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Topological classification of periodic orbits in Lorenz system*

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We systematically investigate the periodic orbits of the Lorenz flow up to certain topological length. As an alternative to Poincaré section map analysis, we propose a new approach for establishing one-dimensional symbolic dynamics based on the topological structure of the orbit. A newly designed variational method is stable numerically for cycle searching, and two orbital fragments can be used as basic building blocks for initialization. The topological classification based on the entire orbital structure is revealed to be effective. The deformation of periodic orbits with the change of parameters provides a chart to the periods of cycles. The current research may provide a methodology for finding and systematically classifying periodic orbits in other similar chaotic flows.

Keywords: Lorenz equations, periodic orbit, variational method, symbolic dynamics

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1. Introduction

The study of chaotic systems began with the first chaotic system, the Lorenz equations, in 1963.[1] As the first chaotic model, the Lorenz system is an important milestone in the history of chaos. Studying the Lorenz system has greatly promoted the development of chaos. To date, researchers have strictly proven the existence of the Lorenz attractor via Conley index theory and strict numerical calculation methods.[2–4]

Many computer-assisted numerical results for Lorenz flow have been presented in subsequent years. A high-precision method of calculating periodic orbits on the Lorenz attractor was discussed in Ref. [5], in which the establishment of symbolic dynamics based on the return map was exploited. In Ref. [6], the bound of the generalized Lorenz system was obtained, based on the Lyapunov function and the Lagrange multiplier method. The complex dynamical behavior of a fractional-order Lorenz-like system and its control were investigated in Ref. [7]. Global bifurcations of the Lorenz manifold were explored in Ref. [8], in which 512 generic heteroclinic orbits were found that are given as the intersection curves of two-dimensional (2D) manifolds. In Ref. [9], time-delayed feedback control was used as a method of stabilizing unstable periodic orbits in the Lorenz equations. Recently, a novel hidden chaotic attractor in the classical Lorenz system was revealed in Ref. [10]. Time series of orbits, Lyapunov exponents, and bifurcations of the hidden chaotic attractor have also also reported. In Ref. [11] topological phase transitions when varying the parameters were studied. For special parameter values, the Lorenz flow will be topologically equivalent to an Anosov flow on a knot complement, and different knots appear for different parameter values. Dynamical analysis of the hyperchaos Lorenz system was investigated in Ref. [12], where the ultimate bound on the trajectories based on Lyapunov stability theory was investigated. Sparrow's monograph provided a detailed discussion of the dynamic behavior of the Lorenz system.[13]

In chaotic systems, owing to the randomness of motion, individual events cannot be predicted with precision in the long run. However, due to the ergodicity of dynamical motion, the average behavior under many initial conditions is still predictable. Cycle expansion is a powerful tool for calculating averages of physical observables in nonlinear dynamical systems by the periodic orbit theory.[14,15] It was first applied to a spatially extended nonlinear system in 1997,[16] in which a multi-point shooting method was used for finding unstable periodic orbits up to certain topological length and yielded several global averages characterizing the chaotic dynamics. For the Lorenz system, periodic orbit theory will continue to play an important role. The key in the application of periodic orbit theory is to establish appropriate symbolic dynamics; otherwise, even when some cycles are found, we still do not know the number of them that should exist and how they are related to each other. Ignoring a short periodic orbit will also impact the accuracy of calculations. In this paper, we put forth a general scheme to locate systematically the cycles of the Lorenz flow, and to build the one-dimensional (1D) symbolic dynamics effectively based on the orbital topology in the phase space for the classification of all periodic orbits.

The rest of this paper is organized as follows. In Section 2, we review basic dynamical properties of the Lorenz

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system. The extraction of periodic orbits in dissipative chaotic flows requires forceful numerical algorithms, and in Section 3 we present the variational method and apply it at our location for unstable periodic orbits of the Lorenz flow. In Section 4, we discuss the establishment of one-dimensional symbolic dynamics, which plays a key role in the classification of all short cycles up to a certain topological length. The evolution of periodic orbits with the change of parameters is also discussed. The results in the paper are summarized in the last section.

2. Lorenz system dynamics

The Lorenz system consists of three ordinary differential equations, which are used to describe the thermal convective instability take the forms

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = a(y-x), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = cx - xz - y, \\ \dfrac{\mathrm{d}z}{\mathrm{d}t} = xy - bz, \end{cases} \tag{1}$$

where a, b, and c are three parameters of the equations, and xz and xy are the nonlinear terms. Since the system (1) has invariance under the transformation $(x,y,z) \to (-x,-y,z)$, the system is on the symmetry of the z axis. In addition, the Lorenz flow is dissipative and strongly contracts the phase-space volumes, by a factor of 10^{-4} per mean turnover time. The trajectories of the flow will eventually be limited to a set of points with a volume of zero, and its asymptotic dynamics behavior is fixed on an attractor.[17] When $c > 1$, the three fixed points of the flow are

$$\begin{cases} S_0 = (0,0,0), \\ S_+ = \left(\sqrt{b(c-1)}, \sqrt{b(c-1)}, c-1\right), \\ S_- = \left(-\sqrt{b(c-1)}, -\sqrt{b(c-1)}, c-1\right). \end{cases} \tag{2}$$

Many studies have been done on the bifurcation behavior of the Lorenz system. The Hopf bifurcation will appear when the parameter $c = 24.74$, and the two fixed points S_- and S_+ will become a strange attractor. At this moment the two equilibria comprise an unstable center, and when the phase space orbit approaches the equilibrium, it will be excluded, extending from the inside to the outside with the spiral. Moreover, due to the dissipative property of the system, the phase space must shrink as a whole. When the phase-space orbit expands to a certain extent, it will accordingly return to one of the fixed points and expand from the inside to outside around it. When it extends to a certain extent, it will suddenly randomly enter one of the fixed points, which causes the trajectories to hover between the two centers and never stop. Figure 1(a) shows the complex trajectory of the Lorenz system in two-dimensional (2D) phase space when we take the parameters for $a = 10$, $b = 8/3$, and $c = 28$.

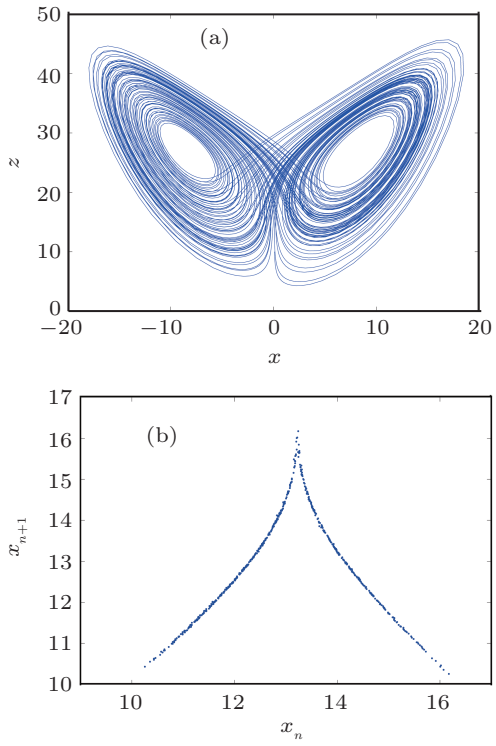


Fig. 1. (color online) (a) 2D projection of a trajectory in the Lorenz flow at time $t = 100$. (b) Return map for the Poincaré section $z = 27$ after reduction to $x > 0$ by virtue of inversion symmetry.

The stability matrix of system (1) under the parameters taken is

$$A = \begin{pmatrix} -10 & 10 & 0 \\ 28-z & -1 & -x \\ y & x & -8/3 \end{pmatrix}. \tag{3}$$

Together with the eigenvalues of the stability matrix, the equilibria yield quite detailed information about the flow. The eigenvalues are obtained from the stability matrix (3):

$$\begin{aligned} &S_0: \ (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}) = (11.83, -2.67, -22.83), \\ &S_-: \ (\mu^{(1)} \pm i\omega^{(1)}, \lambda^{(3)}) = (0.094 \pm 10.19i, -13.85). \end{aligned} \tag{4}$$

In the vicinity of the S_0 fixed point, the extremely strong contraction $\lambda^{(3)} = -22.83$ along the $e^{(3)}$ direction restricts the hyperbolic dynamics near S_0 to the plane spanned by the unstable eigenvector $e^{(1)}$ with $\lambda^{(1)} = 11.83$, and the slowest contraction rate eigenvector $e^{(2)}$ with $\lambda^{(2)} = -2.67$. In the plane, the strong expansion along the $e^{(1)}$ direction prevails over the slow $\lambda^{(2)} = -2.67$ contraction down the z axis, making it extremely unlikely for a random orbit to approach S_0. Therefore, trajectories close to S_0 are rare.

The S_- eigenvalues indicate that the rotation period is $T_{S_-} = 2\pi/\omega^{(1)} = 0.62$, and the turnover timescale is of order 1. The associated multipliers per spiral-out turn are $\Lambda^{(1)} = \exp(\mu^{(1)} T_{S_-}) = 1.06$ and $\Lambda^{(3)} = \exp(\lambda^{(3)} T_{S_-}) = 1.96 \times 10^{-4}$. In the S_- neighborhood, the unstable manifold trajectories

slowly spiral out with a very small radial per-turn expansion multiplier $\Lambda^{(1)} = 1.06$ and a very strong contraction multiplier $\Lambda^{(3)} = 10^{-4}$ onto the unstable manifold.[18] It is obvious that periodic orbits densely covering the strange attractor exist, and the Poincaré section technique can be used in cycle searching, which will probably reduce the 3D flow to a unimodal return map.

The Poincaré section is required since it cannot be tangential to the trajectories; thus, we choose the Poincaré section for a plane $z = 27$. Once the particular section has been selected, we can show the return map. Figure 1(b) displays the return map of the Lorenz system, and from the figure we can see that the return map is unimodal, which indicates the possibility for the establishment of 1D symbolic dynamics by finite Markov partitions,[19] which will help us to encode all the cycles. In order to calculate the unstable periodic orbits systematically, in the next section we introduce a powerful method for extracting them in a chaotic system.

3. Variational method for finding periodic orbits in general flow

The variational method was proposed by Lan in 2004 to explore the periodic orbit in high-dimensional chaotic systems.[20] This method not only preserves the robustness characteristics of most other methods, but it also has the characteristics of fast convergence when the search process is close to the real periodic orbit in practice. The physical idea of the method is that, first, a rough loop guess must be made based on the entire topology for the cycle to be searched, and then the initial guess loop must be driven to evolve toward the real periodic orbit by the variational algorithm. To achieve stability of the numerical method, the Newton descent method is used instead of the Newton–Raphson iteration method.

The variational method uses partial differential equation (5) to describe the evolution process of the initial guess loop toward the real cycle:

$$\frac{\partial^2 \tilde{x}}{\partial s \partial \tau} - \lambda A \frac{\partial \tilde{x}}{\partial \tau} - v \frac{\partial \lambda}{\partial \tau} = \lambda v - \tilde{v}, \tag{5}$$

where the dynamical flow vector field v is defined by $\dot{x} = v(x)$, $\tilde{v} = \mathrm{d}\tilde{x}/\mathrm{d}s$ is the vector tangent to the loop parameterized by $s \in [0, 2\pi]$. λ is a parameter used to control the cycle period T, fictitious time τ parametrizes continuous deformations of the loop, and $A_{ij} = \partial v_i / \partial x_j$ is the gradient matrix of the velocity field.

The important feature of Eq. (5) is that a minimizing cost functional exists monotonically as the loop guess evolves toward the periodic orbit:

$$F^2[(\tilde{x})] \equiv \frac{1}{2\pi} \oint_{L(\tau)} \mathrm{d}\tilde{x}[\tilde{v}(\tilde{x}) - \lambda v(\tilde{x})]^2. \tag{6}$$

For each iteration, the difference between the direction of loop velocity and flow velocity in the dynamical system decreases. The two become consistent when $\tau \to \infty$, so the real periodic orbit of the system is captured. The period of the orbit can be calculated by

$$T_p = \int_0^{2\pi} \lambda(\tilde{x}(s, \infty))\, \mathrm{d}s. \tag{7}$$

Numerical implementation of the variational method requires use of finite-difference methods to discretize the loop guess. In order to ensure the numerical stability, we accurately discretize the loop derivatives:

$$\tilde{v}_n \equiv \left.\frac{\partial \tilde{x}}{\partial s}\right|_{\tilde{x}=\tilde{x}(s_n)} \approx (\widehat{D}\tilde{x})_n. \tag{8}$$

A five-point approximation is used in the numerical work:

$$\widehat{D} = \frac{1}{12h} \begin{pmatrix} 0 & 8 & -1 & & & & & 1 & -8 \\ -8 & 0 & 8 & -1 & & & & & 1 \\ 1 & -8 & 0 & 8 & -1 & & & & \\ & & & & \cdots & & & & \\ & & & & & 1 & -8 & 0 & 8 & -1 \\ -1 & & & & & & 1 & -8 & 0 & 8 \\ 8 & -1 & & & & & & 1 & -8 & 0 \end{pmatrix}, \tag{9}$$

where $h = 2\pi/N$. Each entry in Eq. (9) represents a $d \times d$ matrix with blank spaces filled with zeros. The two $2d \times 2d$ matrices

$$M_1 = \begin{pmatrix} 1 & -8 \\ 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} -1 & 0 \\ 8 & -1 \end{pmatrix} \tag{10}$$

located at the top right and bottom left corners take care of the periodic boundary condition.

The discretized version of Eq. (5) with a fictitious time Euler step $\delta\tau$ is

$$\begin{pmatrix} \widehat{A} & -\hat{v} \\ \hat{a} & 0 \end{pmatrix} \begin{pmatrix} \delta\tilde{x} \\ \delta\lambda \end{pmatrix} = \delta\tau \begin{pmatrix} \lambda\hat{v} - \hat{\tilde{v}} \\ 0 \end{pmatrix}, \tag{11}$$

where

$$\widehat{A} = \widehat{D} - \lambda\, \mathrm{diag}[A_1, A_{2,\ldots,} A_N],$$
$$\hat{v} = (v_1, v_{2,\ldots,} v_N)^t, \text{ and } \hat{\tilde{v}} = (\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_N)^t$$

are the two vector fields that are to be matched everywhere along the loop. $\hat{a}$ is an Nd-dimensional row vector that imposes the constraint on the coordinate variations. In order to solve for the deformation of the loop coordinates $\delta\tilde{x}$ and period $\delta\lambda$, we invert the matrix on the left-hand side of Eq. (11) by the banded LU decomposition on the embedded band diagonal matrix, and we treat the cyclic and border terms by the Woodbury formula.[21]

The iteration of Eq. (11) can yield quickly and robustly the unstable periodic orbits for standard models of low-dimensional dissipative flows, which have an important feature, namely the existence of strange attractors. The method

can also be applied to detect cycles in Hamiltonian flows and for spatio-temporal periodic solutions of partial differential equations. The method is effective for locating periodic orbits in high-dimensional phase space, especially when the low-dimensionality of the strange invariant set is incorporated.

In the previous work, we used the variational method effectively in calculating the periodic orbits in low- and high-dimensional systems, such as the Kuramoto–Sivashinsky equation and its steady-state solutions,[22,23] the Rössler flow,[24] and the Rydberg atom in crossed electromagnetic fields.[25] It is obvious that this method is applicable to the Lorenz flow.

4. Applying the variational method to calculate periodic orbits of the Lorenz system

Here, we first discuss how to classify all the short periodic orbits of the Lorenz system when $a = 10$, $b = 8/3$, and $c = 28$ through the variational method, and then the evolution rule of the cycles with parameter changes is investigated. Two orbital segments can be used as basic building blocks, according to which we established 1D symbolic dynamics successfully to find all the periodic orbits up to certain topological length. The methodology is quite effective for this system and apparently applicable to other analogous dynamical systems, such as the Chen and Lü systems.[26,27] In the numerical examples that follow, the convergence condition is $F^2 < 10^{-6}$.

4.1. Initialization and symbolic dynamics

There are many ways to initialize the loop guess when applying the variational method. As in any other method, a qualitative understanding of the dynamics is a prerequisite for successfully locating periodic orbits. We initialize the loop guess by numerical integration with the dynamical system (1), which can identify the frequently visited regions of the phase space, giving us the first indication of how to initialize a loop. An initial loop guess is crafted by taking the fast Fourier transform of nearly close recurring orbital segments and keeping only the lowest-frequency components. We can obtain a smooth loop by taking the inverse Fourier transform back to the phase space, which can be used as the initial guess.

Utilizing homotopy evolution can also initialize the loop easily. If the dynamical system is related to a parameter, most short periodic orbits of the system may continue to exist when the parameter varies only slightly. Therefore, a periodic orbit of the previous parameter can be used as an initial guess for a nearby cycle surviving a small change. In practice, new periodic orbits can be obtained by only taking several iterations. In the following, we will introduce another initialization method to search for the periodic orbits of the Lorenz system.

In order to locate all the cycles of the Lorenz system up to a certain topological length, we can rely on a sequence of symbolic dynamics to help.[28] This sequence represents a unique classification in the dynamical system, which is very useful in coding the return map. The return map probably leads to approximate finite Markov partitions which divide the phase space of the dynamical system into nice little blocks that map into each other. Each block is labeled by a code, and the dynamics on the phase space maps the codes around the blocks, inducing the symbolic dynamics that encodes all the possible orbits and their topological layout. This enables us to locate unstable cycles in a systematic way. For the Lorenz system, the traditional way to establish symbolic dynamics is to use its 1D unimodal return map and systematically calculate the periodic orbits using the multi-point shooting method. The interval of this type of map is stretched and folded only once, with, at most, two points mapping into a point in the refolded interval; thus, the phase space of the system can be partitioned into various regions, labeling each one with its own unique symbol 0 or 1. As an alternative to the return-map analysis, here we propose a new method of establishing the 1D symbolic dynamics for the system by utilizing the topology of the orbit.

4.2. Topological classification of periodic orbits of the Lorenz system

Using the initialization approach as previously mentioned, we found several short periodic orbits with simple topological structures in the system after a few trials. Figure 2(a) exhibits the simplest periodic orbit found by the variational method. We initialized the guess loop by numerical integration with the dynamical system, keeping the nearly close orbital segments, and then manually connected it into a loop. Although the guessed orbit is not smooth enough, the variational method will make the loop evolution the true periodic orbit of the system, as shown in the figure. By observing the topological structure of the orbit obtained, we established the one-dimensional symbolic dynamics. Figure 2(b) shows the periodic orbit of Fig. 2(a) projected onto the x–z plane. We can mark the orbital fragment winding near the left equilibrium S_- with the symbol 0 and that winding near the right equilibrium S_+ with the symbol 1; therefore, the cycle in Fig. 2 is a 01 orbit. Not all conceivable symbol sequences are actually admissible as a dynamical trajectory according to the kneading theory.[18] We also devised the initial loop guess for a 0 orbit; however, it does not converge to a genuine periodic orbit of the

dynamical flow by the variational algorithm. For a 1 orbit, the same situation arises. Therefore, we did not find the presence of the 0 and 1 orbits, which means that they are pruned.

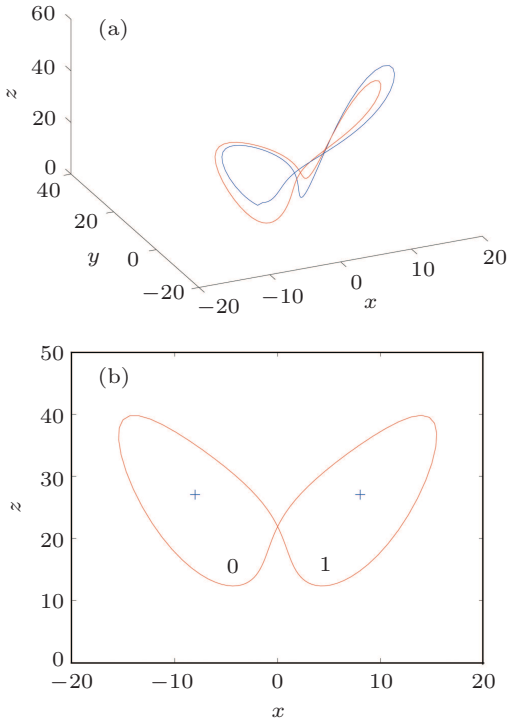


Fig. 2. (color online) (a) Shortest periodic orbit of Lorenz flow found by the variational method; blue line is the loop guess and red line is the periodic orbit. (b) 2D projection of the periodic orbit 01; the sequence of symbols corresponding to the orbit fragments is labeled. The two equilibria S_- and S_+ are marked with "+".

The two orbital fragments can be used as basic building blocks for locating other complicated cycles. Searching for complex cycles with multiple circuits around the two fixed points requires more delicate initial conditions when using the variational method; otherwise, it will probably lead to non-convergence. We can initialize the loop guess for longer cycles constructed by cutting and gluing the short, known ones. For most systems, such a method yields a quite good systematic initial guess for longer cycles. Even if we deform the orbit manually into a closed loop, the variational method still shows its strength for good convergence.

In this way, we can initialize the loop guess corresponding to a longer sequence of symbols utilizing 1D symbolic dynamics. Figure 3(a) shows a 011 orbit with a topological length of 3 that is made of two 1 orbital fragments and one 0 orbital fragment. Figure 3(b) shows a 0011 orbit with a topological length of 4 that is made up of two circles around the two respective fixed points. Figures 3(c) and 3(d) show the cycles with a topological length of 5. Aided by the 1D symbolic dynamics, we can systematically locate all periodic orbits up to a certain topological length by first constructing the initial loop based on the symbolic sequence, and then evolving the loop to a true cycle using the variational method to check whether the corresponding symbolic dynamics is pruned.

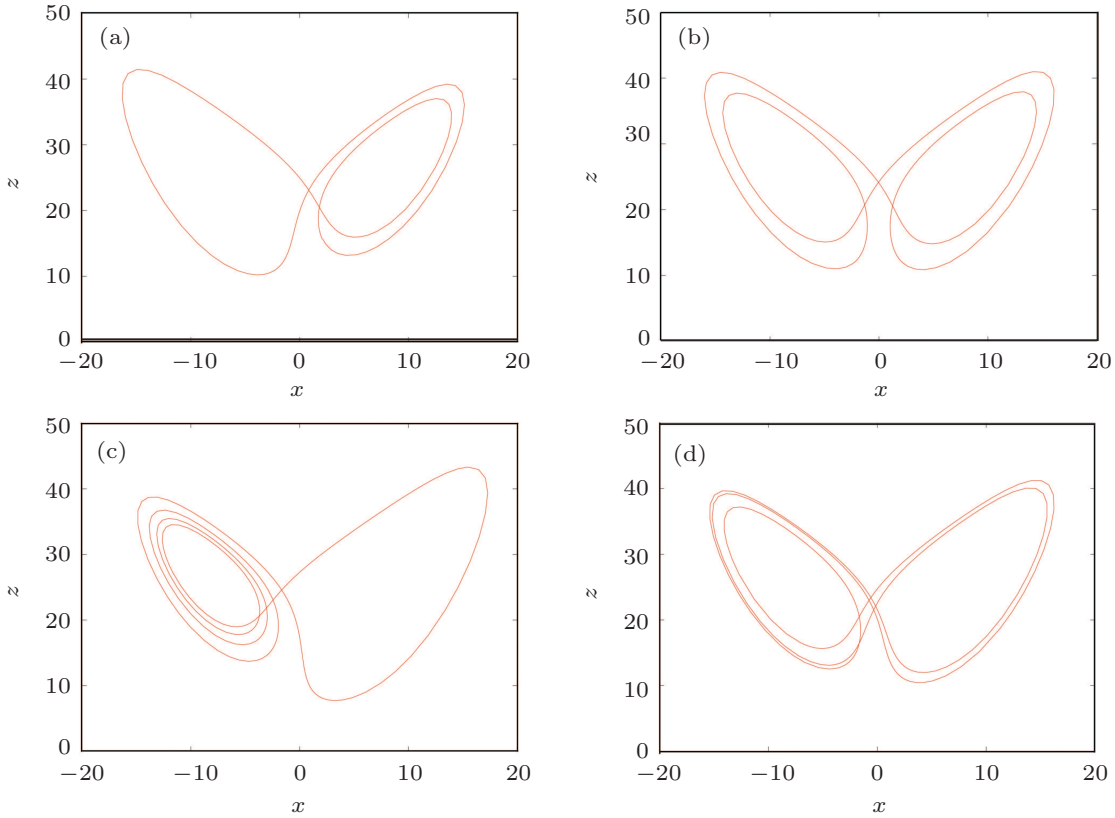


Fig. 3. (color online) Four periodic orbits of the Lorenz system: (a) 011 orbit, (b) 0011 orbit, (c) 00001 orbit, and (d) 00101 orbit.

Table 1. Cycles up to topological length 5 for the Lorenz flow. Listed are the topological lengths, the itineraries p, periods T_p, the largest Lyapunov exponents λ and the coordinates x, y, z of one point on the periodic orbit. The cycles pruned are marked "−".

Length	p	T_p	λ	x	y	z
1	0	–	–	–	–	–
	1	–	–	–	–	–
2	01	1.558615	0.995	−8.606282	0.256056	35.754806
3	001	2.305912	0.961	−2.131913	−3.131322	15.653809
	011	2.305912	0.961	2.131913	3.131322	15.653809
4	0001	3.023466	0.919	−0.107177	0.719410	19.296022
	0011	3.084255	0.969	4.676212	−0.312933	29.448208
	0111	3.023466	0.919	0.107177	−0.719410	19.296022
5	00001	3.725386	0.884	−0.663911	−4.837620	26.272703
	00011	3.820246	0.946	0.869511	2.619909	21.704587
	00101	3.869531	0.978	−5.803396	−9.394599	15.788917
	00111	3.820246	0.946	−0.869511	−2.619909	21.704587
	01011	3.869531	0.978	5.803396	9.394599	15.788917
	01111	3.725386	0.884	0.663911	4.837620	26.272703

We constructed a 1D symbolic dynamics hierarchy for the Lorenz flow in an exhaustive manner. Altogether, we found all 12 admissible orbits up to a topological length of 5 and list them in Table 1. The Lyapunov exponent of a periodic orbit is defined as $\ln|\Lambda|/T_p$, where Λ is the magnitude of its leading characteristic multiplier and T_p is its period. From the table we can see that the cycle 001 has the symmetry partner 011, which has the same period, and that the cycle 00101 has the symmetry partner 01011, while the cycle 0011 is conjugate to itself. This is determined by the z-axis symmetry of the system.

4.3. Evolution of periodic orbits with a change of parameters

Here, we investigate how the periodic orbits evolve upon changing the values of parameters a, b, and c. We first explore the 01 periodic orbit, which deforms with increasing a value. In our calculations, the 01 cycle calculated previously can be used as a initialization loop guess for the next a value; thus, we are able to continuously deform the 01 cycle. Figure 4(a) shows the deformation of the 01 orbit with four different a values and Table 2 lists the evolution. We find that the period of the 01 orbit decreases gradually with increasing a value.

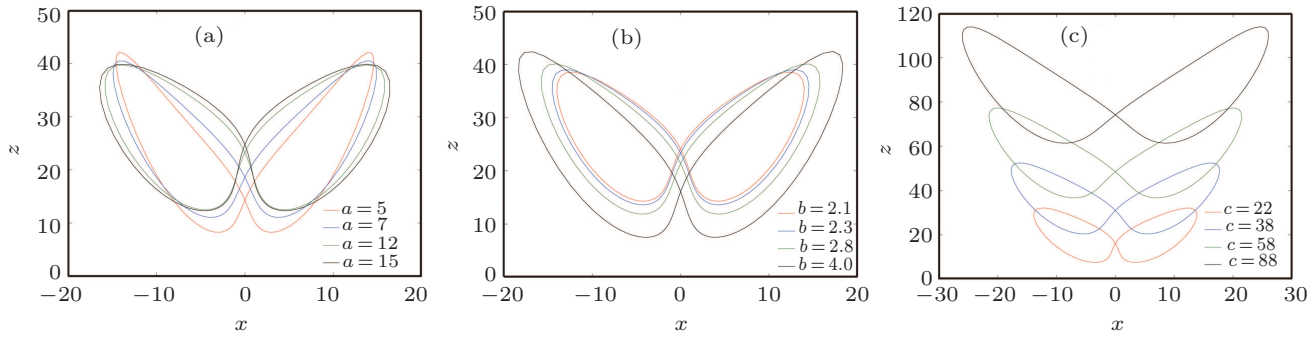


Fig. 4. (color online) (a) Deformation of cycle 01 with four different a values. (b) Four different b values. (c) Four different c values.

We also studied the 01 orbit which deforms with increasing b and c values separately. Figure 4(b) shows the deformation with four different b values, while figure 4(c) shows that of the 01 orbit with four different c values. We find that the period of the 01 orbit also decreases gradually with increasing b and c values as listed in Table 2.

Table 2. The periods T_p of the 01 orbit at different a, b, and c values.

a	T_p	b	T_p	c	T_p
5	1.910341	2.1	1.685825	22	1.877405
7	1.667089	2.3	1.632464	38	1.275800
12	1.533349	2.8	1.538086	58	0.996013
15	1.524488	4.0	1.474188	88	0.793060

5. Conclusions and discussion

In this paper, we proposed a topological classification of the periodic orbits in the Lorenz system. We established 1D symbolic dynamics successfully and employed the variational method for cycle searching due to its numerical stability. Two basic building blocks are used to initialize the loop guess, and we calculated the long cycles by cutting and gluing the short, known blocks according to the symbolic sequence, which can locate all the cycles up to a certain topological length in the system. We also discussed the deformation of the 01 orbit when parameters undergo changes, and the evolution rule of orbital period with parameter changes was obtained.

The Chen and Lorenz systems have similar but different topological structure. They are both 3D continuous dynamical systems with two nonlinear terms, and they cannot transform one to the other through topological transformation. The two systems are dual to each other, and the Chen system has more complex topological structure and more dynamical behavior than the Lorenz system. Thus, how to classify the periodic orbits in the Chen system is an open problem. As demon-

strated in the present paper, the classification method combining topology and the variational method used here could be a promising candidate for such classification.

The Lü system is a bridge between the Lorenz and Chen systems that realizes the transition from one system to the other. The above three systems can be expressed as a unified chaotic system. There are still many unknown problems in the Lü system, such as the organization of its periodic orbits and the existence of connecting orbits. As with the Lorenz flow, we can track the deformation of the orbit continuously in the Chen and Lü system by utilizing the homotopy evolution, which can help in the analysis of the existing range of the orbits under a change of parameters. Thus, a variety of bifurcation properties merit further study.

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