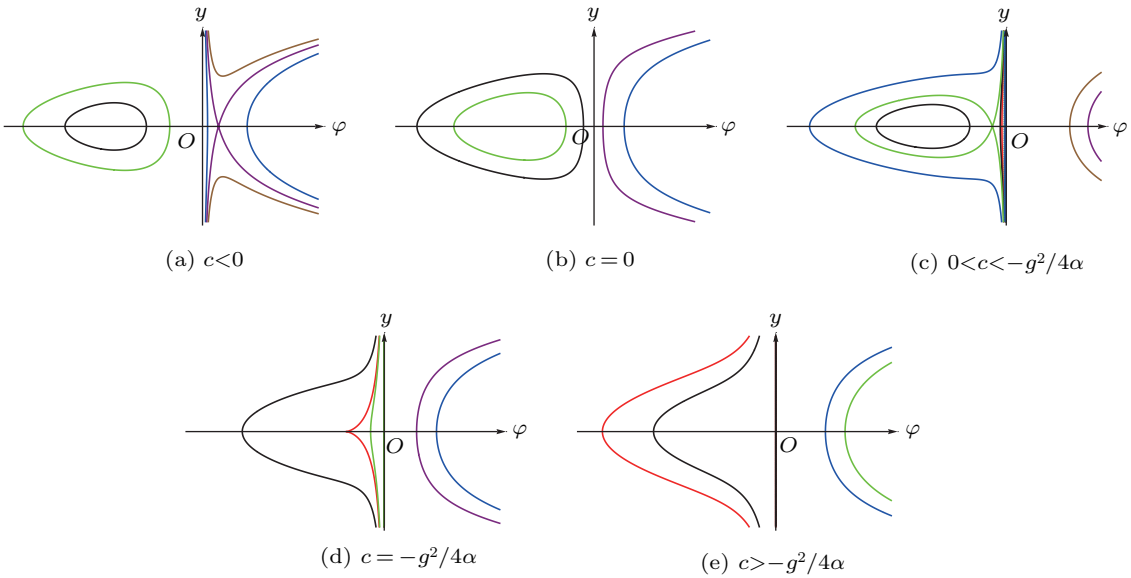
Through qualitative analysis, we obtain all possible bifurcations of phase portraits of system (4) as shown in Figs. 1 and 2.



(a) $c<-g^2/4\alpha$ (b) $c=-g^2/4\alpha$ (c) $-g^2/4\alpha<c<0$

(d) $c=0$ (e) $c>0$

Fig. 1. (color online) The phase portraits of system (4) when $\alpha > 0$.



(a) $c<0$ (b) $c=0$ (c) $0<c<-g^2/4\alpha$

(d) $c=-g^2/4\alpha$ (e) $c>-g^2/4\alpha$

Fig. 2. (color online) The phase portraits of system (4) when $\alpha < 0$.

Remark 1 In this paper, we have supposed that $g > 0$. In fact, when $g < 0$, the phase portraits of system (4) under the conditions $\alpha > 0$ and $\alpha < 0$ will be exactly the same as those shown in Figs. 2 and 1, respectively. Therefore, we only consider the case when $g > 0$.

3. Main results

To state conveniently, we introduce some marks $m = \sqrt{\frac{g}{2\alpha}}$, $a = \sqrt{\varphi_- - \frac{3c}{\alpha\varphi_-}} - \sqrt{\varphi_-}$, $b = \sqrt{\varphi_- - \frac{3c}{\alpha\varphi_-}} + \sqrt{\varphi_-}$, $\rho = (a - \sqrt{\varphi})(b + \sqrt{\varphi})$, $\sigma = (a - \sqrt{\varphi_-})(b + \sqrt{\varphi_-})$.

Our main results will be stated in the following theorem, and the proof follows. Note that we only focus our attention on the case when $\alpha > 0$, because the other case when $\alpha < 0$ can be considered similarly.

Theorem 1 When $\alpha > 0$, equation (1) possesses the following solitary wave solutions, periodic wave solutions, kink-like (antikink-like) wave solutions, and compactons under corresponding parameter conditions.

(i) When $\frac{-g^2}{4\alpha} < c < 0$, there exists a solitary wave solution to Eq. (1),

$$2\sqrt{\rho} - \frac{2\varphi_-}{\sqrt{\sigma}}\left[\ln(a+b) - \ln\left|\frac{2\sqrt{\rho\sigma} + 2\sigma}{\sqrt{\varphi} - \sqrt{\varphi_-}} + a - b - 2\sqrt{\varphi_-}\right|\right] = \sqrt{\frac{4\alpha}{3}}|\xi|. \tag{8}$$

(ii) When $c > \frac{-g^2}{4\alpha}$, there exist a family of periodic wave solutions to Eq. (1).

(iii) When $\frac{-g^2}{4\alpha} \le c < 0$ or $c > 0$, there exist kink-like (antikink-like) wave solutions to Eq. (1). Specifically, when $c = \frac{-g^2}{4\alpha}$, the kink-like (antikink-like) wave solutions possess the expression

$$\frac{-m\sqrt{4m^2 - (\sqrt{\varphi} + m)^2}}{m - \sqrt{\varphi}} + \frac{m\sqrt{4m^2 - (\sqrt{\varphi_0} + m)^2}}{m - \sqrt{\varphi_0}} + 2\sqrt{(m - \sqrt{\varphi})(\sqrt{\varphi} + 3m)} - 2\sqrt{(m - \sqrt{\varphi_0})(\sqrt{\varphi_0} + 3m)} = \sqrt{\frac{4\alpha}{3}}|\xi|, \tag{9}$$

where $\varphi_0 \in (0, \varphi^*)$.

(iv) When $c < \frac{-g^2}{4\alpha}$ or $c > 0$, there exist a family of compactons to Eq. (1). While $\frac{-g^2}{4\alpha} \le c < 0$, there exist two families of compactons to Eq. (1).

Proof (i) When $\frac{-g^2}{4\alpha} < c < 0$, from Fig. 1(c), we see that there is one homoclinic orbit for system (4). In (φ, y)-plane, the expressions of the homoclinic orbit are given as

$$y = \pm\sqrt{\frac{4\alpha}{3\varphi}}(\sqrt{\varphi} - \sqrt{\varphi_-})\sqrt{(a - \sqrt{\varphi})(b + \sqrt{\varphi})}, \quad \varphi_- < \varphi < a^2. \tag{10}$$

Substituting Eq. (10) into $\mathrm{d}\varphi/\mathrm{d}\xi = y$ and integrating them along the homoclinic orbit, it follows that

$$\int_{\varphi}^{a^2} \frac{\sqrt{s}\,\mathrm{d}s}{(\sqrt{s} - \sqrt{\varphi_-})\sqrt{(a - \sqrt{s})(b + \sqrt{s})}} = \sqrt{\frac{4\alpha}{3}}|\xi|. \tag{11}$$

From Eq. (11), we get the implicit expression of solitary wave solution (8), the graphic of which is shown in Fig. 3(a).

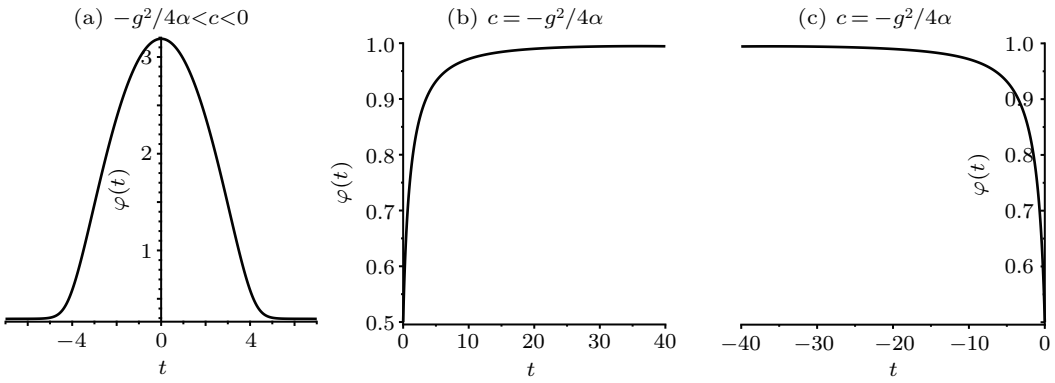


Fig. 3. The (a) solitary wave solution and (b), (c) kink-like (antikink-like) wave solutions to Eq. (1).

(ii) When $c > \frac{-g^2}{4\alpha}$, system (4) has a family of periodic orbits in Figs. 1(c), 1(d), and 1(e), which give rise to a family of periodic wave solutions to Eq. (1).

(iii) The stable and unstable manifolds, on the left side of singular point $(\varphi^*, 0)$ in Fig. 1(b), on the left side of singular point $(\varphi_-, 0)$ in Fig. 1(c), and on the right side of singular point $(\varphi_-, 0)$ in Fig. 1(e), respectively, give rise to kink-like (antikink-like) wave solutions to Eq. (1).[10,20,25–27] Here we illustrate the kink-like (antikink-like) wave solutions when $c = \frac{-g^2}{4\alpha}$, and omit the other cases.

The stable and unstable manifolds on the left side of singular point $(\varphi^*, 0)$ in Fig. 1(b) can be expressed as

$$y = \pm\sqrt{\frac{4\alpha}{3\varphi}}(m - \sqrt{\varphi})\sqrt{(m - \sqrt{\varphi})(\sqrt{\varphi} + 3m)}, \quad 0 < \varphi < \varphi^*. \tag{12}$$

Substituting Eq. (12) into $\mathrm{d}\varphi/\mathrm{d}\xi = y$ and integrating them along the stable and unstable manifolds, it follows that

$$\int_{\varphi_0}^{\varphi} \frac{\sqrt{s}\mathrm{d}s}{(m-\sqrt{s})\sqrt{(m-\sqrt{s})(\sqrt{s}+3m)}} = \sqrt{\frac{4\alpha}{3}}|\xi|. \quad (13)$$

From Eq. (13), we get the kink-like (antikink-like) wave solutions (9), the graphics of which are shown in Figs. 3(b) and 3(c).

(iv) We only illustrate the cases when $c < \frac{-g^2}{4\alpha}$ and $\frac{-g^2}{4\alpha} < c < 0$ in Figs. 1(a) and 1(c), since the other cases can be obtained similarly.

Corresponding to the one family of orbit, passing through the point $(\varphi_0, 0)$ with $\varphi_0 \in (0, +\infty)$ in Fig. 1(a), equation (1) has a family of compactons shown in Fig. 4(a).

Corresponding to the two family of orbits, passing through the point $(\varphi_0, 0)$ with $\varphi_0 \in (0, \varphi_-) \bigcup (a^2, +\infty)$ in Fig. 1(c), equation (1) has two families of compactons shown in Figs. 4(b) and 4(c).

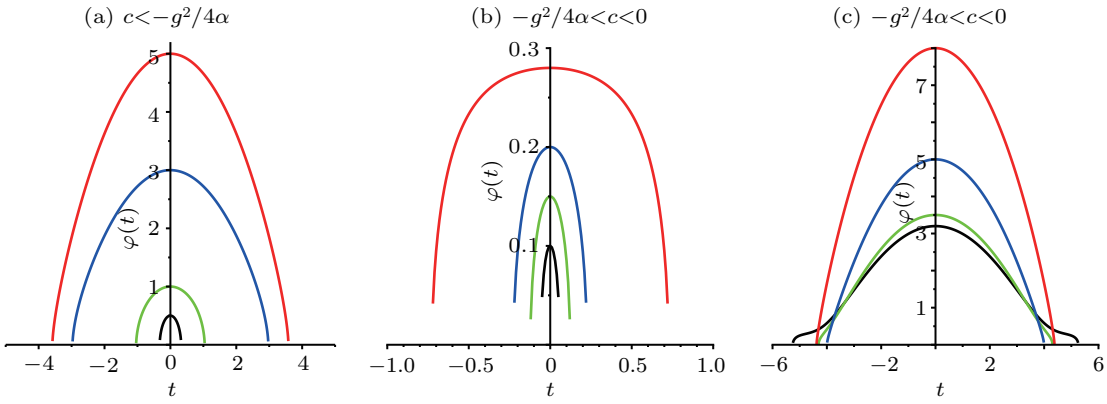


Fig. 4. (color online) The graphics of compactons to Eq. (1).

4. Conclusion

Through all possible bifurcations for the system in three-dimensional parameter space, we not only show the existence of traveling wave solutions including solitary wave solutions, periodic wave solutions, kink-like (antikink-like) wave solutions, and compactons, under corresponding parameters conditions, but also give their exact expressions and simulations.

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