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Correlation method estimation of the modulation signal in the weak equivalence principle test*

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In a test of the weak equivalence principle (WEP) with a rotating torsion pendulum, it is important to estimate the amplitude of the modulation signal with high precision. We use a torsional filter to remove the free oscillation signal and employ the correlation method to estimate the amplitude of the modulation signal. The data analysis of an experiment shows that the uncertainties of amplitude components of the modulation signal obtained by the correlation method are in agreement with those due to white noise. The power spectral density of the modulation signal obtained by the correlation method is about one order higher than the thermal noise limit. It indicates that the correlation method is an effective way to estimate the amplitude of the modulation signal and it is instructive to conduct a high-accuracy WEP test.

Keywords: weak equivalence principle test, correlation method, amplitude estimation, thermal noise limit

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1. Introduction

The Einstein equivalence principle states that there is no difference between a uniformly accelerated reference frame and a homogeneous gravitational field.[1,2] Since Einstein derived his theory of general relativity from the equivalence principle, the weak equivalence principle (WEP) is the basis of Einstein's general theory of relativity.[3,4] Many theories and models attempting to combine the gravitational forces with the other three kinds of interactions require that the WEP is violative,[5,6] such as superstring theory[7–9] and supersymmetry theory.[10,11] Therefore, the theoretical research and experimental test of the equivalence principle has always been a topic of immense interest and a challenging problem for the scientific world.

The earliest WEP test was carried out by Newton using a simple pendulum with an accuracy of the test to Eötvös parameter $\eta \sim 10^{-3}$.[12] Around 1900, Eötvös used the torsion pendulum to test the WEP and achieved $\eta < 10^{-9}$.[13] To overcome the lack of signal modulation, Roll *et al.*[14] and Braginsky *et al.*[15] improved Eötvös experiment in 1964 and 1972, respectively. They replaced the Earth by the Sun as the source of gravity, transforming the WEP violation signal into a sinusoidal signal with a 24-hour period owing to the Earth's rotation. As a result, the accuracy reached 10^{-11} and 10^{-12} levels, individually. Later, the Eöt-Wash group[16–18] raised the period of the modulation signal from 24 h to 1 h and later to 20 min using a rotating torsion pendulum, and they did not find the violative signal of the order of 10^{-13}. In addition, with the rapid development of the atom-interference technique, more and more investigations[19,20] of the possible WEP test with the atoms were performed, which have reached a level of 10^{-8} for the dual species atom interferometer, and 10^{-7} for the different spin-rotation atom interferometer. In this study, we focus on analyzing the modulation signal in the WEP test with a torsion pendulum.

In the WEP test, the modulation signal, which is a weak signal, would be disturbed by the free torsional oscillation and the noise. The correlation method, also called the phase method, with its ability to suppress the influence of higher harmonics, and its insensitivity to the linear drift and weak damping, is one of the most efficient ways for weak signal analysis.[21,22] In this study, we analyze the effect of free torsional oscillation and eliminate the effect by torsional filter.[2] We use the correlation method to estimate the amplitude of the modulation signal, and then we acquire its uncertainties based on the white noise model and thermal noise model, respectively. In addition, a corresponding simulation is performed to verify the correctness of the theoretical derivation of the correlation method. An analysis of a set of data from a preliminary experiment of the WEP test with a rotating pendulum

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conducted by a group in Huazhong University of Science and Technology (HUST) shows that the uncertainties of the amplitude components due to the white noise are in agreement with the uncertainties obtained by the correlation method; further, the power spectral density of the modulation signal is about one order more than the thermal noise limit. The results show that it is reasonable to use the correlation method for estimating the amplitude of the modulation signal and it is significant for the test of the high-accuracy WEP.

2. Equation of motion

In the test of the WEP with the rotating pendulum, the turntable is rotated with a constant angular speed, ω. The equation of motion of the pendulum in the rotating frame can be expressed as[2]

$$I\ddot{\theta}+k\theta+2I\gamma\dot{\theta}=\tau, \tag{1}$$

where I is the moment of inertia of the pendulum about the suspension fiber, k is the torsional constant of the fiber, γ is the damping coefficient, and $\tau=\tau_n\sin\omega_n t$ is the external torque on the pendulum, in which τ_n denotes the amplitude of the external torque, $\omega_n=n\omega$ and n is an integer.

Setting $\theta(0)=\theta_0$ and $\dot{\theta}(0)=v_0$, the solution of Eq. (1) can be expressed as

$$\theta(t)=A\,\mathrm{e}^{-\gamma t}\sin(\omega_{\mathrm{d}}t+\varphi_{\mathrm{d}})+B_n\sin(\omega_n t-\varphi_n), \tag{2}$$

where

$$\begin{cases} A=\dfrac{(\theta_0+B_n\sin\varphi_n)}{\sin\varphi_{\mathrm{d}}}, \quad \omega_{\mathrm{d}}=\sqrt{\omega_0^2-\gamma^2}, \\ \varphi_{\mathrm{d}}=\arctan\dfrac{\omega_{\mathrm{d}}(\theta_0+B_n\sin\varphi_n)}{v_0-B_n\omega_n\cos\varphi_n+\gamma\theta_0+\gamma B_n\sin\varphi_n}, \\ B_n=\dfrac{\tau_n}{I\sqrt{(\omega_0^2-\omega_n^2)^2+(2\gamma\omega_n)^2}}, \\ \omega_0=\sqrt{\dfrac{k}{I}}, \quad \varphi_n=\arctan\dfrac{2\gamma\omega_n}{\omega_0^2-\omega_n^2}. \end{cases}$$

The first term on the right of the equality sign in Eq. (2) is the free torsional oscillation $\theta_{\mathrm{d}}(t)$. The second term is the modulation signal $\theta_{\mathrm{t}}(t)$. We need to consider the fundamental frequency harmonic component of $\theta_{\mathrm{t}}(t)$, since fundamental frequency is the same as the excepted violative signal.[1,2,23] In other words, we take $n=1$. Then we have

$$\begin{aligned}\theta(t) &= \theta_{\mathrm{d}}(t)+\theta_{\mathrm{t}}(t)+\varepsilon(t) \\ &= A\,\mathrm{e}^{-\gamma t}\sin(\omega_{\mathrm{d}}t+\varphi_{\mathrm{d}})+B_1\sin(\omega_1 t+\varphi_1)+\varepsilon(t),\end{aligned} \tag{3}$$

where B_1 and ω_1 denote the amplitude and angular frequency of the modulation signal, respectively, and $\varepsilon(t)$ is the random noise.

3. The correlation method

By expansion of the trigonometric function, equation (3) can be rewritten as

$$\begin{aligned}\theta(t) &= \mathrm{e}^{-\gamma t}[a_{\mathrm{d}}\cos(\omega_{\mathrm{d}}t)+b_{\mathrm{d}}\sin(\omega_{\mathrm{d}}t)] \\ &\quad +[a_1\cos(\omega_1 t)+b_1\sin(\omega_1 t)]+\varepsilon(t),\end{aligned} \tag{4}$$

where $A=\sqrt{a_{\mathrm{d}}^2+b_{\mathrm{d}}^2}$, $B_1=\sqrt{a_1^2+b_1^2}$, $\varphi_{\mathrm{d}}=\arctan(b_{\mathrm{d}}/a_{\mathrm{d}})$, $\varphi_1=\arctan(b_1/a_1)$.

To obtain the amplitude B_1 by the correlation method, the free torsional oscillation $\theta_{\mathrm{d}}(t)$ needs to be removed by a torsional filter. The torsional filter is used to subtract pairs of data points with frequency ω_{d} and separated by 2π phase. Due to the damping coefficient γ, we need to repeat the torsional filter numerous times. The filtered signal $\tilde{\theta}(t)$ is

$$\begin{aligned}\tilde{\theta}(t) &= \tilde{\theta}_{\mathrm{t}}(t)+\tilde{\varepsilon}(t)+\vartheta(t) \\ &= \tilde{a}_1\cos(\omega_1 t)+\tilde{b_1}\sin(\omega_1 t)+\tilde{\varepsilon}(t)+\vartheta(t),\end{aligned} \tag{5}$$

where $\tilde{\theta}_{\mathrm{t}}(t)$ denotes the filtered modulation signal, $\tilde{\varepsilon}(t)$ denotes the filtered noise, and $\vartheta(t)$ denotes the residual term of $\theta_{\mathrm{d}}(t)$ after the torsional filter. This term will be discussed in detail in the next section. The filtered amplitude components $(\tilde{a}_1,\tilde{b}_1)$ of the filtered modulation signal $\tilde{\theta}_{\mathrm{t}}(t)$ are

$$\begin{cases} \tilde{a}_1=(2-2\cos\omega_1 T_{\mathrm{d}})^{m/2}(a_1\cos m\varphi+b_1\sin m\varphi) \\ \tilde{b}_1=(2-2\cos\omega_1 T_{\mathrm{d}})^{m/2}(b_1\cos m\varphi-a_1\sin m\varphi) \end{cases}, \tag{6}$$

where $\varphi=\arctan[\sin\omega_1 T_{\mathrm{d}}/(1-\cos\omega_1 T_{\mathrm{d}})]$, $T_{\mathrm{d}}=2\pi/\omega_{\mathrm{d}}$ which is the period of free torsional oscillation $\theta_{\mathrm{d}}(t)$, and m denotes the number of times the torsional filter is employed.

From Eq. (6), we can get the relation between the amplitude components (a_1,b_1) of the modulation signal $\theta_{\mathrm{t}}(t)$ and the amplitude components $(\tilde{a}_1,\tilde{b}_1)$ of the filtered modulation signal $\tilde{\theta}_{\mathrm{t}}(t)$. The relation can be expressed as

$$\begin{bmatrix}\tilde{a}_1\\ \tilde{b}_1\end{bmatrix}=(2-2\cos\omega_1 T_{\mathrm{d}})^{m/2}\begin{bmatrix}\cos m\varphi & \sin m\varphi\\ -\sin m\varphi & \cos m\varphi\end{bmatrix}\begin{bmatrix}a_1\\ b_1\end{bmatrix}. \tag{7}$$

After the filtered modulation signal $\tilde{\theta}_{\mathrm{t}}(t)$ is obtained as in Eq. (5), we divide the $\tilde{\theta}_{\mathrm{t}}(t)$ sequence into some smaller components $\tilde{\theta}_{\mathrm{ti}}(t)$. Each small component contains one period $T_1=2\pi/\omega_1$ of the modulation signal $\theta_{\mathrm{t}}(t)$. The amplitude components $(\tilde{a}_{1i},\tilde{b}_{1i})$ of the filtered modulation signal $\tilde{\theta}_{\mathrm{t}}(t)$ in the i-th small component are calculated using the following equation.

$$\begin{aligned}\tilde{a}_{1i}&=\frac{2}{T_1}\int_{iT_1}^{(i+1)T_1}\tilde{\theta}_{\mathrm{ti}}(t)\cos(\omega_1 t)\mathrm{d}t, \\ \tilde{b}_{1i}&=\frac{2}{T_1}\int_{iT_1}^{(i+1)T_1}\tilde{\theta}_{\mathrm{ti}}(t)\sin(\omega_1 t)\mathrm{d}t.\end{aligned} \tag{8}$$

In a similar manner, the amplitude components $(\tilde{a}_{1i},\tilde{b}_{1i})$ of the filtered modulation signal $\tilde{\theta}_{\mathrm{t}}(t)$ and the amplitude components

(a_{1i}, b_{1i}) of the modulation signal $\theta_t(t)$ follow the relation expressed in Eq. (7).

$$\begin{bmatrix} \tilde{a}_{1i} \\ \tilde{b}_{1i} \end{bmatrix} = (2-2\cos\omega_n T_d)^{m/2} \begin{bmatrix} \cos m\varphi & \sin m\varphi \\ -\sin m\varphi & \cos m\varphi \end{bmatrix} \begin{bmatrix} a_{1i} \\ b_{1i} \end{bmatrix}. \quad (9)$$

We can take the mean value of (a_{1i}, b_{1i}) as the amplitude components (a_1, b_1)

$$a_1 = \frac{\sum_{i=1}^{N} a_{1i}}{N}, \quad b_1 = \frac{\sum_{i=1}^{N} b_{1i}}{N}, \quad (10)$$

where N is the number of T_1 periods during the whole measurement.

The standard deviation of the amplitude components (a_1, b_1) of the modulation signal $\theta_t(t)$ can be obtained from the following expressions.

$$\delta(a_1) = \sqrt{\frac{\sum_{i=1}^{N} (a_i - a_1)^2}{N-1}}, \quad \delta(b_1) = \sqrt{\frac{\sum_{i=1}^{N} (b_i - b_1)^2}{N-1}}. \quad (11)$$

We can get the corresponding uncertainties of the amplitude components (a_1, b_1) from the expressions given below.

$$u(a_1) = \frac{\delta(a_1)}{\sqrt{N}}, \quad u(b_1) = \frac{\delta(b_1)}{\sqrt{N}}. \quad (12)$$

The amplitude of the modulation signal B_1 and its uncertainty can be expressed as follows.

$$B_1 = \sqrt{a_1^2 + b_1^2}, \quad u(B_1) = \sqrt{\left[\frac{a_1}{B_1} u(a_1)\right]^2 + \left[\frac{b_1}{B_1} u(b_1)\right]^2}. \quad (13)$$

We can get the corresponding power spectral density of the modulation signal from the following equation.

$$S(B_1) = u(B_1) \cdot \sqrt{NT_1}, \quad (14)$$

where N is the number of T_1 periods during the whole measurement.

4. The effect of $\vartheta(t)$ on amplitude estimation of modulation signal

Due to the damping coefficient γ, the torsional filter cannot remove the free torsional oscillation $\theta_d(t)$ completely. We can reduce the effect of residual term $\vartheta(t)$ by repeating the torsional filter m times.

The residual term, $\vartheta(t)$, after employing torsional filter m times in the i-th small component can be expressed as

$$\vartheta_i(t) = (1 - e^{-\gamma T_d})^m e^{-\gamma t} (a_d \cos\omega_d t + b_d \sin\omega_d t). \quad (15)$$

Substituting Eq. (15) into Eq. (8), the effect of $\vartheta_i(t)$ on the amplitude components $(\tilde{a}_{1i}, \tilde{b}_{1i})$ of the filtered modulation signal $\tilde{\theta}_t(t)$ in the i-th small component can be calculated using the following expressions.

$$\Delta\tilde{a}_{1j} = \frac{2}{T_1} \int_{jT_1}^{(j+1)T_1} \vartheta_i(t) \cos(\omega_1 t) dt,$$
$$\Delta\tilde{b}_{1j} = \frac{2}{T_1} \int_{jT_1}^{(j+1)T_1} \vartheta_i(t) \sin(\omega_1 t) dt. \quad (16)$$

The effects of $\vartheta_i(t)$ in the i-th small component are given as

$$\Delta\tilde{a}_{1i} = \frac{2}{T_1}(1 - e^{-\gamma T_d})^m \left\{ \frac{a_d}{2} \left\{ \frac{1}{\sqrt{\gamma^2 + (\omega_d + \omega_1)^2}} \left\{ \begin{array}{l} -e^{-i\gamma T_1} \sin(i\omega_d T_1 + \phi_1) \\ +e^{-(i+1)\gamma T_1} \sin[(i+1)\omega_d T_1 + \phi_1] \end{array} \right. \right. \right.$$
$$\left. \left. + \frac{1}{\sqrt{\gamma^2 + (\omega_d - \omega_1)^2}} \left\{ \begin{array}{l} -e^{-i\gamma T_1} \sin(i\omega_d T_1 + \phi_2) \\ +e^{-(i+1)\gamma T_1} \sin[(i+1)\omega_d T_1 + \phi_2] \end{array} \right. \right.$$
$$\left. + \frac{b_d}{2} \left\{ \frac{1}{\sqrt{\gamma^2 + (\omega_d + \omega_1)^2}} \left\{ \begin{array}{l} e^{-i\gamma T_1} \cos(i\omega_d T_1 + \phi_1) \\ -e^{-(i+1)\gamma T_1} \cos[(i+1)\omega_d T_1 + \phi_1] \end{array} \right. \right.$$
$$\left. \left. + \frac{1}{\sqrt{\gamma^2 + (\omega_d - \omega_1)^2}} \left\{ \begin{array}{l} e^{-i\gamma T_1} \cos(i\omega_d T_1 + \phi_2) \\ -e^{-(i+1)\gamma T_1} \cos[(i+1)\omega_d T_1 + \phi_2], \end{array} \right. \right. \right. \quad (17)$$

$$\Delta\tilde{b}_{1i} = \frac{2}{T_1}(1 - e^{-\gamma T_d})^m \left\{ \frac{a_d}{2} \left\{ \frac{1}{\sqrt{\gamma^2 + (\omega_d + \omega_1)^2}} \left\{ \begin{array}{l} e^{-i\gamma T_1} \cos(i\omega_d T_1 + \phi_1) \\ -e^{-(i+1)\gamma T_1} \cos[(i+1)\omega_d T_1 + \phi_1] \end{array} \right. \right. \right.$$
$$\left. \left. + \frac{1}{\sqrt{\gamma^2 + (\omega_d - \omega_1)^2}} \left\{ \begin{array}{l} -e^{-i\gamma T_1} \cos(i\omega_d T_1 + \phi_2) \\ +e^{-(i+1)\gamma T_1} \cos[(i+1)\omega_d T_1 + \phi_2] \end{array} \right. \right.$$
$$\left. + \frac{b_d}{2} \left\{ \frac{1}{\sqrt{\gamma^2 + (\omega_d + \omega_1)^2}} \left\{ \begin{array}{l} e^{-i\gamma T_1} \sin(i\omega_d T_1 + \phi_1) \\ -e^{-(i+1)\gamma T_1} \sin[(i+1)\omega_d T_1 + \phi_1] \end{array} \right. \right.$$
$$\left. \left. + \frac{1}{\sqrt{\gamma^2 + (\omega_d - \omega_1)^2}} \left\{ \begin{array}{l} -e^{-i\gamma T_1} \sin(i\omega_d T_1 + \phi_2) \\ +e^{-(i+1)\gamma T_1} \sin[(i+1)\omega_d T_1 + \phi_2], \end{array} \right. \right. \right. \quad (18)$$

where $\phi_1 = \arctan[-\gamma/(\omega_d + \omega_1)]$, $\phi_2 = \arctan[-\gamma/(\omega_d - \omega_1)]$. This effect would have a correction according to Eq. (9), and then the effect $(\Delta a_{1i}, \Delta b_{1i})$ of $\vartheta_i(t)$ on the amplitude components (a_{1i}, b_{1i}) of modulation signal $\theta_t(t)$ can be obtained using the expressions below

$$\begin{bmatrix} \Delta a_{1i} \\ \Delta b_{1i} \end{bmatrix} = (2 - 2\cos\omega_1 T_d)^{-m/2} \begin{bmatrix} \cos m\varphi & \sin m\varphi \\ -\sin m\varphi & \cos m\varphi \end{bmatrix}^{-1} \times \begin{bmatrix} \Delta\tilde{a}_{1i} \\ \Delta\tilde{b}_{1i} \end{bmatrix}. \tag{19}$$

From Eq. (16) and Eq. (17), it is clear that as the number of times the torsional filter is employed increases, the effect of $\vartheta_i(t)$ decreases.

5. Uncertainty of amplitude estimation based on white noise model

In general, the noise $\varepsilon(t)$ can be modeled as the white noise, which satisfies the statistical properties given below[24]

$$\langle \varepsilon(t) \rangle = 0, \quad \langle \varepsilon(t_1)\varepsilon(t_2) \rangle = \sigma^2 \delta(t_1 - t_2), \tag{20}$$

where $\langle\ \rangle$ is the expectation operator representing population mean and σ is the standard deviation of the filtered noise. The power spectral density of the filtered noise can be obtained from $\varepsilon(\omega) = \sigma \cdot \Delta t^{1/2}$,[25] where Δt is the sampling interval. From Eq. (8), we can calculate the standard deviation $(\delta a_{1i}, \delta b_{1i})$ of the amplitude components (a_{1i}, b_{1i}) of the filtered modulation signal for the i-th small component using the following equations.

$$\begin{cases} (\delta a_{1i})^2 = \left\langle \dfrac{2}{T_1} \displaystyle\int_{iT_d}^{(i+1)T_d} \varepsilon(t_1)\cos(\omega_1 t_1)\mathrm{d}t_1 \cdot \dfrac{2}{T_1} \int_{iT_d}^{(i+1)T_d} \varepsilon(t_2)\cos(\omega_1 t_2)\mathrm{d}t_2 \right\rangle, \\ (\delta b_{1i})^2 = \left\langle \dfrac{2}{T_1} \displaystyle\int_{iT_d}^{(i+1)T_d} \varepsilon(t_1)\sin(\omega_1 t_1)\mathrm{d}t_1 \cdot \dfrac{2}{T_1} \int_{iT_d}^{(i+1)T_d} \varepsilon(t_2)\sin(\omega_1 t_2)\mathrm{d}t_2 \right\rangle. \end{cases} \tag{21}$$

Combining Eq. (20) with Eq. (21), the results are given as follows:[26]

$$\delta_{wh}^2 = (\delta a_{1i})^2 = (\delta b_{1i})^2 = \frac{2\sigma^2 \cdot \Delta t}{T_1} = \frac{2\varepsilon^2(\omega)}{T_1}, \tag{22}$$

where δ_{wh} represents the standard deviation of amplitude components (a_1, b_1) caused by the white noise and the $\varepsilon(\omega)$ is the power spectral density of the noise. Finally, the uncertainties of the amplitude components (a_1, b_1) caused by white noise can be expressed as

$$u_{wh}(a_1) = u_{wh}(b_1) = \frac{\delta_{wh}}{\sqrt{N}}, \tag{23}$$

where N is the number of T_1 periods during the whole measurement.

6. The thermal noise limit in the test of WEP

Thermal noise originates from Brownian motion, where the white noise model is usually used,[27,28] and is one of the most fundamental limits to the sensitivity in the test of the WEP with a rotating torsional pendulum. $T_{th}(t)$ is the thermal torque on the torsion pendulum. It satisfies the statistical properties given below[28]

$$\langle T_{th}(t) \rangle = 0, \quad \langle T_{th}(t_1)T_{th}(t_2) \rangle = \frac{2k_B T I \omega_0}{Q}\delta(t_1 - t_2), \tag{24}$$

where k_B is the Boltzmann constant, T is absolute temperature, and Q is the quality factor. The autocorrelation function of the torsion pendulum response to thermal noise is as follows:[28]

$$\begin{aligned} R_\theta(\tau) &\equiv \langle \theta_{th}(t)\theta_{th}(t + t') \rangle \\ &= \frac{k_B T}{k} \mathrm{e}^{-\gamma|t'|} \left(\cos\omega_d |t'| + \frac{\gamma}{\omega_d} \sin\omega_d |t'| \right), \end{aligned} \tag{25}$$

where $\theta_{th}(t)$ is the torsion pendulum response to thermal noise, t' denotes the lag time.

The power spectral density of the thermal noise limit can be expressed as[29]

$$\theta_{th}(\omega) = \sqrt{\frac{4k_B T}{k\omega_d Q} \frac{1}{(1 - \omega_1^2/\omega_d^2)^2 + [\omega_1/(\omega_d Q)]^2}}. \tag{26}$$

According to Eq. (10) and Eq. (11), the standard deviation $(\delta_{th} a_{1i}, \delta_{th} b_{1i})$ of the amplitude components (a_{1i}, b_{1i}) of the modulation signal for the i-th small component can be obtained using the following expressions:

$$\begin{cases} \langle \delta_{th} a_{1i}^2 \rangle = \dfrac{4}{T_1^2} \left\langle \displaystyle\int_{iT_1}^{(i+1)T_1} [\theta_{th}(t_1)\cos(\omega_1 t_1)]\,\mathrm{d}t_1 \int_{iT_1}^{(i+1)T_1} [\theta_{th}(t_2)\cos(\omega_1 t_2)]\,\mathrm{d}t_2 \right\rangle, \\ \langle \delta_{th} b_{1i}^2 \rangle = \dfrac{4}{T_1^2} \left\langle \displaystyle\int_{iT_1}^{(i+1)T_1} [\theta_{th}(t_1)\sin(\omega_1 t_1)]\,\mathrm{d}t_1 \int_{iT_1}^{(i+1)T_1} [\theta_{th}(t_2)\sin(\omega_1 t_2)]\,\mathrm{d}t_2 \right\rangle. \end{cases} \tag{27}$$

On substituting Eq. (25) into Eq. (27), the standard deviations are given as shown below.

$$\delta^2_{\text{th}-\text{a}} = \langle \delta_{\text{th}} a^2_{1i} \rangle \approx \frac{2k_{\text{B}}T}{T_1^2 k\left[\left(\frac{\omega_{\text{d}}\omega_1}{Q}\right)^2 + (\omega_1^2 - \omega_{\text{d}}^2)^2\right]} \times \left[\frac{(-3\omega_1^4 + 2\omega_{\text{d}}^4 + 6\omega_1^2\omega_{\text{d}}^2)\omega_{\text{d}}^2/Q}{\left(\frac{\omega_{\text{d}}\omega_1}{Q}\right)^2 + (\omega_1^2 - \omega_{\text{d}}^2)^2} - \frac{2T_1\omega_{\text{d}}^3}{Q} + 2(\omega_1^2 - \omega_{\text{d}}^2 + \omega_{\text{d}}^2/Q)\right], \quad (28)$$

$$\delta^2_{\text{th}-\text{b}} = \langle \delta_{\text{th}} b^2_{1i} \rangle \approx \frac{2k_{\text{B}}T}{T_1^2 k\left[\left(\frac{\omega_{\text{d}}\omega_1}{Q}\right)^2 + (\omega_1^2 - \omega_{\text{d}}^2)^2\right]} \times \left[\frac{(-3\omega_1^4 + 2\omega_{\text{d}}^4 + 6\omega_1^2\omega_{\text{d}}^2)\omega_{\text{d}}^2/Q}{\left(\frac{\omega_{\text{d}}\omega_1}{Q}\right)^2 + (\omega_1^2 - \omega_{\text{d}}^2)^2} - \frac{2T_1\omega_{\text{d}}^3}{Q} - 2(\omega_1^2 - \omega_{\text{d}}^2 + \omega_{\text{d}}^2/Q)\right], \quad (29)$$

where $\delta_{\text{th}-\text{a}}$ and $\delta_{\text{th}-\text{b}}$ represent the standard deviations of thermal noise on the amplitude components (a_1, b_1), respectively. The final uncertainties of (a_1, b_1) caused by thermal noise can be expressed as follows.

$$u_{\text{th}}(a_1) = \frac{\delta_{\text{th}-\text{a}}}{\sqrt{N}}, \quad u_{\text{th}}(b_1) = \frac{\delta_{\text{th}-\text{b}}}{\sqrt{N}}, \quad (30)$$

where N is the number of T_1 periods during the whole measurement.

7. Data processing

7.1. Numerical simulation

In order to verify the correctness of the theoretical derivation above, a corresponding numerical simulation is performed. In this simulation, we set the total time and the sampling interval as 3 days and 1 s, respectively. The parameters for Eq. (4) are set as follows: $\gamma = 1.8 \times 10^{-6}$ rad/s, $a_{\text{d}} = 3.0 \times 10^{-4}$ rad, $b_{\text{d}} = 4.0 \times 10^{-4}$ rad, $\omega_{\text{d}} = 1.1 \times 10^{-2}$ rad/s, $a_1 = 1.9 \times 10^{-4}$ rad, $b_1 = 2.7 \times 10^{-4}$ rad, $\omega_1 = 5.2 \times 10^{-3}$ rad/s, and the basic noise level as 3.5×10^{-6} rad/Hz$^{1/2}$, based on the preliminary experiment of the WEP test by the HUST group. In their preliminary experiments, a gradiometer pendulum was used to measure the gradient of the gravitational field.[23] The power spectral density of the simulation signal is shown in Fig. 1.

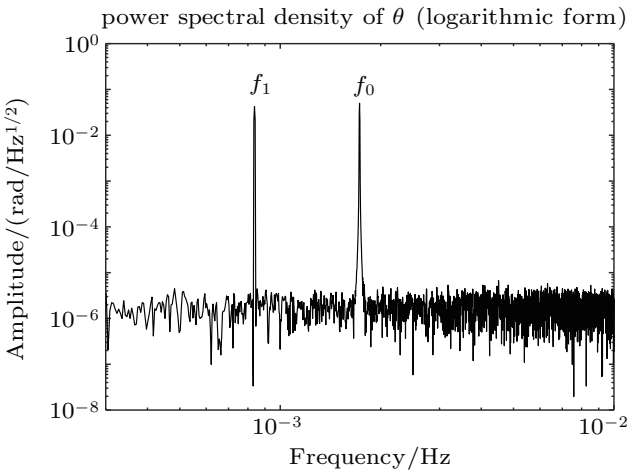


Fig. 1. The power spectral density of the simulation signal of angular deflection, $\theta(t)$, of the pendulum. The peak f is the free torsional oscillation $\theta_{\text{d}}(t)$. The peak f_1 is the modulation signal.

In order to arrive at a relation between the number of times the torsional filter is employed and the effect of the free torsional oscillation signal $\theta_{\text{d}}(t)$ on the amplitude estimation, the noise is temporarily not considered. The simulation results are shown in Table 1. The theoretical value is calculated using the theoretical derivation above and the actual value is obtained using the simulation. It is seen from Table 1 that the theoretical value calculated using Eq. (19) is in agreement with the actual value calculated using Eq. (11). When $m = 4$, the theoretical value is different from the actual value; this difference resulted from the limited computation accuracy that can only reach an order of 10^{-17} rad. With the increase in the number of times, m, that the torsional filter is employed, the effect of $\theta_{\text{d}}(t)$ on the estimation of the amplitude components (a_1, b_1) is smaller. In this simulation, when $m = 3$, the effect of the $\theta_{\text{d}}(t)$ is found to be less than 1 ppm, which is small enough to be ignored.

Table 1. The effect of the $\theta_{\text{d}}(t)$ on the estimation of the amplitude components (a_1, b_1) without noise.

Number of torsional filter m	Δa/rad		Δb/rad	
	Theoretical value	Actual value	Theoretical value	Actual value
0	2.5×10^{-5}	2.5×10^{-5}	1.2×10^{-5}	1.2×10^{-5}
1	1.3×10^{-8}	1.3×10^{-8}	6.2×10^{-9}	6.2×10^{-9}
2	6.8×10^{-12}	6.8×10^{-12}	3.3×10^{-12}	3.3×10^{-12}
3	3.5×10^{-15}	3.5×10^{-15}	1.8×10^{-15}	1.8×10^{-15}
4	1.8×10^{-18}	1.6×10^{-17}	9.5×10^{-19}	1.1×10^{-17}

After checking the theoretical derivation of the effect of $\theta_d(t)$, the influence of white noise on the amplitude estimation of the modulation signal is simulated. The power spectral density of the white noise, $\varepsilon(\omega)$, is 3.5×10^{-6} rad/Hz$^{1/2}$ in this simulation. In order to control the variables, only the modulation signal and the white noise should be considered regardless of the free torsional oscillation signal. Except the white noise, the thermal noise would have an effect on the amplitude estimation of the modulation, which sets a fundamental limit to it. Similarly, only the modulation signal and the thermal noise are considered, when we study the effect of the thermal noise. For $Q = 3000$, $k = 6.2\times10^{-9}$ N·m/rad, we can get the standard deviation of the amplitude components (a_1, b_1) caused by the white noise and the thermal noise using Eq. (22) and Eqs. (28) and (29), respectively, which should be in agreement with the value calculated by Eq. (11). Since the white noise and the thermal noise are random, we randomly generate 1000 sets of white noise data and thermal noise data each to avoid the contingency. The results are presented in Table 2. As expected, the theoretical value is consistent with the actual value. That is to say, the theoretical derivation above is correct and the correlation method is effective for the simulation data.

Table 2. The effect of the white noise and the thermal noise on the estimation of the amplitude components (a_1, b_1).

Noise	δ_a/rad		δ_b/rad	
	Theoretical value	Actual value	Theoretical value	Actual value
White noise	1.4×10^{-7}	1.4×10^{-7}	1.4×10^{-7}	1.4×10^{-7}
Thermal noise	1.1×10^{-8}	2.2×10^{-8}	1.1×10^{-8}	1.6×10^{-8}

7.2. Experiment data processing

Since the HUST group have not finished their experiments for the WEP test, to check the effectiveness of the correlation method, a set of data from their preliminary experiments of measurement for the gravitational field gradient is used. One typical angle-time data set of the pendulum is about 2 days, the sample interval $\Delta t = 1$ s, the free oscillation period $T_d \approx 580$ s, and the period of the modulation signal $T_1 = 1200$ s.

The power spectral density of the angular deflection of the pendulum $\theta(t)$ is shown in Fig. 2(a). The short peak f_1 in the figure is the modulation signal, which is due to the gravitational gradient effect. The peak f is the free torsional oscillation signal. According to the above simulation, the free torsional oscillation signal has an effect on the amplitude estimation of the modulation signal, which should be eliminated before we use the correlation method to estimate the amplitude of the modulation signal. An earlier simulation highlights that when the time of torsional filter $m = 3$, the effect of $\theta_d(t)$ can be reduced to a level of 10^{-6} nrad from which we can conclude that the effect of the free torsional oscillation is small enough to be ignored. The power spectral density of $\theta_d(t)$, after removing the free torsional oscillation signal, is shown in Fig. 2(b). As figure 2(b) depicts, the free oscillation signal is removed by the torsional filter three times and the peak f_1 is higher than it is in Fig. 2(a), which is caused by the torsional filter. Therefore, the amplitude components (a_{1i}, b_{1i}) in each small component estimated by the correlation method should have a correction in accordance with Eq. (9). The results are displayed in Fig. 3. The amplitude components (a_1, b_1) are 192.31(3) μrad and 272.22(2) μrad, respectively, as calculated using Eq. (10) and Eq. (12).

In this experiment, the torsional constant of the fiber, $k = 6.2\times10^{-9}$ N·m/rad and $Q = 3000$. According to Eq. (30), both the $u_{th}(a_1)$ and $u_{th}(b_1)$ due to the thermal noise are about 1.7×10^{-9} rad, where the number of the time period of the modulation signal is $N = 132$. Equation (14) yields the power spectral density of the modulation signal amplitude $S(B_1)$ as about 8.3×10^{-6} rad/Hz$^{1/2}$, which is about one order higher than the power spectral density of the thermal noise limit $\theta_{th}(\omega) \approx 3.7\times10^{-7}$ rad/Hz$^{1/2}$ as calculated using Eq. (26).

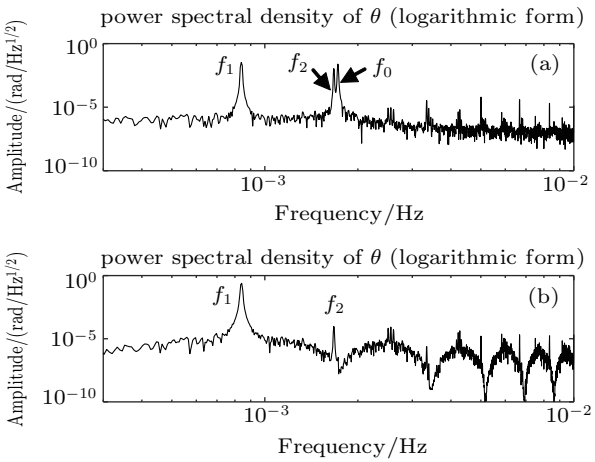


Fig. 2. Power spectral density of the angular deflection of the pendulum $\theta(t)$. (a) The power spectral density of the raw angular deflection. The peak f is the free torsional oscillation $\theta_d(t)$. The peaks f_1 and f_2 are the fundamental frequency and second harmonic components of the modulation signal while the other peaks are higher harmonic components of the modulation signal. (b) The power spectral density of the angular deflection after removing the free torsion oscillation, where the peak f is gone.

The basic noise spectral level at the modulation signal is about 4.3×10^{-6} rad/Hz$^{1/2}$ as shown in Fig. 2(a). The autocorrelation function of the basic noise is shown in Fig. 4 and it can be considered as the white noise. In other words, the power spectral density of the noise is about 4.3×10^{-6} rad/Hz$^{1/2}$.

According to Eq. (23), the uncertainties of both the amplitude components (a_1, b_1) caused by the white noise are 0.02 μrad, which is in agreement with the uncertainties 0.03 μrad and 0.02 μrad calculated using Eq. (12). It proves that the correlation method is an effective approach to determine the amplitude of the modulation signal, and the white noise model is reasonable.

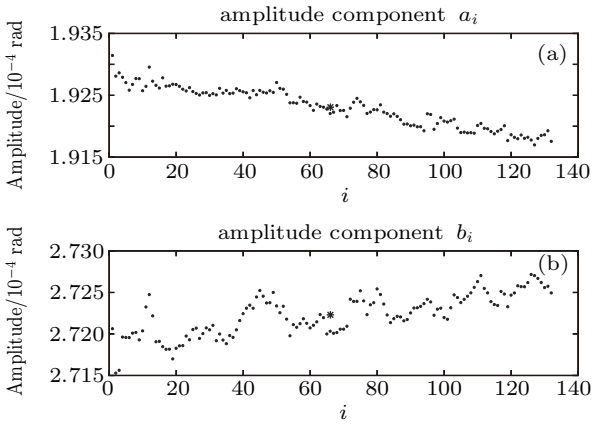


Fig. 3. Each point denotes the amplitude component of a small component estimated by the correlation method. The top panel (a) and bottom panel (b) are the cosine component and sinusoidal component of the modulation signal in each part, respectively. The asterisks denote the final value of the amplitude components.

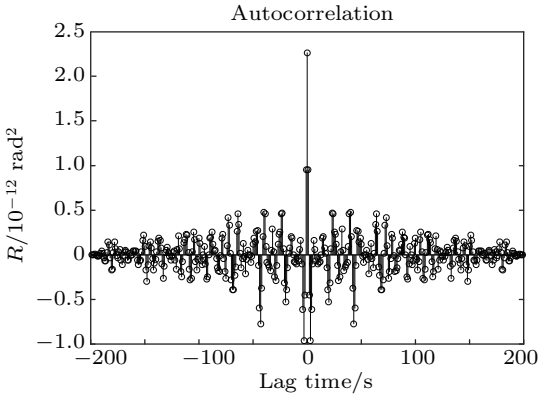


Fig. 4. The autocorrelation function for the basic noise of the experimental data. It can be confirmed that the basic noise satisfies the statistical properties of white noise generally, which can be regarded as the white noise model.

8. Summary

In the experiment of the WEP test with a rotating torsion balance, a high accuracy estimation of the amplitude of the modulation signal is highly important. The correlation method is often used for weak signal analysis and it can suppress the influence of higher harmonics, linear drift, and weak damping. The free torsional oscillation signal would influence the accurate extraction of the modulation signal amplitude while using the correlation method to estimate the amplitude of the modulation signal. We analyze the effects of free torsional oscillation, white noise, and thermal noise on the amplitude estimation of the modulation signal and verified the corresponding theoretical derivation using simulation. Later, the experimental data of the WEP test carried out by the HUST group is analyzed and the free oscillation signal is removed using the torsional filter three times. With the effect of the free oscillation has been eliminated, we estimate the amplitude of the modulation signal with its uncertainties using the correlation method and find that the power spectral density of the modulation signal amplitude is about one order higher than the thermal noise limit. The results prove that the correlation method is effective in determining the amplitude of the modulation signal, and it is instructive of the high accuracy of the WEP test. Except for the noise, the modulation signal would be disturbed by many factors including the change in the environment temperature and the electromagnetic field. Much work still needs to be carried out in this field.

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