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# Mean-square composite-rotating consensus of second-order systems with communication noises*

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We study the mean-square composite-rotating consensus problem of second-order multi-agent systems with communication noises, where all agents rotate around a common center and the center of rotation spins around a fixed point simultaneously. Firstly, a time-varying consensus gain is introduced to attenuate to the effect of communication noises. Secondly, sufficient conditions are obtained for achieving the mean-square composite-rotating consensus. Finally, simulations are provided to demonstrate the effectiveness of the proposed algorithm.

**Keywords:** multi-agent systems, mean-square consensus, composite-rotating, communication noises

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## 1. Introduction

Recently, multi-agent systems have attracted increased research attention because of their characteristics of autonomy, distribution, and coordination, and their extensive applications in practical engineering, such as aerospace, sensor networks, social networks, and so on.[1,2]

As the most fundamental and basic issue on the research of multi-agent systems, the consensus problem has been deeply studied and a great deal of results have been reported.[3–6] Consensus means that the agents reach some common states such as position and velocity by using a distributed protocol. However, the vast majority of the existing results are about parallel consensus, which means that all agents will move along a linear path. In nature, there are a lot of circular motion phenomena, such as the celestial motion and earth orbiting satellite. Fiorelli *et al*. made use of autonomous underwater vehicles (AUVs) to collect oceanographic measurements and described a methodology for the cooperative control of multiple vehicles to complete missions such as gradient climbing and feature tracking in an uncertain environment.[7] Following the work of Fiorelli *et al*., Sepulchre *et al*. proposed a methodology to stabilize isolated relative equilibria in an all-to-all communication system where coupled identical particles move in the plane at unit speed;[8] this was extended to a general communication framework.[9] Ren made a further research on the consensus algorithm for double-integrator dynamics by introducing a rotation matrix.[10] Then Lin *et al*. investigated the collective rotating motions of second-order multi-agent systems, including the rotation consensus and the rotation formation problems.[11,12] Mo *et al*. dealt with the finite-time distributed rotating encirclement control problem of multi-agent systems.[13] As a slightly more complicated motion than rotating motion, the composite rotating motions exist extensively in nature, for example, the earth moves around the sun, and the moon moves around the earth. Hence, it is worth studying the composite rotating motion problem for multi-agent systems. Recently, the composite-rotating consensus problems were considered in Refs. [14] and [15] by designing proper control laws, where all agents move in a circular orbit and the circle center moves along a horizontal circumference.

All results mentioned above of rotation motion problems were solved for multi-agent systems without any noise. However, the real physical systems are often under uncertain communication environments, information exchanges among the agents cannot be carried out accurately. Therefore, it is necessary and meaningful to study the composite rotation motion problems of multi-agent systems under communication noises. For multi-agent systems with communication noises, the mean-square consensus problems were studied perfectly in Refs. [16]–[19]. However, the methods in the references cannot be extended directly to solve the mean-square composite-rotating consensus problem because the system matrix is a complex matrix and the traditional Lyapunov function method is invalid.

Motivated by the works mentioned above, we consider the mean-square composite-rotating consensus problems of second-order systems with communication noises, which can describe the actual physical system more precisely. A time-varying consensus gain is introduced to the distributed protocol and sufficient conditions for achieving the mean-square

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composite-rotation consensus are obtained. There exist extensive applications about composite rotating consensus, such as vehicles control, satellite control, multi-robot systems, and so on. The main contributions of this paper are summarized as follows. (i) In contrast with Ref. [14], where the composite-rotation consensus problem was solved for systems without noises, this paper considers multi-agent systems with communication noises, which is more reasonable and realistic. (ii) In contrast with Ref. [17], the method used in Ref. [17] cannot solve the mean-square composite-rotating problem, because the state transition matrix of the system in that paper cannot be obtained accurately. (iii) The method used in this paper is also different from that in Refs. [16]–[19], here, we make a new linear nonsingular transformation on the system equation firstly, then by solving the system equation and using the property of complex, the sufficient conditions for achieving the mean-square composite-rotating consensus are obtained.

**Notions** $\mathbb{R},\mathbb{C}$ represent the sets of real number and complex number; $\otimes$ represents the Kronecker product, j represents the imaginary unit; $\boldsymbol{I}_n$ represents the $n$-dimensional unit matrix, and $\mathbf{1}_n$ represents the vector that all elements are 1; $\boldsymbol{A}^*$ represents the conjugate transpose of matrix $\boldsymbol{A}$; for complex vector $\boldsymbol{x}\in\mathbb{C}^n$, $||\boldsymbol{x}||=\boldsymbol{x}^*\boldsymbol{x}$; $\mathrm{tr}(\boldsymbol{P})$ represents the trace of $\boldsymbol{P}$.

## 2. Preliminaries

An undirected graph $\mathcal{G}$ consists of two sets $V_{\mathcal{G}}=\{1,\ldots,n\}$ and $\mathcal{E}_{\mathcal{G}}=\{(i,j)|i,j\in V_{\mathcal{G}}\}$. If $(i,j)\in\mathcal{E}$, then we say node $j$ is the neighbor of $i$, and the neighbors of $i$ are defined as $N_i=\{j|(i,j)\in\mathcal{E}\}$. The adjacency matrix of $\mathcal{G}$ is defined as $\boldsymbol{A}_{n\times n}$, and if $(i,j)\in\mathcal{E}_{\mathcal{G}}$, $a_{ij}>0$; otherwise, $a_{ij}=0$. The degree matrix of $\mathcal{G}$ is defined as $\boldsymbol{D}=\mathrm{diag}\{d_1,d_2,\ldots,d_n\}$, where $d_i=\Sigma_{j\in N_i}a_{ij}$, $i=1,2,\ldots,n$. The Laplacian matrix of $\mathcal{G}$ is defined as $\boldsymbol{L}=\boldsymbol{D}-\boldsymbol{A}$. A path in $\mathcal{G}$ is given by a sequence of edges $(i_{s-1},\mathrm{j}),(i_{s-2},i_{s-1}),\ldots,(i,i_1)$, and the graph $\mathcal{G}$ is connected if at least one path exists between any two nodes. By the definition of a Laplasse matrix, we can obtain the spectrum of the Laplacian matrix of a connected and undirected graph, which can be ordered as $0=\lambda_1<\lambda_2\leq\cdots\leq\lambda_N$. Also $\mathbf{1}_n$ is the eigenvector associated with the zero eigenvalue $\lambda_1$.[20]

**Lemma 1**[6] If $\mathcal{G}$ is undirected and connected, there exists an orthogonal matrix $\boldsymbol{U}$ with last column $\frac{1}{\sqrt{n}}\mathbf{1}_n$ such that $\boldsymbol{U}^{\mathrm{T}}\boldsymbol{L}\boldsymbol{U}=\mathrm{diag}\{0,\lambda_2,\ldots,\lambda_n\}$. Let $\boldsymbol{U}=\left(\frac{1}{\sqrt{n}}\mathbf{1}_n\ \bar{\boldsymbol{U}}\right)$, where $\bar{\boldsymbol{U}}\in\mathbb{R}^{n\times n-1}$, then, $\bar{\boldsymbol{U}}^{\mathrm{T}}\boldsymbol{L}\bar{\boldsymbol{U}}=\tilde{\boldsymbol{L}}=\mathrm{diag}\{\lambda_2,\ldots,\lambda_n\}$.

## 3. System model

Consider a second order multi-agent system consisting of $n$ agents, the dynamic of each agent is as follows:

$$\begin{aligned}\dot{x}_i(t)&=v_i(t),\\ \dot{v}_i(t)&=u_i(t),\end{aligned}\tag{1}$$

where $x_i(t)\in\mathbb{C}$, $v_i(t)\in\mathbb{C}$, and $u_i(t)\in\mathbb{C}$ represent the position, velocity, and control input of the $i$-th agent, respectively.

Assume that the angular velocities of all agents are $\omega_1$, the information exchange among the agents is affected by communication noises, and the $i$-th agent received information from its neighbor agents is

$$\boldsymbol{Y}_{ik}(t)=\begin{pmatrix}Y_{xik}(t)\\ Y_{vik}(t)\end{pmatrix}=\begin{pmatrix}x_k(t)+\sigma_{xik}\eta_{xik}\\ v_k(t)+\sigma_{vik}\eta_{vik}\end{pmatrix},\quad k\in N_i,\tag{2}$$

where $\{\eta_{xik},\eta_{vik}\}$ represent independent standard white noises, and $\{\sigma_{xik}\in\mathbb{C},\sigma_{vik}\in\mathbb{C},i,k=1,2,\ldots,n,\}$ are the noises intensities.

**Definition 1** The mean-square composite-rotating consensus of the multi-agent system is said to be achieved if for $i=1,2,\ldots,n$,

$$\lim_{t\to\infty}E||x_i(t)-x_k(t)||^2=0,\tag{3}$$

$$\lim_{t\to\infty}E||v_i(t)-v_k(t)||^2=0,\tag{4}$$

$\lim\limits_{t\to\infty}\mathrm{Var}[x_i(t)]<\infty$, $\lim\limits_{t\to\infty}\mathrm{Var}[v_i(t)]<\infty$, and for given $\omega_1>0,\omega_2>0$,

$$\lim_{t\to\infty}E||\dot{x}_i(t)+\mathrm{j}\omega_1^{-1}\dot{v}_i(t)-\mathrm{j}\omega_2(x_i(t)+\mathrm{j}w_1^{-1}v_i(t))||^2=0.\tag{5}$$

**Remark 1** In Definition 1, equations (3) and (4) mean that the positions and velocities of all agents reach consensus eventually in the sense of mean square. The terms $\mathrm{j}\omega_1^{-1}v_i(t)$ and $x_i(t)+\mathrm{j}\omega_1^{-1}v_i(t)$ represent the rotating radius and the rotation center of the $i$-th agent, respectively. Equation (5) means that the circle centers $x_i(t)+\mathrm{j}\omega_1^{-1}v_i(t),i=1,2,\ldots,n$ eventually rotate around the origin of the complex plane with angular velocity $\omega_2$ in the sense of mean square.

## 4. Mean-square composite-rotating consensus

In this section, we study the mean-square composite-rotating consensus problem of multi-agent system (1), the following protocol is adopted:

$$\begin{aligned}u_i(t)&=u_{i1}(t)+u_{i2}(t),\\ u_{i1}(t)&=\mathrm{j}\omega_1v_i(t)+\omega_1\omega_2\left(x_i(t)+\mathrm{j}\frac{v_i(t)}{\omega_1}\right),\\ u_{i2}(t)&=a(t)\sum_{k\in N_i}a_{ik}(Y_{vik}(t)-v_i(t))\\ &\quad+a(t)\sum_{k\in N_i}a_{ik}\left[Y_{xik}(t)+\mathrm{j}\frac{Y_{vik}(t)}{\omega_1}-\left(x_i+\mathrm{j}\frac{v_i(t)}{\omega_1}\right)\right],\end{aligned}\tag{6}$$

where $k_1>0,k_2>0$ are constants to be determined, $a(t)$ is the time-varying consensus gain to attenuate the effect of the noises.

Let $\boldsymbol{c}(t)=[c_1(t),c_2(t),\ldots,c_n(t)]^{\mathrm{T}}$, $c_i(t)=x_i(t)+\mathrm{j}\frac{v_i(t)}{\omega_1}$, and $\boldsymbol{\xi}(t)=(x_1(t),c_1(t),\ldots,x_n(t),c_n(t))^{\mathrm{T}}$. Then, system (1) with protocol (6) can be written as

$$\dot{\boldsymbol{\xi}}(t)=(\boldsymbol{I}_n\otimes\boldsymbol{A}+a(t)\boldsymbol{L}\otimes\boldsymbol{B})\boldsymbol{\xi}(t)+a(t)\boldsymbol{\Sigma}\boldsymbol{\eta}(t),\tag{7}$$

where

$$\boldsymbol{\Sigma}=\mathrm{diag}\{\boldsymbol{\Sigma}_1,\boldsymbol{\Sigma}_2,\ldots,\boldsymbol{\Sigma}_n\},$$
$$\boldsymbol{\Sigma}_i=\left(a_{i1}\begin{pmatrix}0&0\\ \sigma_{xi1}&\frac{1+j}{\omega_1}\sigma_{vi1}\end{pmatrix},\ldots,a_{in}\begin{pmatrix}0&0\\ \sigma_{xin}&\frac{1+j}{\omega_1}\sigma_{vin}\end{pmatrix}\right),$$

$\boldsymbol{\eta}(t)=(\sigma_{x11},\sigma_{v11},\ldots,\sigma_{x1n},\sigma_{v1n},\ \ldots,\ \sigma_{xn1},\sigma_{vn1},\ \ldots,\ \sigma_{xnn},\sigma_{vnn})^{\mathrm{T}}$, $\boldsymbol{A}=\begin{pmatrix}\mathrm{j}\omega_1&-\mathrm{j}\omega_1\\ 0&\mathrm{j}\omega_2\end{pmatrix}$, $\boldsymbol{B}=\begin{pmatrix}0&0\\ -1&1+\mathrm{j}/\omega_1\end{pmatrix}$.

Take $\boldsymbol{\delta}(t)=(\boldsymbol{I}_n\otimes\boldsymbol{S})\boldsymbol{\xi}(t)$, where $\boldsymbol{S}=\begin{pmatrix}1&-\frac{\omega_1}{\omega_1-\omega_2}\\ 0&\frac{\omega_1}{\omega_1-\omega_2}\end{pmatrix}$, system (7) can be transformed into the following equation:

$$\dot{\boldsymbol{\delta}}(t)=(\boldsymbol{I}_n\otimes\bar{\boldsymbol{A}}+a(t)\boldsymbol{L}\otimes\bar{\boldsymbol{B}})\boldsymbol{\delta}(t)+a(t)(\boldsymbol{I}_n\otimes\boldsymbol{S})\boldsymbol{\Sigma}\boldsymbol{\eta}(t),\quad(8)$$

where $\bar{\boldsymbol{A}}=\begin{pmatrix}\mathrm{j}\omega_1&0\\ 0&\mathrm{j}\omega_2\end{pmatrix}$, and

$$\bar{\boldsymbol{B}}=\begin{pmatrix}\frac{\omega_1}{\omega_1-\omega_2}&\frac{\omega_1}{\omega_1-\omega_2}-1-\frac{\mathrm{j}}{\omega_1}\\ -\frac{\omega_1}{\omega_1-\omega_2}&-\frac{\omega_1}{\omega_1-\omega_2}+1+\frac{\mathrm{j}}{\omega_1}\end{pmatrix}.$$

To proceed, we make the following assumptions:[16]

(A1) the topology corresponding to the multi-agent system (1) is connected;

(A2) convergence condition $\int_0^\infty a(s)\mathrm{d}s=\infty$;

(A3) robust condition: $\int_0^\infty a^2(s)\mathrm{d}s<\infty$.

**Remark 2** Assumption (A1) assures that the communication graph is connected, which means that the information exchanges among the agents are unobstructed. The physical implications of time-varying gain $a(t)$ are as follows. First, Assumption (A2) guarantees all agents can achieve composite rotating motion with a proper rate, which cannot decay too fast. Second, Assumption (A3) is to make the composite rotating consensus control law against measurement noises. Third, the purpose of time-varying is to suppress the communication noises and make protocol (6) robust to communication noises, while the protocol with constant gain is not robust to communication noises, which will be shown in the simulation section.

**Lemma 2** The eigenvalues of $\bar{\boldsymbol{A}}-\lambda\bar{\boldsymbol{B}}$ have negative real parts, where $\lambda>0$.

**Proof** According to Lemma 3 of Ref. [14], the eigenvalues of $\boldsymbol{A}+\lambda\boldsymbol{B}$ have negative real parts. Since $\bar{\boldsymbol{A}}-\lambda\bar{\boldsymbol{B}}=\boldsymbol{S}(\boldsymbol{A}+\lambda\boldsymbol{B})\boldsymbol{S}^{-1}$, $\bar{\boldsymbol{A}}-\lambda\bar{\boldsymbol{B}}$ has the same eigenvalues with $\boldsymbol{A}+\lambda\boldsymbol{B}$. So the eigenvalues of $\bar{\boldsymbol{A}}-\lambda\bar{\boldsymbol{B}}$ have negative real parts.

**Theorem 1** Consider multi-agent system (1), the mean-square composite-rotating consensus can be achieved by protocol (6) if Assumptions (A1)–(A3) are satisfied.

**Proof** Let $\boldsymbol{\varphi}(t)=(\boldsymbol{U}^{\mathrm{T}}\otimes\boldsymbol{I}_2)\boldsymbol{\delta}(t)$, where $\boldsymbol{U}$ is defined in Lemma 1, $\boldsymbol{\varphi}(t)=(\boldsymbol{\varphi}_1^{\mathrm{T}}(t),\boldsymbol{\varphi}_2^{\mathrm{T}}(t),\ldots,\boldsymbol{\varphi}_n^{\mathrm{T}}(t))^{\mathrm{T}}$, $\bar{\boldsymbol{\varphi}}^{\mathrm{T}}(t)=\boldsymbol{\varphi}_1(t)$, $\tilde{\boldsymbol{\varphi}}^{\mathrm{T}}(t)=(\boldsymbol{\varphi}_2^{\mathrm{T}}(t),\ldots,\boldsymbol{\varphi}_n^{\mathrm{T}}(t))^{\mathrm{T}}$, then $\boldsymbol{\varphi}(t)=(\bar{\boldsymbol{\varphi}}^{\mathrm{T}}(t),\tilde{\boldsymbol{\varphi}}^{\mathrm{T}}(t))^{\mathrm{T}}$, and

$$\mathrm{d}\bar{\boldsymbol{\varphi}}(t)=A\bar{\boldsymbol{\varphi}}(t)\mathrm{d}t+\frac{1}{\sqrt{n}}a(t)(\boldsymbol{1}_n^{\mathrm{T}}\otimes\boldsymbol{S})\boldsymbol{\Sigma}\mathrm{d}\boldsymbol{W}(t),\quad(9)$$
$$\mathrm{d}\tilde{\boldsymbol{\varphi}}(t)=(\boldsymbol{I}_{n-1}\otimes\boldsymbol{A}-a(t)\tilde{\boldsymbol{L}}\otimes\boldsymbol{B})\tilde{\boldsymbol{\varphi}}(t)\mathrm{d}t+a(t)(\bar{\boldsymbol{U}}^{\mathrm{T}}\otimes\boldsymbol{S})\boldsymbol{\Sigma}\mathrm{d}\boldsymbol{W}(t).\quad(10)$$

The solution of Eq. (10) can be written in form

$$\tilde{\boldsymbol{\varphi}}(t)=\mathrm{e}^{\int_0^t(\boldsymbol{I}_{n-1}\otimes\boldsymbol{A}-a(s)\tilde{\boldsymbol{L}}\otimes\boldsymbol{B})\mathrm{d}s}\tilde{\boldsymbol{\varphi}}(0)+\int_0^t a(s)\,\mathrm{e}^{\int_\tau^t(\boldsymbol{I}_{n-1}\otimes\boldsymbol{A}-a(s)\tilde{\boldsymbol{L}})\mathrm{d}\tau}\times(\bar{\boldsymbol{U}}^{\mathrm{T}}\otimes\boldsymbol{S})\boldsymbol{\Sigma}\mathrm{d}\boldsymbol{W}(s).\quad(11)$$

Then

$$\begin{aligned}\lim_{t\to\infty}E||\tilde{\boldsymbol{\varphi}}(t)||^2&=\lim_{t\to\infty}E||\,\mathrm{e}^{\int_0^t(\boldsymbol{I}_{n-1}\otimes\bar{\boldsymbol{A}}-a(s)\tilde{\boldsymbol{L}}\otimes\bar{\boldsymbol{B}})\mathrm{d}s}\tilde{\boldsymbol{\varphi}}(0)+\int_0^t a(s)\,\mathrm{e}^{\int_s^t(\boldsymbol{I}_{n-1}\otimes\bar{\boldsymbol{A}}-a(\tau)\tilde{\boldsymbol{L}}\otimes\bar{\boldsymbol{B}})\mathrm{d}\tau}(\bar{\boldsymbol{U}}^{\mathrm{T}}\otimes\boldsymbol{S})\boldsymbol{\Sigma}\mathrm{d}\boldsymbol{W}(s)||^2\\&=\lim_{t\to\infty}E||\,\mathrm{e}^{\int_0^t(\boldsymbol{I}_{n-1}\otimes\bar{\boldsymbol{A}}-a(s)\tilde{\boldsymbol{L}}\otimes\bar{\boldsymbol{B}})\mathrm{d}s}\tilde{\boldsymbol{\varphi}}(0)||^2+\lim_{t\to\infty}E||\int_0^t a(s)\,\mathrm{e}^{\int_s^t(\boldsymbol{I}_{n-1}\otimes\bar{\boldsymbol{A}}-a(\tau)\tilde{\boldsymbol{L}}\otimes\bar{\boldsymbol{B}})\mathrm{d}\tau}(\bar{\boldsymbol{U}}^{\mathrm{T}}\otimes\boldsymbol{S})\boldsymbol{\Sigma}\mathrm{d}\boldsymbol{W}(s)||^2\\&=\lim_{t\to\infty}E||\boldsymbol{\theta}_1||^2+\lim_{t\to\infty}E||\boldsymbol{\theta}_2||^2,\end{aligned}\quad(12)$$

where $\boldsymbol{\theta}_1=\mathrm{e}^{\int_0^t(\boldsymbol{I}_{n-1}\otimes\bar{\boldsymbol{A}}-a(s)\tilde{\boldsymbol{L}}\otimes\bar{\boldsymbol{B}})\mathrm{d}s}\tilde{\boldsymbol{\varphi}}(0)$ and $\boldsymbol{\theta}_2=\int_0^t a(s)\,\mathrm{e}^{\int_s^t(\boldsymbol{I}_{n-1}\otimes\bar{\boldsymbol{A}}-a(\tau)\tilde{\boldsymbol{L}}\otimes\bar{\boldsymbol{B}})\mathrm{d}\tau}(\bar{\boldsymbol{U}}^{\mathrm{T}}\otimes\boldsymbol{S})\boldsymbol{\Sigma}\mathrm{d}\boldsymbol{W}(s)$.

Since Assumptions (A1)–(A3) hold,

$$\begin{aligned}\lim_{t\to\infty}E||\boldsymbol{\theta}_1||^2&=\lim_{t\to\infty}E||\,\mathrm{e}^{(\boldsymbol{I}_{n-1}\otimes\bar{\boldsymbol{A}})\int_0^t(1-a(s))\mathrm{d}s}\,\mathrm{e}^{\int_0^t a(s)(\boldsymbol{I}_{n-1}\otimes\bar{\boldsymbol{A}}-\tilde{\boldsymbol{L}}\otimes\bar{\boldsymbol{B}})\mathrm{d}s}\tilde{\boldsymbol{\varphi}}(0)||^2\\&\leqslant\lim_{t\to\infty}E||\,\mathrm{e}^{\int_0^t a(s)(\boldsymbol{I}_{n-1}\otimes\bar{\boldsymbol{A}}-\tilde{\boldsymbol{L}}\otimes\bar{\boldsymbol{B}})\mathrm{d}s}\tilde{\boldsymbol{\varphi}}(0)||^2\\&=\lim_{t\to\infty}E[\tilde{\boldsymbol{\varphi}}^*(0)(\mathrm{e}^{\int_0^t a(s)(\boldsymbol{I}_{n-1}\otimes\bar{\boldsymbol{A}}-\tilde{\boldsymbol{L}}\otimes\bar{\boldsymbol{B}})\mathrm{d}s})^*\,\mathrm{e}^{\int_0^t a(s)(\boldsymbol{I}_{n-1}\otimes\bar{\boldsymbol{A}}-\tilde{\boldsymbol{L}}\otimes\bar{\boldsymbol{B}})\mathrm{d}s}\tilde{\boldsymbol{\varphi}}(0)]\\&\leq\lim_{t\to\infty}\mathrm{e}^{2\mathrm{Real}(\lambda_{\max}(\boldsymbol{I}_{n-1}\otimes\bar{\boldsymbol{A}}-\tilde{\boldsymbol{L}}\otimes\bar{\boldsymbol{B}}))\int_0^t a(s)\mathrm{d}s}E[||\tilde{\boldsymbol{\varphi}}(0)||^2]=0.\end{aligned}\quad(13)$$

For any given $\varepsilon>0$, from Assumption (A3), there exists a $t_0$ such that $\int_{t_0}^\infty a^2(s)\mathrm{d}s<\varepsilon$, so

$$\int_0^t a^2(s)\,\mathrm{e}^{2\mathrm{Real}(\lambda_{\max}(\boldsymbol{I}_{n-1}\otimes\bar{\boldsymbol{A}}-\tilde{\boldsymbol{L}}\otimes\bar{\boldsymbol{B}}))\int_s^t a(\tau)\mathrm{d}\tau}\mathrm{d}s$$

$$
\begin{aligned}
&= \int_0^{t_0} a^2(s)\,\mathrm{e}^{2\mathrm{Real}(\lambda_{\max}(\boldsymbol{I}_{n-1}\otimes\bar{\boldsymbol{A}}-\tilde{\boldsymbol{L}}\otimes\bar{\boldsymbol{B}}))\int_s^t a(\tau)\mathrm{d}\tau}\mathrm{d}s + \int_{t_0}^{t} a^2(s)\,\mathrm{e}^{2\mathrm{Real}(\lambda_{\max}(\boldsymbol{I}_{n-1}\otimes\bar{\boldsymbol{A}}-\tilde{\boldsymbol{L}}\otimes\bar{\boldsymbol{B}}))\int_s^t a(\tau)\mathrm{d}\tau}\mathrm{d}s\\
&\le \mathrm{e}^{2\mathrm{Real}(\lambda_{\max}(\boldsymbol{I}_{n-1}\otimes\bar{\boldsymbol{A}}-\tilde{\boldsymbol{L}}\otimes\bar{\boldsymbol{B}}))\int_{t_0}^t a(\tau)\mathrm{d}\tau}\int_0^{t_0} a^2(s)\mathrm{d}s + \int_0^{t_0} a^2(s)\mathrm{d}s\\
&\le \mathrm{e}^{2\mathrm{Real}(\lambda_{\max}(\boldsymbol{I}_{n-1}\otimes\bar{\boldsymbol{A}}-\tilde{\boldsymbol{L}}\otimes\bar{\boldsymbol{B}}))\int_{t_0}^t a(\tau)\mathrm{d}\tau}\int_0^{\infty} a^2(s)\mathrm{d}s + \int_0^{t_0} a^2(s)\mathrm{d}s\\
&\le \mathrm{e}^{2\mathrm{Real}(\lambda_{\max}(\boldsymbol{I}_{n-1}\otimes\bar{\boldsymbol{A}}-\tilde{\boldsymbol{L}}\otimes\bar{\boldsymbol{B}}))\int_{t_0}^t a(\tau)\mathrm{d}\tau}\int_0^{\infty} a^2(s)\mathrm{d}s + \varepsilon.
\end{aligned}
$$

By Assumption (A2) and Lemma 2,

$$
\lim_{t\to\infty}\mathrm{e}^{2\mathrm{Real}(\lambda_{\max}(\boldsymbol{I}_{n-1}\otimes\bar{\boldsymbol{A}}-\tilde{\boldsymbol{L}}\otimes\bar{\boldsymbol{B}}))\int_{t_0}^t a(\tau)\mathrm{d}\tau} = 0,
$$

$$
\begin{aligned}
\lim_{t\to\infty}E[\boldsymbol{\theta}_2\boldsymbol{\theta}_2^*] &\le \lim_{t\to\infty}E\int_0^t a^2(s)\,\mathrm{e}^{\int_s^t(\boldsymbol{I}_{n-1}\otimes\bar{\boldsymbol{A}}-a(\tau)\tilde{\boldsymbol{L}}\otimes\bar{\boldsymbol{B}})\mathrm{d}\tau}\boldsymbol{\Xi}(\mathrm{e}^{\int_s^t(\boldsymbol{I}_{n-1}\otimes\bar{\boldsymbol{A}}-a(\tau)\tilde{\boldsymbol{L}}\otimes\bar{\boldsymbol{B}})\mathrm{d}\tau})^*\mathrm{d}s\\
&\le \lambda_{\max}(\boldsymbol{\Xi})\lim_{t\to\infty}E\int_0^t a^2(s)\,\mathrm{e}^{\int_s^t a(\tau)(\boldsymbol{I}_{n-1}\otimes\bar{\boldsymbol{A}}-\tilde{\boldsymbol{L}}\otimes\bar{\boldsymbol{B}})\mathrm{d}\tau}(\mathrm{e}^{\int_s^t a(\tau)(\boldsymbol{I}_{n-1}\otimes\bar{\boldsymbol{A}}-\tilde{L}\otimes\bar{\boldsymbol{B}})\mathrm{d}\tau})^*\mathrm{d}s\\
&\le \lambda_{\max}(\boldsymbol{\Xi})\lim_{t\to\infty}E\int_0^t (n-1)a^2(s)\,\mathrm{e}^{2\mathrm{Real}(\lambda_{\max}(\boldsymbol{I}_{n-1}\otimes\bar{\boldsymbol{A}}-\tilde{\boldsymbol{L}}\otimes\bar{\boldsymbol{B}}))\int_s^t a(\tau)\mathrm{d}\tau}\mathrm{d}s = 0, \qquad (14)
\end{aligned}
$$

where $\boldsymbol{\Xi} = (\bar{\boldsymbol{U}}^{\mathrm{T}}\otimes\boldsymbol{S})\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{\mathrm{T}}(\bar{\boldsymbol{U}}\otimes\boldsymbol{S}^{\mathrm{T}})$.

Hence $\lim_{t\to\infty}E||\boldsymbol{\theta}_2||^2 = \lim_{t\to\infty}E[\mathrm{tr}(\boldsymbol{\theta}_2\boldsymbol{\theta}_2^*)] = 0$.

Equation (13) together with equation (14) implies that $\lim_{t\to\infty}E[\tilde{\boldsymbol{\varphi}}(t)] = 0$. Besides,

$$
\bar{\boldsymbol{\varphi}}(t) = \mathrm{e}^{\bar{\boldsymbol{A}}t}\bar{\boldsymbol{\varphi}}(0) + \int_0^t \frac{1}{\sqrt{n}}a(s)\,\mathrm{e}^{\boldsymbol{A}(t-s)}(\boldsymbol{1}_n^{\mathrm{T}}\otimes\boldsymbol{S})\boldsymbol{\Sigma}\mathrm{d}\boldsymbol{W}(s). \qquad (15)
$$

So

$$
\lim_{t\to\infty}\boldsymbol{\delta}(t) = \left(\frac{1}{\sqrt{n}}\boldsymbol{1}_n\otimes\boldsymbol{I}_2\right)\lim_{t\to\infty}\bar{\boldsymbol{\varphi}}(t),
$$

$$
\begin{aligned}
\lim_{t\to\infty}\boldsymbol{\xi}(t) &= \lim_{t\to\infty}(\boldsymbol{I}_n\otimes\boldsymbol{S}^{-1})\boldsymbol{\delta}(t)\\
&= \left(\frac{1}{\sqrt{n}}\boldsymbol{1}_n\otimes\boldsymbol{S}^{-1}\right)\lim_{t\to\infty}\bar{\boldsymbol{\varphi}}(t).
\end{aligned}
$$

Then,

$$
\boldsymbol{\xi}_i(t) = \begin{pmatrix} x_i(t)\\ v_i(t)\end{pmatrix} = \frac{1}{\sqrt{n}}\boldsymbol{S}^{-1}\bar{\boldsymbol{\varphi}}(t),
$$

which implies $\lim_{t\to\infty}E||x_i(t)-x_k(t)||^2 = 0$, $\lim_{t\to\infty}E||v_i(t)-v_k(t)||^2 = 0$, and $\lim_{t\to\infty}E||\dot{c}_i(t)-\mathrm{j}\omega_2 c_i(t)||^2 = \lim_{t\to\infty}a^2(t)\mathrm{tr}(\boldsymbol{\Sigma}^{\mathrm{T}}\boldsymbol{\Sigma}) = 0$. Therefore, the composite-rotating consensus of multi-agent system (1) is achieved, the proof is completed.

**Remark 3** Since the communication topology is undirected and connected, we obtain

$$
\frac{1}{n}\sum_{i=1}^n\dot{c}_i(t) = \frac{1}{n}\mathrm{j}\omega_2\sum_{i=1}^n c_i(t) + a(t)\tilde{\Sigma}\tilde{\eta}(t).
$$

Hence

$$
\begin{aligned}
\frac{1}{n}\mathrm{j}\omega_2\sum_{i=1}^n c_i(t) &= \mathrm{e}^{\mathrm{j}\omega_2 t}\frac{1}{n}\mathrm{j}\omega_2\sum_{i=1}^n c_i(0)\\
&\quad + \int_0^t \mathrm{e}^{\mathrm{j}\omega_2(t-s)}a(s)\tilde{\Sigma}\tilde{\eta}(s)\mathrm{d}s,
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{n}\sum_{i=1}^n\dot{r}_i(t) &= \frac{1}{n}\sum_{i=1}^n\mathrm{j}\omega_1 r_i(t) - \frac{1}{n}\sum_{i=1}^n\mathrm{j}\omega_1 c_i(t),\\
\frac{1}{n}\sum_{i=1}^n r_i(t) &= \mathrm{e}^{\mathrm{j}\omega_1 t}\frac{1}{n}\sum_{i=1}^n r_i(0)\\
&\quad + \frac{\omega_1}{\omega_1+\omega_2}(\mathrm{e}^{\mathrm{j}\omega_2 t}-\mathrm{e}^{\mathrm{j}\omega_1 t})\frac{1}{n}\sum_{i=1}^n c_i(0)\\
&\quad + \int_0^t \mathrm{e}^{\mathrm{j}\omega_1(t-s)}\mathrm{j}\omega_1\int_0^s \mathrm{e}^{\mathrm{j}\omega_2(s-\tau)}a(\tau)\tilde{\Sigma}\tilde{\eta}\mathrm{d}(\tau)\mathrm{d}s.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&E\left[\frac{1}{n}\sum_{i=1}^n c_i(t) - \frac{1}{n}\sum_{i=1}^n r_i(t)\right]\\
&= \mathrm{e}^{\mathrm{j}\omega_2 t}\left(\mathrm{e}^{-\mathrm{j}\omega_1 t-\mathrm{j}\omega_2 t}\frac{1}{n}\sum_{i=1}^n c_i(0) + \frac{\omega_1}{\omega_1+\omega_2}\frac{1}{n}\sum_{i=1}^n c_i(0)\right.\\
&\quad \left. + \frac{\omega_1}{\omega_1+\omega_2}\mathrm{e}^{-\mathrm{j}\omega_1 t-\mathrm{j}\omega_2 t} - \frac{1}{n}\sum_{i=1}^n c_i(0)\right).
\end{aligned}
$$

Therefore, the expectation of $\mathrm{e}^{-\mathrm{j}\omega_1 t-\mathrm{j}\omega_2 t}\frac{1}{n}\sum_{i=1}^n c_i(0) + \frac{\omega_1}{\omega_1+\omega_2}\frac{1}{n}\sum_{i=1}^n c_i(0) + \frac{\omega_1}{\omega_1+\omega_2}\mathrm{e}^{-\mathrm{j}\omega_1 t-\mathrm{j}\omega_2 t} - \frac{1}{n}\sum_{i=1}^n c_i(0)$ rotates around the point $\frac{-\omega_2}{\omega_1+\omega_2}\frac{1}{n}\sum_{i=1}^n c_i(0)$ with radius $\frac{1}{n}(\sum_{i=1}^n r_i(0) + \frac{\omega_1}{\omega_1+\omega_2}\frac{1}{n}\sum_{i=1}^n c_i(0))$.

**Remark 4** Assumption (A1) is also necessary for achieving the mean-square composite consensus. If (A1) does not hold, there exist at least two vectors $\boldsymbol{f}_1, \boldsymbol{f}_2$ that $\boldsymbol{f}_1^{\mathrm{T}}\boldsymbol{L} = 0, \boldsymbol{f}_2^{\mathrm{T}}\boldsymbol{L} = 0$, and $\boldsymbol{f}_1 \ne \boldsymbol{f}_2$, then, $\lim_{t\to\infty}E[(\boldsymbol{f}_1^{\mathrm{T}}\otimes\boldsymbol{I}_2)\boldsymbol{\delta}(t)] = \lim_{t\to\infty}E\,\mathrm{e}^{\bar{\boldsymbol{A}}t}(\boldsymbol{f}_1^{\mathrm{T}}\otimes\boldsymbol{I}_2)\boldsymbol{\delta}(0)$, $\lim_{t\to\infty}E[\boldsymbol{f}_2^{\mathrm{T}}\boldsymbol{\delta}(t)] = \lim_{t\to\infty}E[\mathrm{e}^{\bar{\boldsymbol{A}}t}(\boldsymbol{f}_2^{\mathrm{T}}\otimes\boldsymbol{I}_2)\boldsymbol{\delta}(0)]$, which means that $((\boldsymbol{f}_1^{\mathrm{T}}-\boldsymbol{f}_2)^{\mathrm{T}}\otimes\boldsymbol{I}_2)\boldsymbol{\delta}(t) = 0$. From the arbitrariness of the initial state, we obtain $((\boldsymbol{f}_1^{\mathrm{T}}-\boldsymbol{f}_2)^{\mathrm{T}}\otimes\boldsymbol{I}_2) = 0$, which is a contradiction.

**Remark 5** Assumption (A3) is necessary for achieving the mean-square composite consensus. If Assumption (A3) does not hold, according to Eq. (13), $\boldsymbol{\delta}(t)$ cannot achieve the

mean-square composite-rotating consensus with finite second-order moment, which is verified in Figs. 5 and 6. The proof of the necessity of Assumption (A2) is challenging work, but we show the necessity by simulations, which are shown in Figs. 7 and 8.

**Remark 6** Cheng *et al*. obtained the sufficient and necessary condition for achieving the mean-square consensus of second-order multi-agent systems with communication noises by solving the state transition matrix of the system equation accurately.[17] However, the method therein cannot be used to solve the mean-square composite-rotating consensus in this paper, because the state transition matrix of the system equation in this paper cannot be obtained precisely. In this paper, we transform the closed-loop system into an equivalent system by introducing a linear nonsingular transformation, then, by using the properties of complex matrix and matrix inequalities, sufficient conditions for achieving the mean-square composite-rotating consensus are obtained. The system in Ref. [14] is the special case of this paper where all the noises intensities $\eta_{xij}, \eta_{vij}, i, j = 1, \ldots, n$, are zero.

## 5. Simulations

Consider a multi-agent system consisting of 4 agents, the topology structure of the agents is shown in Fig. 1, the noise intensities are all $1+i$, and the time-varying consensus gain $a(t) = \frac{2}{1+t}$. Take the initial state $\boldsymbol{x}(0) = (-3-5\mathrm{j}, 2+4\mathrm{j}, -1+6\mathrm{j}, 3+5\mathrm{j})^{\mathrm{T}}, \boldsymbol{v}(0) = (2,3,-1,2)$, and $\omega_1 = 0.8, \omega_2 = 0.02$.

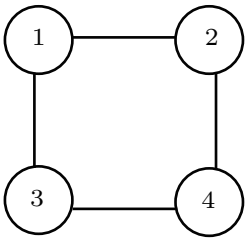



**Fig. 1.** Network topology.

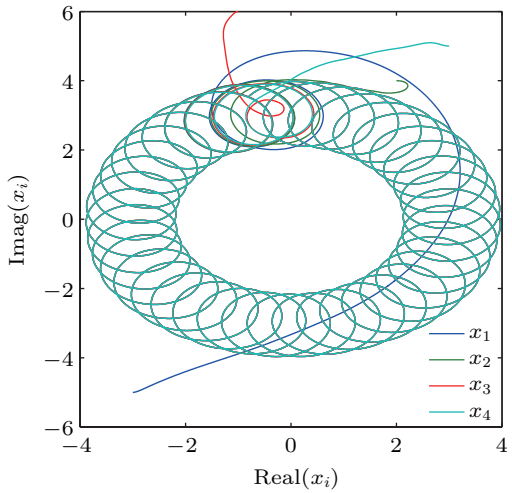



**Fig. 2.** (color online) The positions of all agents on complex plane.

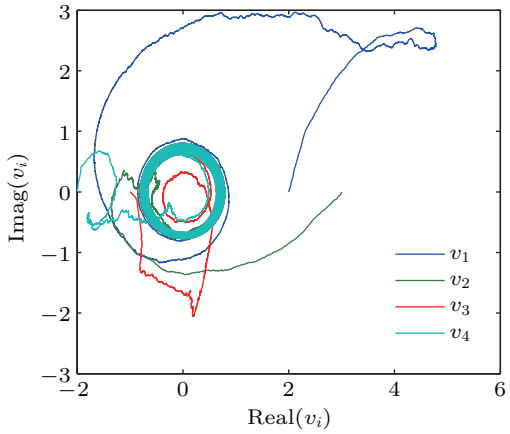



**Fig. 3.** (color online) The velocities of all agents on complex plane.

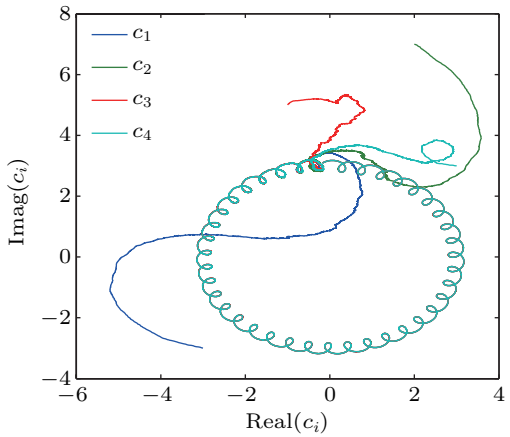



**Fig. 4.** (color online) The trajectories of all circle centers on complex plane.

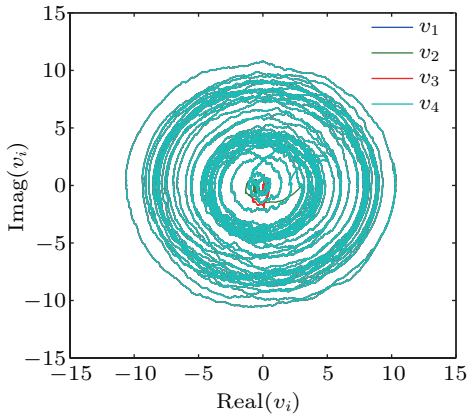



**Fig. 5.** (color online) The velocities of all agents on complex plane ($a(t) = 2$).

The evolution of positions, velocities, and the center of rotating are presented in Figs. 2–4, which show that the positions and velocities of all agents can achieve composite rotation consensus and the center of rotation can spin around (0,0), i.e., the mean-square composite-rotating consensus can be achieved. Figures 5 and 6 present the trajectories of the velocities and circle centers of all agents when $a(t) = 2$, which

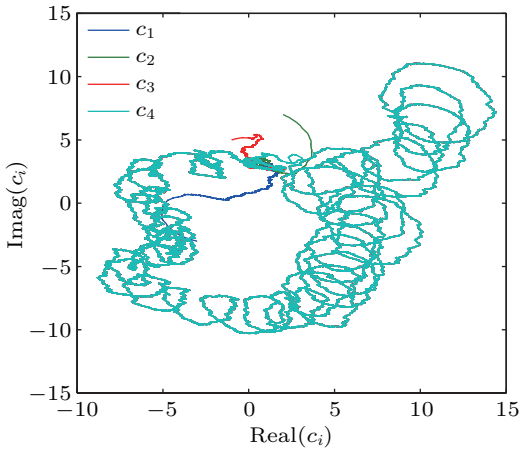



**Fig. 6.** (color online) The trajectories of all circle centers on complex plane ($a(t) = 2$).

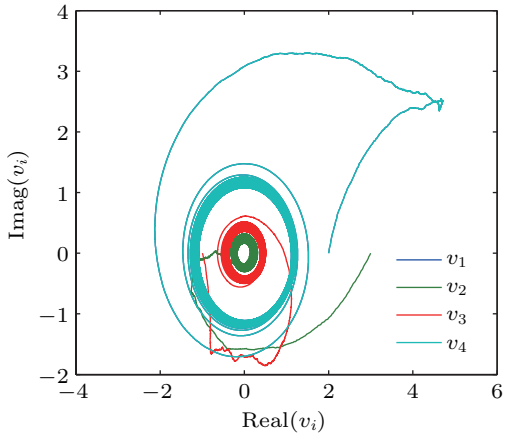



**Fig. 7.** (color online) The velocities of all agents on complex plane ($a(t) = \frac{2}{(1+t)^2}$).

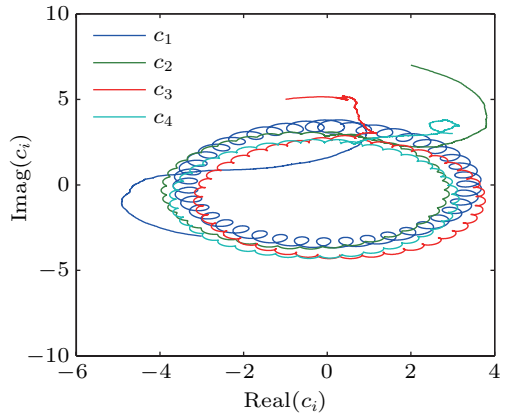



**Fig. 8.** (color online) The trajectories of all circle centers on complex plane ($a(t) = \frac{2}{(1+t)^2}$).

show that the effect of communication noises cannot be attenuated in this situation. Figures 7 and 8 present the trajectories of the velocities and circle centers of all agents when $a(t) = \frac{2}{(1+t)^2}$, which show that the mean-square composite-rotating consensus cannot also be achieved. Hence, Assumptions (A2) and (A3) are necessary for achieving the mean-square composite-rotating consensus.

## 6. Conclusion

We study the mean-square composite-rotating consensus problem of second-order multi-agent systems with communication noises. A distributed protocol is proposed and a time-varying consensus gain is introduced to attenuate the effect of communication noises. By using a linear nonsingular transformation, the closed-loop system is transformed into an equivalent system firstly. Then, sufficient conditions are obtained for achieving the mean-square composite-rotating consensus by analyzing the system equation. In our future work, event-trigger composite-rotation control of multi-agent systems with communication noises will be considered.

## References

[1] Fax J A and Murray R M 2004 *IEEE Trans. Autom. Control* **49** 1465
[2] Ren W and Atkins E 2007 *International Journal of Robust and Nonlinear Control* **17** 1002
[3] Vicsek T, Czirok A and Benjacob E 1995 *Phys. Rev. Lett.* **75** 1226
[4] Jadbabaie A, Lin J and Morse A S 2003 *IEEE Trans. Autom. Control* **48** 988
[5] Olfati-Saber R and Murray R M 2004 *IEEE Trans. Autom. Control* **49** 1520
[6] Ren W and Beard R W 2005 *IEEE Trans. Autom. Control* **50** 655
[7] Fiorelli E, Leonard N E and Bhatta P 2006 *IEEE/OES Autonomous Underwater Vehicles* 134
[8] Sepulchre R, Paley D A and Leonard N E 2007 *IEEE Trans. Autom. Control* **52** 811
[9] Sepulchre R, Paley D A and Leonard N E 2008 *IEEE Trans. Autom. Control* **53** 706
[10] Ren W 2009 *IEEE Trans. Autom. Control* **54** 1330
[11] Lin P and Jia Y 2010 *Systems & Control Letters* **59** 587
[12] Lin P, Qin K and Li Z 2011 *Systems & Control Letters* **60** 365
[13] Mo L and Yu Y 2017 *Acta Automatica Sin.* **43** 1665
[14] Lin P, Lu W and Song Y 2014 *Chin. Phys. B* **23** 040503
[15] Huang W, Zheng W and Mo L 2017 *International Journal of Distributed Sensor Networks* **13** 1
[16] Li T and Zhang J 2009 *Automatica* **45** 1929
[17] Cheng L, Hou Z and Tan M 2011 *IEEE Trans. Autom. Control* **56** 1958
[18] Cheng L, Hou Z and Tan M 2014 *IEEE Trans. Autom. Control* **59** 261
[19] Guo S, Mo L and Yu Y 2018 *Journal of Franklin Institute* **355** 3717
[20] Godsil C and Royle G 2001 *Algebraic Graph Theory* (New York: Springer-Verlag)