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Radiating frequency of three-loop mesoscopic LC circuit with mutual inductance obtained by IEO method*

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Instead of normally tackling electric circuits by virtue of the Kirchhoff's theorem whose aim is to derive voltage, electric current, and electric impedence, our aim in this paper is to derive the characteristic frequency of a three-loop mesoscopic *LC* circuit with three mutual inductances, e.g., for the radiating frequency of the three-loop *LC* oscillator, we adopt the invariant eigen-operator (IEO) method to realize our aim.

Keywords: IEO method, three-loop *LC* mesoscopic circuit, radiating frequency

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1. Introduction

In condensed matter physics theory, when the transport dimension reaches a characteristic dimension, namely, the charge carrier inelastic coherence length, quantum effects in electric circuits must be taken into account and the circuit is named mesoscopic. The trend in the miniaturization of integrated circuits and devices towards atomic scale dimensions stimulates theoretical progress in quantization of mesoscopic circuits.[1] In history, a single *LC* (inductance–capacitance) circuit, as a fundamental cell in electric circuits, was quantized by Louisell in 1973;[2] he considered the charge q as a dynamic canonical coordinate, while taking electric current $I = \mathrm{d}q/\mathrm{d}t$ multiplied by the inductance L as canonical momentum, $p = L(\mathrm{d}q/\mathrm{d}t)$, by further quantizing (q, p) with the quantization condition $[q, p] = \mathrm{i}h$, the Hamiltonian of the *LC* circuit is quantized. In this work we consider what the energy level of a quantized two-loop mesoscopic *LC* circuit with mutual inductance M is, in other words, we want to know the characteristic frequency of two mesoscopic *LC* circuits when mutual inductance exists between them, this is equivalent to knowing the emitting frequency of this complex *L*–*C* oscillator. To the best of our knowledge, such a topic has not been touched on in the literature before. In the following we shall solve this problem by virtue of the invariant eigen-operator (IEO) method, which we briefly review in Section 2.

2. Brief review of the invariant eigen-operator method

Recall the original idea of Schrödinger's quantization scheme,[3] he took the identification: $\hat{H}$ (Hamiltonian) $\leftrightarrow$ $\mathrm{i}(\mathrm{d}/\mathrm{d}t)$ (in this paper we assume $\hbar = 1$ for simplicity), so $\mathrm{i}(\mathrm{d}/\mathrm{d}t)$ is named as the Schrödinger operator in many references. Thus one naturally has

$$\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \longleftrightarrow \widehat{H}^n,$$

now we set up an equation for an operator $\widehat{O}_\mathrm{e}$:

$$\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \widehat{O}_\mathrm{e} = \lambda \widehat{O}_\mathrm{e}, \tag{1}$$

when $n = 1$, it looks similar in form to the Schrödinger equation $\mathrm{i}(\mathrm{d}/\mathrm{d}t)\psi = \widehat{H}\psi$. Thus we call Eq. (1) as n-order invariant eigen-operator equation, and λ as n-order eigenvlaue. Further, using the Heisenberg equation:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\widehat{O}_\mathrm{e} = \left[\widehat{O}_\mathrm{e}, \widehat{H}\right] \tag{2}$$

we can write Eq. (1) as

$$\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \widehat{O}_\mathrm{e} = \left[\cdots\left[\left[\widehat{O}_\mathrm{e}, \widehat{H}\right], \widehat{H}\right]\cdots, \widehat{H}\right] = \lambda \widehat{O}_\mathrm{e}. \tag{3}$$

If we can find such a $\widehat{O}_\mathrm{e}$ that satisfies Eq. (3) we say that $\sqrt[n]{\lambda}$ is the energy-level gap. To clarify this point of view, we take $n = 2$ in Eq. (3) for example, and assuming $|\psi_a\rangle$ and $|\psi_b\rangle$ are two different stationary eigen-states of the Hamiltonian $\widehat{H}$ with eigen-values E_a and E_b, respectively, then we have:

$$\begin{aligned}\left\langle \psi_a\middle|\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 \widehat{O}_\mathrm{e}\middle|\psi_b\right\rangle &= \langle \psi_a|\left[\left[\widehat{O}_\mathrm{e}, \widehat{H}\right], \widehat{H}\right]|\psi_b\rangle \\ &= (E_b - E_a)^2 \langle \psi_a|\widehat{O}_\mathrm{e}|\psi_b\rangle \\ &= \lambda \langle \psi_a|\widehat{O}_\mathrm{e}|\psi_b\rangle \end{aligned} \tag{4}$$

whenever $\langle \psi_a|\widehat{O}_\mathrm{e}|\psi_b\rangle$ is a nonzero matrix element, the energy gap between $|\psi_a\rangle$ and $|\psi_b\rangle$ can be obtained as $|E_a - E_b| = \sqrt{\lambda}$. We name

$$\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \widehat{O}_\mathrm{e} = \lambda \widehat{O}_\mathrm{e}$$

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as the invariant eigen-operator equation, and this IEO method can be used to derive the energy eigenvalues of some quantum systems.[2–5] The IEO equation is a natural result of combining both the Schrödinger operator and the Heisenberg equation. In the following we apply the IEO method to solving the energy level of a two-loop *LC* mesoscopic circuit with mutual inductance according to its quantized Hamiltonian.

3. Quantization of the two-loop *LC* mesoscopic circuit with mutual inductance

Supposing there is a mesoscopic circuit as shown in Fig. 1, where m represents the mutual inductance between the two loops.

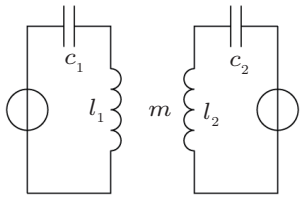


Fig. 1. Two *LC* mesoscopic circuits with mutual inductance.

The classical Lagrangian of the whole electric system is

$$L=\frac{1}{2}\left(l_1I_1^2+l_2I_2^2\right)+mI_1I_2-\frac{1}{2}\left(\frac{q_1^2}{c_1}+\frac{q_2^2}{c_2}\right)+\varepsilon_1(t)q_1+\varepsilon_2(t)q_2, \tag{5}$$

where $\varepsilon_1(t)$ is an external source in loop 1, mI_1I_2 denotes the mutual inductance between l_1 and l_2, in the case of leaking magnetism, $0<m<\sqrt{l_1l_2}$. Taking the charge q_1, q_2 as canonical coordinates, their conjugates are

$$P_1=\frac{\partial L}{\partial\dot{q}_1}=l_1I_1+mI_2,$$
$$P_2=\frac{\partial L}{\partial\dot{q}_2}=l_2I_2+mI_1. \tag{6}$$

The Hamiltonian is

$$\begin{aligned}H&=P_1\dot{q}_1+P_2\dot{q}_2-L\\&=\frac{1}{2}\left(l_1I_1^2+l_2I_2^2\right)+mI_1I_2+\frac{1}{2}\left(\frac{q_1^2}{c_1}+\frac{q_2^2}{c_2}\right)\\&\quad-\varepsilon_1(t)q_1-\varepsilon_2(t)q_2\\&=\frac{l_1l_2}{2(l_1l_2-m^2)}\left(\frac{P_1^2}{l_1}+\frac{P_2^2}{l_2}\right)-\frac{m}{l_1l_2-m^2}P_1P_2\\&\quad+\frac{1}{2}\left(\frac{q_1^2}{c_1}+\frac{q_2^2}{c_2}\right)-\varepsilon_1q_1-\varepsilon_2q_2\\&=H_0-\varepsilon_1q_1-\varepsilon_2q_2,\end{aligned} \tag{7}$$

where

$$H_0=\frac{1}{2F}\left(\frac{P_1^2}{l_1}+\frac{P_2^2}{l_2}\right)-\frac{m}{Fl_1l_2}P_1P_2+\frac{1}{2}\left(\frac{q_1^2}{c_1}+\frac{q_2^2}{c_2}\right) \tag{8}$$

with

$$F=1-\frac{m^2}{l_1l_2},\quad m^2<l_1l_2. \tag{9}$$

Imposing the quantization condition $[q_i,P_j]=\mathrm{i}\hbar\delta_{ij}$, H is an operator. The term $(m/Al_1l_2)P_1P_2$ may causes quantum entanglement.

4. Characteristic frequency of a two-loop *LC* mesoscopic circuit with mutual inductance

We now employ the IEO method to derive the characteristic frequency of the two-loop *LC* mesoscopic circuit with mutual inductance. Assuming the invariant eigen-operator is

$$O_\mathrm{e}=P_1+gP_2, \tag{10}$$

where g is to be determined. Using the Heisenberg equation we have

$$\begin{aligned}\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}O_\mathrm{e}&=[O_\mathrm{e},H_0]\\&=\left[(P_1+gP_2),\frac{1}{2}\left(\frac{q_1^2}{c_1}+\frac{q_2^2}{c_2}\right)\right]\\&=-\mathrm{i}\frac{q_1}{c_1}-\mathrm{i}g\frac{q_2}{c_2}\end{aligned} \tag{11}$$

and the corresponding IEO equation is

$$\begin{aligned}\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\right)^2O_\mathrm{e}&=\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\left[-\mathrm{i}\frac{q_1}{c_1}-\mathrm{i}g\frac{q_2}{c_2}\right]=\left[-\mathrm{i}\frac{q_1}{c_1}-\mathrm{i}g\frac{q_2}{c_2},\frac{1}{2F}\left(\frac{P_1^2}{l_1}+\frac{P_2^2}{l_2}\right)-\frac{m}{Fl_1l_2}P_1P_2\right]\\&=\left(\frac{1}{c_1Fl_1}-\frac{gm}{c_2Fl_1l_2}\right)P_1+\left(\frac{g}{c_2Fl_2}-\frac{m}{c_1Fl_1l_2}\right)P_2=\omega^2O_\mathrm{e}.\end{aligned} \tag{12}$$

Comparing Eq. (12) with Eq. (10) we see

$$\frac{1}{g}=\frac{1/c_1l_1-gm/c_2l_1l_2}{g/c_2l_2-m/c_1l_1l_2}=\frac{c_2l_2-gmc_1}{gc_1l_1-mc_2}, \tag{13}$$

which leads to the following expression

$$g^2mc_1-g(c_2l_2-c_1l_1)-mc_2=0. \tag{14}$$

Its solution is

$$g=\frac{c_2l_2-c_1l_1\pm\sqrt{(c_2l_2-c_1l_1)^2+4m^2c_1c_2}}{2mc_1}. \tag{15}$$

For convenience, we let

$$(c_2l_2-c_1l_1)^2+4m^2c_2c_1=\Delta. \tag{16}$$

Substituting Eq. (15) into the right-hand side of Eq. (12) we know

$$\begin{aligned}\omega_{\pm}^2 &= \frac{1}{c_1 F l_1} - \frac{m}{c_2 F l_1 l_2}\frac{c_2 l_2 - c_1 l_1 \pm \sqrt{\Delta}}{2mc_1} \\ &= \frac{c_2 l_2 + c_1 l_1 \pm \sqrt{\Delta}}{2c_1 c_2 F l_1 l_2} \\ &= \frac{c_2 l_2 + c_1 l_1 \pm \sqrt{\Delta}}{2c_1 c_2 (l_1 l_2 - m^2)},\end{aligned} \tag{17}$$

this is the square of characteristic frequency of the two-loop *LC* mesoscopic circuit with mutual inductance.

Consider the identical circuits which means $l_1 = l_2 \equiv l$ and $c_1 = c_2 \equiv c$, we have

$$\omega_{\pm} = \sqrt{\frac{1}{c(l \pm m)}}. \tag{18}$$

5. Characteristic frequency of a three-loop *LC* mesoscopic circuit with mutual inductance

As an extension of the two-loop *LC* mesoscopic circuit, we consider the three-loop *LC* circuit with mutual inductance as shown in the following Fig. 2.

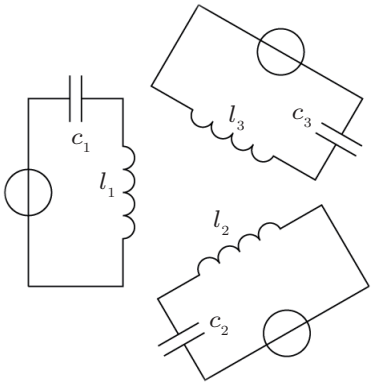


Fig. 2. Three *LC* mesoscopic circuits with mutual inductance.

The classical Lagrangian of the whole electric system is

$$L = \frac{1}{2}\sum_{i=1}^{3}\left(l_i I_i^2 - \frac{q_i^2}{c_i} + \varepsilon q_i\right) + m\sum_{1\le i<j\le 3} I_i I_j. \tag{19}$$

For the convenience of calculations, we consider the identical loops from the very beginning, letting $l_1 = l_2 = l_3 \equiv l$ and $c_1 = c_2 = c_3 \equiv c$.

Taking the charge q_1, q_2, q_3 as canonical coordinate, their conjugates are

$$\begin{aligned}P_1 &= \frac{\partial L}{\partial \dot{q}_1} = lI_1 + m(I_2 + I_3), \\ P_2 &= \frac{\partial L}{\partial \dot{q}_2} = lI_2 + m(I_1 + I_3), \\ P_3 &= \frac{\partial L}{\partial \dot{q}_3} = lI_3 + m(I_1 + I_2).\end{aligned} \tag{20}$$

The Hamiltonian is

$$H = P_1\dot{q}_1 + P_2\dot{q}_2 + P_3\dot{q}_3 - L = H_0 - \varepsilon\sum_{i=1}^{3} q_i, \tag{21}$$

where

$$\begin{aligned}H_0 = &\frac{(l+m)\left(P_1^2 + P_2^2 + P_3^2\right) - 2m(P_1P_2 + P_2P_3 + P_1P_3)}{2(l-m)(l+2m)} \\ &+ \frac{1}{2c}\left(q_1^2 + q_2^2 + q_3^2\right).\end{aligned} \tag{22}$$

For convenience we make a variable substitution:

$$\begin{aligned}\alpha &= \frac{l+m}{(l-m)(l+2m)}, \\ \beta &= \frac{1}{c}, \\ \gamma &= -\frac{m}{(l-m)(l+2m)},\end{aligned} \tag{23}$$

which makes

$$\begin{aligned}H_0 = &\frac{\alpha}{2}\left(P_1^2 + P_2^2 + P_3^2\right) + \frac{\beta}{2}\left(q_1^2 + q_2^2 + q_3^2\right) \\ &+ \gamma(P_1P_2 + P_2P_3 + P_1P_3).\end{aligned} \tag{24}$$

We now employ the IEO method to derive the energy level gap of H_0. Assuming the invariant eigen-operator is

$$O_{\mathrm{e}} = P_1 + g_1P_2 + g_2P_3, \tag{25}$$

where g_1, g_2 is to be determined. Using the Heisenberg equation we have

$$\begin{aligned}\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}O_{\mathrm{e}} &= [O_{\mathrm{e}}, H_0] \\ &= \left[(P_1 + g_1P_2 + g_2P_3), \frac{\beta}{2}\left(q_1^2 + q_2^2 + q_3^2\right)\right] \\ &= -\mathrm{i}\beta(q_1 + g_1q_2 + g_2q_3)\end{aligned} \tag{26}$$

and the corresponding IEO equation is

$$\begin{aligned}\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 O_{\mathrm{e}} &= \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\left[-\mathrm{i}\beta(q_1 + g_1q_2 + g_2q_3)\right] \\ &= (\alpha\beta + g_1\beta\gamma + g_2\beta\gamma)P_1 \\ &\quad + (g_1\alpha\beta + \beta\gamma + g_2\beta\gamma)P_2 \\ &\quad + (g_2\alpha\beta + \beta\gamma + g_1\beta\gamma)P_2 \\ &= \omega^2 O_{\mathrm{e}}.\end{aligned} \tag{27}$$

Comparing Eq. (25) with Eq. (27) we obtain two coefficient equations:

$$\begin{cases}g_1^2 + g_1g_2 - g_2 - 1 = 0, \\ g_2^2 + g_1g_2 - g_1 - 1 = 0.\end{cases} \tag{28}$$

There are two solutions of this equation set:

$$g_1 = g_2 = 1, \qquad \text{or} \qquad g_1 = -1 - g_2. \tag{29}$$

Each solution corresponds to an energy level gap. For $g_1 = g_2 = 1$, which can be substituted into Eq. (27), we have

$$\omega_1 = \sqrt{\beta(\alpha + 2\gamma)}. \tag{30}$$

For $g_1 = -1 - g_2$, we have

$$\omega_2 = \sqrt{\beta\,(\alpha - \gamma)}. \tag{31}$$

Using Eq. (23) to express ω_1 and ω_2 in terms of the original variables, we have

$$\omega_1 = \sqrt{\frac{1}{c\,(l+2m)}}, \qquad \omega_2 = \sqrt{\frac{1}{c\,(l-m)}}. \tag{32}$$

This is the characteristic frequency of the three-loop *LC* mesoscopic circuit with mutual inductance.

Since deriving characteristc frequency means energy quantization, the method introduced in this paper can be applied to quantizing other more complex mesoscopic electric circuits.

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