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Decaying solitary waves propagating in one-dimensional damped granular chain*

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We numerically investigate the nonlinear waves propagating in a one-dimensional particle chain when the damping effect is taken into account. It is found that decaying solitary waves exist, in which the amplitude of the wave decreases exponentially as time increases. Meanwhile, the velocity of the solitary wave also slows down as time goes. This result implies that the damping coefficient is an important parameter in such a nonlinear system. Theoretical analysis has also been done by the reductive perturbation method. The result indicates that the nonlinear waves propagating in such a system can be described by the damped KdV equation.

Keywords: damping granular chain, solitary wave, damped KdV equation

PACS: 45.05.+x, 05.45.–a, 46.40.Cd, 45.70.–n **DOI:** 10.1088/1674-1056/27/7/074501

1. Introduction

The propagation of solitary wave in a non-prestressed one-dimensional nonlinear granular chain was first proposed by Nesterenko in 1984[1] and was verified lately.[2] In Refs. [1] and [2], the authors assumed that the interaction between two contacted particles follows the Hertzian law, and they found that the interaction forces propagating in the one-dimensional granular chain forms a solitary wave. This solitary wave has strong nonlinearity.[3–5] Nesterenko noticed that the propagation of a perturbation in a granular chain with Hertzian contact has the nature of a soliton.[6–9] The existence of this kind of soliton pulses has also been confirmed by both theoretical and experimental studies.[10–14] Despite the large amount of recent work on this subject,[11,12,14–23] the granular system in physics still needs a great deal of challenging new research.

In Refs. [24]–[26], the researchers also found that the solitary wave in a prestressed one-dimensional granular chain is quite different from the characteristics of a sound wave for the non-prestressed case, implying that the prestress probably plays an important role in the properties of solitary waves. However, the damping effect was neglected in most of the references mentioned above. In this paper, we find that the properties of solitary waves will be changed when the damping effect is taken into account. In fact, the solitary waves propagating in such a system can be described by the damped KdV equation instead of the KdV equation. It is also found numerically that the amplitude of the solitary wave decreases exponentially as time increases.

2. Model

We consider the case that N identical balls (made of steel), confined in a polytetrafluoroethylene (PTFE) tube, are placed horizontally to form a one-dimensional granular chain, as shown in Fig. 1. The first particle of the chain, driven by an external force, oscillates harmonically along the horizontal direction. The last ball is fixed, which causes all the particles to move around the equilibrium position and the displacement cannot be very large. The i-th ball ($i = 1, 2, 3, \ldots, N$) is labeled by i. We assume that all the particles have the same radius R, the same mass m, and the same material parameters, such as Young's modulus Y and Poisson's ratio σ. The potential between two contacted particles is assumed to be Hertz's potential, expressed as $U = \frac{1}{n}k\delta^n$, where δ is the overlap deformation between the two contacted particles, $k = (R/2)^{1/2}/D(Y,\sigma)$, and $D(Y,\sigma) = \frac{3(1-\sigma^2)}{2Y}$.[27] In general, the value of n depends upon the contact geometry between two contacted particles and typically varies between $n = 5/2$ and 3.[14,28] In our work, n is taken to be $5/2$. For the i-th particle ($i = 2, 3, 4, \ldots, N-1$), the damping force is also taken into account, and assumed to be proportional to the i-th particle's velocity. Thus, the dynamic of the granular chain can be described by a set of N coupled nonlinear equations, in which the motion equation of the i-th particle can be expressed as

$$m\frac{\mathrm{d}^2u_i}{\mathrm{d}t^2} = nk[(2R+u_{i-1}-u_i)^{n-1} - (2R+u_i-u_{i+1})^{n-1}] - \beta v_i(t), \quad i = 2,3,\ldots,N-1, \tag{1}$$

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where $u_i = u_i(t)$ is the position of the i-th particle at time t, $v_i(t) = \frac{du_i}{dt}$ is its velocity, and β is the coefficient of the structural damping of the material.

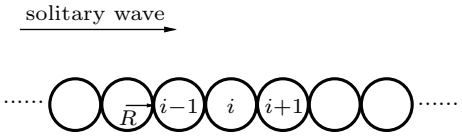


Fig. 1. Illustration of a one-dimensional granular chain.

The first particle oscillates harmonically along the horizontal direction. Thus the motion equation can be given easily as

$$u_1(t) = A\sin\omega t, \quad v_1(t) = A\omega\cos\omega t, \tag{2}$$

where A is the amplitude, ω is the frequency, and both of them are constants. For the N-th particle located at the most far-right of the chain, the motion equation is

$$v_N(t) = 0, \tag{3}$$

because it is fixed.

3. Numerical simulation

We now numerically investigate the solitary waves propagating in the one-dimensional granular chain shown in Fig. 1. We place a granular chain composed of 200 particles ($N = 200$). In the experiment, we use steel balls. The parameters are taken as Young's modulus $Y = 2.06 \times 10^{11}$ Pa, Poisson's ratio $\sigma = 0.3$, the density $\rho = 7.9 \times 10^3$ kg/m^3, the radius $R = 0.006$ m, and the damping coefficient $\beta = 10$ kg/s. So we can get the mass $m = \frac{4}{3}\rho\pi R^3 = 7.144 \times 10^{-3}$ kg and the $k = \frac{Y}{3(1-\sigma^2)}\sqrt{2R} = 8.266 \times 10^9$ kg$\cdot$m$^{-1/2}\cdot$s^{-2}. In the experiment, one can apply an external force on the first particle to generate solitary waves. In our study, we assume that the first particle is driven by an external force. The first particle oscillates harmonically along the horizontal direction, with the frequency ω and a very small amplitude A. Both A and ω are adjustable constants. In our simulation, they are taken as $A = 2.5 \times 10^{-4}$ m and $\omega = 1.0 \times 10^5$ rad/s. We find that solitary waves can be generated under such circumstances.

Figure 2 displays the profiles of solitary waves propagating in a one-dimensional granular chain under different parameters at the same time. The red line is the solitary wave (called undamped soliton) when there is no damping force ($\beta = 0$ kg/s), while the black line shows the solitary wave (called damped soliton) when there is damping force ($\beta = 10$ kg/s). It is remarkable that for a finite granular system, because the particle located at the most far-right of the granular chain is fixed, the reflection wave will be formed when the solitary wave arrives near the last particle. In the numerical experiment, it is also noted that the reflection wave (if it exists), which propagates along the left direction, is still a solitary one. However, here we only consider the case that the solitary wave is far away from the last particle. Thus, the reflected wave is so weak that can be omitted. It can be seen clearly from Fig. 2 that the amplitude of the damped soliton is smaller than the undamped soliton because of damping. Furthermore, the undamped soliton propagates a longer distance than the damped soliton during the same time, which implies that the average speed of the undamped soliton is faster than the damped soliton. Under this case, the average speeds of damped soliton and undamped soliton are $V_{\text{damp}} = 1.820 \times 10^3$ m/s and $V_{\text{undamp}} = 2.016 \times 10^3$ m/s, respectively. A detailed observation indicates that the amplitude of the undamped soliton is nearly invariant when time goes, while the amplitude of the damped soliton decreases exponentially as time increases, which is shown in Fig. 3.

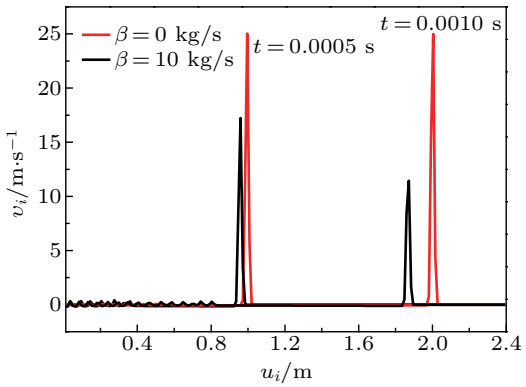


Fig. 2. (color online) Profiles of solitary waves propagating in a one-dimensional granular chain. Red line: no damping, i.e., damping coefficient $\beta = 0$ kg/s. Black line: damping coefficient $\beta = 10$ kg/s.

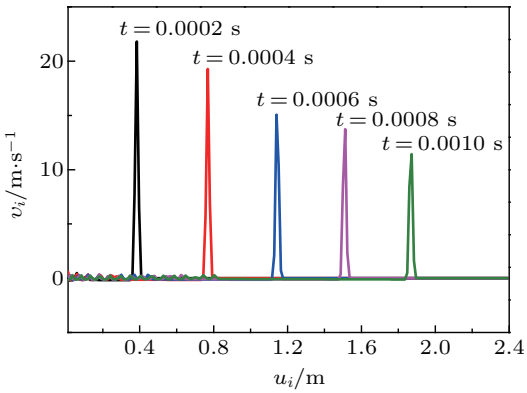


Fig. 3. (color online) Profiles of the damped soliton propagating in a one-dimensional granular chain at different times $t = 0.0002$ s, 0.0004 s, 0.0006 s, 0.0008 s, and 0.0010 s.

In Fig. 3, one can see that the damped soliton travels along the right direction and its amplitude decreases monotonously as time increases. This implies that there exists kinetic energy loss because of the damping force applied on the granular

chain. The damping force in such a system has great influences on the propagation of solitary waves in the granular chain. In Fig. 4, the profiles of damped soliton under different damping coefficients are shown for a certain time $t = 0.001$ s.

We can see from Fig. 4 that the smaller the damping coefficient is, the larger the amplitude is, which suggests that the kinetic energy loss increases when the damping coefficient increases. It can also be seen from Fig. 4 that a damped soliton travels a shorter distance than the one for a smaller damping coefficient, which implies that the average speed of the damped soliton is lower than that one. In order to make it more clear, the maximum average speed, which can be regarded as the amplitude of the solitary wave, is computed as

$$\overline{v}_{\max} = \max\left\{\frac{1}{5}\sum_{j=-2}^{2} v_{i+j} \,|\, i = i_{\mathrm{B}}, i_{\mathrm{B}}+1, \ldots, N-2\right\},$$

where i_{B} should be large enough to avoid the influence of the particles located near the most-left of the chain. In practice, we take $i_{\mathrm{B}} = 10$.

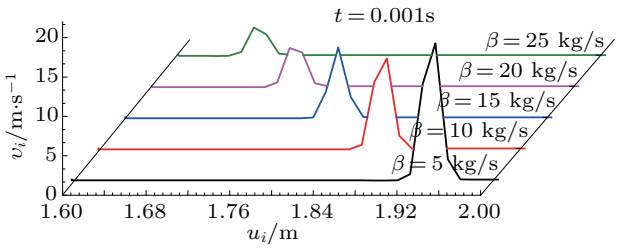


Fig. 4. (color online) Profiles of damped soliton under different damping coefficients $\beta = 5$ kg/s, 10 kg/s, 15 kg/s, 20 kg/s, and 25 kg/s at time $t = 0.001$ s.

Figure 5 shows the maximum average speed $\overline{v}_{\max}$ versus time t for different damping coefficients β. It can be seen evidently from Fig. 5 that the amplitude of the solitary wave decreases as time t increases for a given β. In order to see that $\overline{v}_{\max}$ decreases exponentially indeed, a fitting function for $\overline{v}_{\max}$ when $\beta = 20$ kg/s is achieved as $\overline{v}_{\max} \approx 8.3\,\mathrm{e}^{-1407t}$, which is shown as a solid line in Fig. 5. It can also be seen clearly that $\overline{v}_{\max}$ decays faster and faster as the damping coefficient β increases.

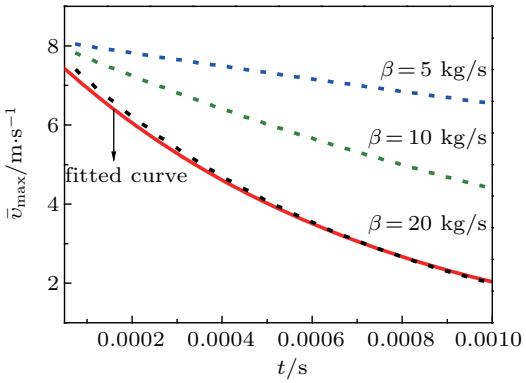


Fig. 5. (color online) $\overline{v}_{\max}$ versus time t under different damping coefficients $\beta = 5$ kg/s, 10 kg/s, and 20 kg/s with dotted lines. Solid line is the fitted curve when $\beta = 20$ kg/s.

Meanwhile, we find numerically that the radius of the particles R also has obvious effects on the solitary waves. Figure 6 shows the maximum average speed $\overline{v}_{\max}$ versus time t for different particle radii R for a fixed $\beta = 10$ kg/s. One can see from Fig. 6 that at the beginning as the solitary wave is formed, the amplitude of the solitary wave for a smaller R is larger than that one with bigger R. However, it decreases even faster than that one with a bigger radius as time increases, which tells us that a bigger radius is to the benefit of the propagating of a solitary wave. It is also noted that the amplitude of the solitary waves decreases exponentially, just like those shown in Fig. 5. The solid line in Fig. 6 is a fitting function $\overline{v}_{\max} \approx 9.32\,\mathrm{e}^{-1123t}$ for the case of $R = 0.005$ m, indicating that $\overline{v}_{\max}$ indeed decreases exponentially as time goes on.

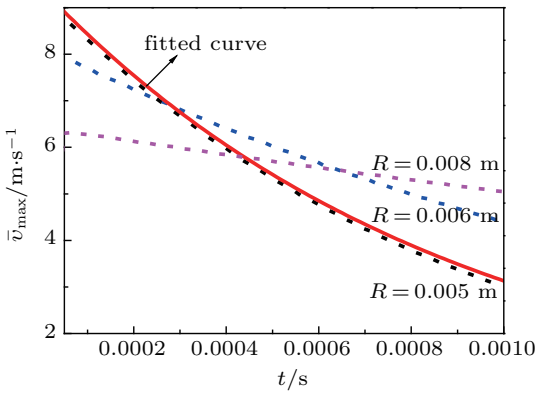


Fig. 6. (color online) $\overline{v}_{\max}$ versus time t under different particle radii $R = 0.005$ m, 0.006 m, and 0.008 m with dotted lines. The solid line is the fitted curve when $R = 0.005$ m. Here we take $\beta = 10$ kg/s.

4. Theoretical analysis

To understand the above nonlinear phenomena clearly, we now perform a theoretical analysis on Eq. (1). The absolute position of the i-th particle at time t can be rewritten as $u_i(t) = (2i-1)R' + s_i(t)$, where $s_i(t)$ is the displacement from its equilibrium position at time t and R' is the effective radius of the particle. At initial time $t = 0$ s, the particle chain is prestressed by an external force, which leads to that the effective radius of each particle will become smaller. We assume that the overlaps between two contacted particles are the same, noted by δ_0. Then R' can be written as $R' = R - \frac{\delta_0}{2}$. Now, the equation (1) can be rewritten as

$$m\frac{\mathrm{d}^2 s_i}{\mathrm{d}t^2} = nk\left[(\delta_0 + s_{i-1} - s_i)^{n-1} - (\delta_0 + s_i - s_{i+1})^{n-1}\right] - \beta\frac{\mathrm{d}s_i}{\mathrm{d}t}, \quad i = 2, 3, \ldots, N-1. \tag{4}$$

If the prestress is large enough, i.e. $\delta_0 \gg s_{i-1} - s_i$ and $\delta_0 \gg s_i - s_{i+1}$, one can expand the two terms in the bracket of Eq. (4) by their Taylor series. Then the equation (4) can be approximated as

$$m\frac{\mathrm{d}^2 s_i}{\mathrm{d}t^2} = n(n-1)k\delta_0^{n-2}[(s_{i-1} - s_i) - (s_i - s_{i+1})]$$

$$+\frac{n(n-1)(n-2)}{2}k\delta_0^{n-3}$$
$$\times[(s_{i-1}-s_i)^2-(s_i-s_{i+1})^2]-\beta\frac{\mathrm{d}s_i}{\mathrm{d}t}, \quad (5)$$

where the cubic nonlinear term and other higher-order terms have been omitted.

Under long-wavelength approximation, s_i varies slowly in space. One can expand $s_{i\pm1}$ with the continuous approximation

$$s_{i\pm1}=s_i\pm\frac{\mathrm{d}s_i}{\mathrm{d}i}+\frac{1}{2}\frac{\mathrm{d}^2s_i}{\mathrm{d}i^2}\pm\frac{1}{6}\frac{\mathrm{d}^3s_i}{\mathrm{d}i^3}+\frac{1}{24}\frac{\mathrm{d}^4s_i}{\mathrm{d}i^4}\pm\cdots, \quad (6)$$

where i is assumed to be a continuous variable.[27] Equation (5) becomes

$$\frac{\mathrm{d}^2s_i}{\mathrm{d}t^2}=B_1\left(\frac{\mathrm{d}^2s_i}{\mathrm{d}i^2}+\frac{1}{12}\frac{\mathrm{d}^4s_i}{\mathrm{d}i^4}\right)+B_2\frac{\mathrm{d}s_i}{\mathrm{d}i}\frac{\mathrm{d}^2s_i}{\mathrm{d}i^2}-\frac{\beta}{m}\frac{\mathrm{d}s_i}{\mathrm{d}t}, \quad (7)$$

where

$$B_1=\frac{n(n-1)k\delta_0^{n-2}}{m},\quad B_2=-\frac{n(n-1)(n-2)k\delta_0^{n-3}}{m},$$

and $s_i=s_i(t)=s(i,t)$.

Following the reductive perturbation method,[29–32] we make the coordinate transformation

$$\begin{cases}\xi=\varepsilon(i-\lambda t),\\ \tau=\varepsilon^3 i,\end{cases}$$

where $\varepsilon\ll k_c$ (k_c the wave number) is a small parameter ($\varepsilon\ll1$),[30] i.e., the long wavelength approximation, and λ is a constant which will be determined later. We also make the expansion

$$s=\varepsilon' s_1(\xi,\tau)+\varepsilon'^3 s_2(\xi,\tau)+\cdots,$$

where ε' is also a small parameter with the same order of ε ($\varepsilon'=\alpha\varepsilon$), i.e., the small amplitude approximation. We choose $\alpha=1$ in the system since its value does not affect the final results.[30] Then we substitute these expansions into Eq. (7). At the lowest order, one can obtain $\lambda=\sqrt{B_1}$. As an assumption, it is proposed that the damping coefficient in Eq. (7) is $\beta=p\varepsilon^3$, where p is a constant. Substituting these expressions into Eq. (7) yields

$$\frac{\partial^2 s_1}{\partial\xi\partial\tau}+\frac{B_2}{2B_1}\frac{\partial s_1}{\partial\xi}\frac{\partial^2 s_1}{\partial\xi^2}+\frac{1}{24}\frac{\partial^4 s_1}{\partial\xi^4}+\frac{p\sqrt{B_1}}{m}\frac{\partial s_1}{\partial\xi}=0. \quad (8)$$

If one introduces the transformation $w=\frac{\partial s_1}{\partial\xi}$, then equation (8) becomes the damped KdV equation

$$\frac{\partial w}{\partial\tau}+\gamma_1 w\frac{\partial w}{\partial\xi}+\gamma_2\frac{\partial^3 w}{\partial\xi^3}+\gamma_3 w=0, \quad (9)$$

where $\gamma_1=B_2/2B_1$, $\gamma_2=1/24$, and $\gamma_3=p\sqrt{B_1}/m$. In Eq. (9), the last item $\gamma_3 w$ represents the damping effect.

For the above damped KdV equation (9), the solitary wave solution with a generalized form is given as[33]

$$w=a(t)\,\mathrm{sech}^2\left[\left(\frac{\gamma_1 a(t)}{12\gamma_2}\right)^{\frac{1}{2}}\zeta\right],$$
$$\zeta=\xi-\frac{\gamma_1 a_0}{4\gamma_3}\left(1-\mathrm{e}^{-\frac{4}{3}\gamma_3 t}\right)+l, \quad (10)$$

where $a(t)$ is the amplitude of the solitary wave, and

$$a(t)=a_0\,\mathrm{e}^{-\frac{4}{3}\int_{t_0}^{t}\gamma_3\,\mathrm{d}\tau}=a_0\,\mathrm{e}^{-\frac{4}{3}\gamma_3(t-t_0)}, \quad (11)$$

a_0 is the amplitude of the wave at initial time $t=t_0$, and l is an arbitrary constant which only affects the phase of the wave. Furthermore, one can get

$$\upsilon_\mathrm{p}=\frac{\gamma_1 a_0}{3}\mathrm{e}^{-\frac{4}{3}\gamma_3 t},\quad K\propto\left(\frac{\gamma_1 a_0}{12\gamma_2}\right)^{\frac{1}{2}}\mathrm{e}^{-\frac{2}{3}\gamma_3 t},$$
$$\chi\propto\sqrt{\frac{12\gamma_2}{\gamma_1 a_0}}\,\mathrm{e}^{\frac{2}{3}\gamma_3 t}, \quad (12)$$

where υ_p is the phase velocity of the solitary wave, K the wave number, and χ the width of the solitary wave. From Eqs. (11) and (12), we find that the amplitude of the solitary wave decreases exponentially, however, the width of the solitary wave increases, as time t increases when $\gamma_3>0$. It is somewhat reversed that the amplitude of the solitary wave increases exponentially but the width of the solitary wave decreases as time goes when $\gamma_3<0$. Meanwhile, the velocity of the solitary wave also varies as the wave propagates.[33]

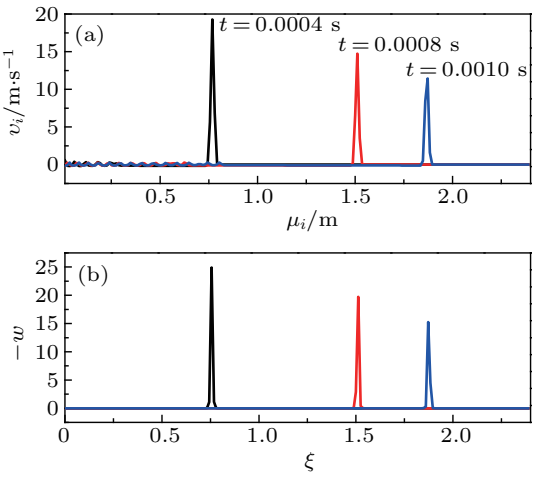


Fig. 7. (color online) Wave frames for (a) the numerical results and (b) the analytical results.

From the above analysis, we know that $u_i(t)=(2i-1)R'+s_i(t)$ and $w=\frac{\partial s_1}{\partial\xi}$, which results in that the velocity of the i-th particle is $\upsilon_i=\frac{\partial u_i}{\partial t}=-\lambda\varepsilon^2\frac{\partial s_1}{\partial\xi}=-\lambda\varepsilon^2 w$. Figure 7(a) shows the profiles of a decaying solitary wave at different time t by solving Eq. (1) directly and numerically. The parameters are chosen as $\beta=10$ kg/s and $R=0.006$ m. Other parameters are the same as before. Figure 7(b) is the graph of an analytical solution of the damped KdV equation (9), given

by Eq. (10). In Fig. 7(b), for the sake of simplicity, $-w$ is instead of w when making the graph. We can see from Fig. 7 that the analytical results are rather similar to the numerical results, implying that it is indeed that the damped KdV equation can be used to describe the solitary waves propagating in a one-dimensional granular chain when the damping effect is taken into account. We also computed the numerical results and analytical solutions under lots of different parameters. The graphs for those results are omitted because they are similar to Fig. 7.

5. Conclusion

We numerically investigate the nonlinear wave propagating in a one-dimensional particle chain when the damping effect is taken into account. It is found that decaying solitary waves exist which can be described by the damped KdV equation. The numerical results indicate that the amplitude of the solitary wave decreases exponentially as time increases. Meanwhile, the velocity of the solitary wave also slows down as time goes on. This result also implies that the damping coefficient is an important parameter in such a nonlinear system. Finally, comparing the numerical solution to the analytical solution of the damped KdV equation shows that the numerical solution and analytical solution are consistent qualitatively with each other.

References

[1] Nesterenko V F 1983 *J. Appl. Mech. Tech. Phys.* **5** 733
[2] Lazaridi A N and Nesterenko V F 1985 *J. Appl. Mech. Tech. Phys.* **26** 405
[3] Wen Z Y, Wang S J, Zhang X M and Li L 2007 *Chin. Phys. Lett.* **24** 2887
[4] Xia J H, Wang P J and Liu C S 2012 *Chin. Phys. B* **21** 024501
[5] Ponson L, Boechler N, Lai Y M, Porter M A, Kevrekidis P G and Daraio C 2010 *Phys. Rev. E* **82** 021301
[6] Huang K, Miao G Q, Zhang P, Yun Y and Wei R J 2006 *Phys. Rev. E* **73** 041302
[7] Wang P J, Xia J H, Li Y D and Liu C S 2007 *Phys. Rev. E* **76** 041305
[8] Zheng H P, Jiang Y M and Peng Z 2013 *Chin. Phys. B* **22** 040511
[9] Zhou Z G, Jiang Y M and Hou M Y 2017 *Chin. Phys. B* **26** 084502
[10] Friesecke G and Wattis J A D 1994 *Commun. Math. Phys.* **161** 391
[11] Sen S and Sinkovits R S 1996 *Phys. Rev. E* **54** 6857
[12] Coste C, Falcon E and Fauve S 1997 *Phys. Rev. E* **56** 6104
[13] Sen S, Manciu M and Wright J D 1998 *Phys. Rev. E* **57** 2386
[14] Hinch E J and Saint-Jean S 1999 *Proc. R. Soc. A* **455** 3201
[15] Hong J and Xu A 2001 *Phys. Rev. E* **63** 061310
[16] Manciu M, Sen S and Hurd A J 2001 *Physica* (*Amsterdam*) **157D** 226
[17] Sen S and Manciu M 2001 *Phys. Rev. E* **64** 056605
[18] Lee J, Park S and Yu I 2003 *Phys. Rev. E* **67** 066607
[19] Kenkre V M and Lindenberg K 2003 *AIP Conf. Proc.* (New York: American Institute of physics) p. 658
[20] Nakagawa M, Agui J H, Wu D T and Extramiana D V 2003 *Granul. Matter* **4** 167
[21] Rosas A and Lindenberg K 2004 *Phys. Rev. E* **69** 037601
[22] Sen S and Hunt M L 2001 *MRS Symposia Proceedings* (Pittsburgh: Material Research Society) p. 627
[23] Daraio C, Nesterenko V F, Herbold E and Jin S 2005 *Phys. Rev. E* **72** 016603
[24] Hertz H 1881 *J. Reine Angew. Math.* **92** 156
[25] Nesterenko V F 1994 *J. Phys. IV France* **04** C8-729
[26] Nesterenko V F, Lazaridi A N and Sibiryakov E B 1995 *J. Appl. Mech. Tech. Phys.* **36** 166
[27] Liu S W, Yang Y Y, Duan W S and Yang L 2015 *Phys. Rev. E* **92** 013202
[28] Spence D A 1968 *Proc. R. Soc. A* **305** 55
[29] Leblond H 2008 *J. Phys. B* **41** 043001
[30] Zhang J, Yang Y, Xu Y X, Yang L, Qi X and Duan W S 2014 *Phys. Plasmas* **21** 103706
[31] Kakutani T, Oni H, Taniuti T and Wei C C 1968 *J. Phys. Soc. Jpn.* **24** 941
[32] Kakutani T, Oni H, Taniuti T and Wei C C 1968 *J. Phys. Soc. Jpn.* **24** 1159
[33] Hilmi D 2002 *Appl. Math. Comp.* **132** 643