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Uncertainty relations in the product form*

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We study the uncertainty relation in the product form of variances and obtain some new uncertainty relations with weight, which are shown to be tighter than those derived from the Cauchy–Schwarz inequality.

Keywords: uncertainty relation, observables, generalized Cauchy–Schwarz inequality

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1. Introduction

The uncertainty relations have played a fundamental role in the development of quantum theory not only in the foundation but also in recent investigations of quantum information and quantum communication, in particular in the areas such as entanglement detection,[1,2] security analysis of quantum key distribution in quantum cryptography,[3] quantum metrology, and quantum speed limit.[4–6] Usually the uncertainty relations are expressed in terms of the product of variances of the measurement results of two incompatible observables. Other forms of the uncertainty relations include entropic uncertainty principle,[7–9] variance-based sum uncertainty relation,[10–12] the skew information,[13,14] the majorization technique,[15–17] and the recent weighted uncertainty relation.[18] Recently, several experimental investigations were also performed to check the corresponding relations.[19,20]

In 1927, Heisenberg[21] analyzed the observation of an individual electron with photons and obtained the famous uncertainty principle

$$(\Delta P)^2(\Delta Q)^2 \geq \left(\frac{\hbar}{2}\right)^2, \tag{1}$$

where $(\Delta P)^2$ and $(\Delta Q)^2$ are the variances of the position P and momentum Q, respectively. The variance or standard deviation of the observable A with respect to the state ρ is defined by $(\Delta A)^2 = \langle A^2\rangle - \langle A\rangle^2$, where $\langle A\rangle = \mathrm{tr}\rho A$ is the mean value of the observable A.

The inequality (1) shows that the uncertainties in the position and momentum of a quantum particle are inversely proportional to each other: a particle's position and momentum cannot be known simultaneously. Thus the accuracy of quantum measurement is limited by the uncertainty principle. This principle uncovers a fundamental and peculiar feature in the atomic world, and is considered as one of the cornerstones of quantum mechanics.

Robertson[22] formulated the uncertainty relation for arbitrary pair of non-commuting observables A and B (with bounded spectra) as follows:

$$(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}|\langle [A,B]\rangle|^2, \tag{2}$$

where $\langle [A,B]\rangle = \mathrm{Tr}\rho[A,B]$, the expectation value of the commutator $[A,B] = AB - BA$.

Robertson's uncertainty relation was further generalized by Schrödinger to be[3]

$$(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}|\langle [A,B]\rangle|^2 + \frac{1}{4}|\langle \{A,B\}\rangle - \langle A\rangle\langle B\rangle|^2. \tag{3}$$

This relation is evidently stronger than Heisenberg's uncertainty relation, and it also shows that the commutator reveals incompatibility while the anticommutator encodes correlation between observables A and B.

The goal of this note is to give a family of generalized Schrödinger uncertainty relations using a stronger Cauchy–Schwarz inequality.

2. Generalized uncertainty relations

We start with a quantum system in the quantum state $\rho = |\Psi\rangle\langle\Psi|$ on the Hilbert space with the inner product $\langle\ |\ \rangle$ and consider observables A and B. Define the operator $\bar{A} = A - \langle A\rangle I$ associated with a given operator A.

Let $\{|\varphi_i\rangle\}$ be an orthonormal basis of the Hilbert space and write $\bar{A}|\Psi\rangle = \sum_i \alpha_i|\varphi_i\rangle$, $\bar{B}|\Psi\rangle = \sum_i \beta_i|\varphi_i\rangle$ so that $\langle\bar{A}\bar{B}\rangle = \langle\alpha|\beta\rangle$, where $\langle\alpha|\beta\rangle$ is the usual inner product (linear in the second argument) for the vectors $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$. We will not distinguish the two inner products as long as it is clear from the context.

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The variance of observable A can be expressed as $(\Delta A)^2 = \langle A^2\rangle - \langle A\rangle^2 = \langle \bar{A}^2\rangle = \langle\alpha|\alpha\rangle$, thus

$$(\Delta A)^2(\Delta B)^2 = \langle\alpha|\alpha\rangle\langle\beta|\beta\rangle = \sum_{i=1}^{n}|\alpha_i|^2\sum_{i=1}^{n}|\beta_i|^2.$$

Theorem 1 For observables A and B, we have the generalized uncertainty relation in the variance-based product form

$$(\Delta A)^2(\Delta B)^2 \geq \sum_{i=1}^{n}|\alpha_i|^{1+\lambda}|\beta_i|^{1-\lambda}\sum_{i=1}^{n}|\alpha_i|^{1-\lambda}|\beta_i|^{1+\lambda}, \quad (4)$$

where λ is any real number $\in[0,1]$, $\alpha_i = \langle\varphi_i|\bar{A}|\Psi\rangle$, and $\beta_i = \langle\varphi_i|\bar{B}|\Psi\rangle$.

Proof For any real number $\lambda\in[0,1]$, one has the following Callebaut inequality:[24]

$$\Big(\sum_{i=1}^{n}a_ib_i\Big)^2 \leq \sum_{i=1}^{n}a_i^{1+\lambda}b_i^{1-\lambda}\sum_{i=1}^{n}a_i^{1-\lambda}b_i^{1+\lambda} \leq \sum_{i=1}^{n}a_i^2\sum_{i=1}^{n}b_i^2, \quad (5)$$

where $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ are two sequences of non-negative real numbers. Then

$$\begin{aligned}(\Delta A)^2(\Delta B)^2 &= \sum_{i=1}^{n}|\alpha_i|^2\sum_{i}^{n}|\beta_i|^2\\ &\geq \sum_{i=1}^{n}|\alpha_i|^{1+\lambda}|\beta_i|^{1-\lambda}\sum_{i=1}^{n}|\alpha_i|^{1-\lambda}|\beta_i|^{1+\lambda},\end{aligned}$$

so we obtain the inequality in Theorem 1.

Until now we have considered only a pure state $|\Psi\rangle$ of the system, this relation can be extended to the case of mixed states $\rho = \sum p_j|\Psi_j\rangle\langle\Psi_j|$, $p_j\geq 0$ and $\sum_j p_j = 1$. Since $|\alpha_i| = |\langle\Psi|\bar{A}|\varphi_i\rangle| = |\langle\varphi_i|\bar{A}|\Psi\rangle|$, $|\beta_i| = |\langle\Psi|\bar{A}|\varphi_i\rangle|$, and we just need to consider the terms in the sum for both $|\alpha_i|\neq 0$ and $|\beta_i|\neq 0$, so we can rewrite inequality (4) as

$$\begin{aligned}(\Delta A)^2(\Delta B)^2 &\geq \sum_i|\langle\Psi|\bar{A}|\varphi_i\rangle\langle\varphi_i|\bar{B}|\Psi\rangle|\left|\frac{\langle\Psi|\bar{A}|\varphi_i\rangle}{\langle\Psi|\bar{B}|\varphi_i\rangle}\right|^{\lambda}\\ &\quad\times\sum_i|\langle\Psi|\bar{A}|\varphi_i\rangle\langle\varphi_i|\bar{B}|\Psi\rangle|\left|\frac{\langle\Psi|\bar{B}|\varphi_i\rangle}{\langle\Psi|\bar{A}|\varphi_i\rangle}\right|^{\lambda}\\ &= \sum_i|\langle\Psi|(\bar{A}\bar{B})^{\varphi_i}|\Psi\rangle|\left|\frac{\langle\Psi|(\bar{A}\bar{B})^{\varphi_i}|\Psi\rangle}{\langle\Psi|(\bar{B}\bar{B})^{\varphi_i}|\Psi\rangle}\right|^{\lambda}\\ &\quad\times\sum_i|\langle\Psi|(\bar{A}\bar{B})^{\varphi_i}|\Psi\rangle|\left|\frac{\langle\Psi|(\bar{A}\bar{B})^{\varphi_i}|\Psi\rangle}{\langle\Psi|(\bar{A}\bar{A})^{\varphi_i}|\Psi\rangle}\right|^{\lambda},\end{aligned}$$

where $(\bar{A}\bar{B})^{\varphi_i} = \bar{A}|\varphi_i\rangle\langle\varphi_i|\bar{B}$.

Since the mean value of observable A can be expressed in the form of trace, $\langle\Psi|A|\Psi\rangle = \mathrm{Tr}A|\Psi\rangle\langle\Psi|$, we rewrite the inequality above as

$$\begin{aligned}&(\Delta A)^2(\Delta B)^2\\ &\geq \sum_i|\mathrm{Tr}(\bar{A}\bar{B})^{\varphi_i}|\Psi\rangle\langle\Psi||\left|\frac{\mathrm{Tr}(\bar{A}\bar{B})^{\varphi_i}|\Psi\rangle\langle\Psi|}{\mathrm{Tr}(\bar{B}\bar{B})^{\varphi_i}|\Psi\rangle\langle\Psi|}\right|^{\lambda}\\ &\quad\times\sum_i|\mathrm{Tr}(\bar{A}\bar{B})^{\varphi_i}|\Psi\rangle\langle\Psi||\left|\frac{\mathrm{Tr}(\bar{A}\bar{B})^{\varphi_i}|\Psi\rangle\langle\Psi|}{\mathrm{Tr}(\bar{A}\bar{A})^{\varphi_i}|\Psi\rangle\langle\Psi|}\right|^{\lambda}. \quad (6)\end{aligned}$$

For each state $|\Psi_j\rangle$, we have

$$\begin{aligned}&(\Delta A)_j^2(\Delta B)_j^2\\ &\geq \sum_i|\mathrm{Tr}(\bar{A}\bar{B})^{\varphi_i}|\Psi_j\rangle\langle\Psi_j||\left|\frac{\mathrm{Tr}(\bar{A}\bar{B})^{\varphi_i}|\Psi_j\rangle\langle\Psi_j|}{\mathrm{Tr}(\bar{B}\bar{B})^{\varphi_i}|\Psi_j\rangle\langle\Psi_j|}\right|^{\lambda}\\ &\quad\times\sum_i|\mathrm{Tr}(\bar{A}\bar{B})^{\varphi_i}|\Psi_j\rangle\langle\Psi_j||\left|\frac{\mathrm{Tr}(\bar{A}\bar{B})^{\varphi_i}|\Psi_j\rangle\langle\Psi_j|}{\mathrm{Tr}(\bar{A}\bar{A})^{\varphi_i}|\Psi_j\rangle\langle\Psi_j|}\right|^{\lambda}, \quad (7)\end{aligned}$$

where $(\Delta A)_j^2$ and $(\Delta B)_j^2$ are the variances calculated on $|\Psi_j\rangle$. As the varicance of the observables is convex for the mixed state, that is, $(\Delta A)^2\geq\sum_j p_j(\Delta A)_j^2$. By multiplying members by p_j^2 and summing over j, one can obtain the inequality based on the mixed state ρ

$$\begin{aligned}&(\Delta A)^2(\Delta B)^2\\ &\geq \sum_j p_j^2(\Delta A)_j^2(\Delta B)_j^2\\ &= \sum_{i,j}|\mathrm{Tr}(\bar{A}\bar{B})^{\varphi_i}p_j|\Psi_j\rangle\langle\Psi_j||\left|\frac{\mathrm{Tr}(\bar{A}\bar{B})^{\varphi_i}p_j|\Psi_j\rangle\langle\Psi_j|}{\mathrm{Tr}(\bar{B}\bar{B})^{\varphi_i}p_j|\Psi_j\rangle\langle\Psi_j|}\right|^{\lambda}\\ &\quad\times\sum_{i,j}|\mathrm{Tr}(\bar{A}\bar{B})^{\varphi_i}p_j|\Psi_j\rangle\langle\Psi_j||\left|\frac{\mathrm{Tr}(\bar{A}\bar{B})^{\varphi_i}p_j|\Psi_j\rangle\langle\Psi_j|}{\mathrm{Tr}(\bar{A}\bar{A})^{\varphi_i}p_j|\Psi_j\rangle\langle\Psi_j|}\right|^{\lambda}. \quad (8)\end{aligned}$$

Therefore, we obtain the mixed-state extension of inequality (4)

$$\begin{aligned}(\Delta A)^2(\Delta B)^2 &\geq \sum_i|\langle(\bar{A}\bar{B})^{\varphi_i}\rangle|\left|\frac{\langle(\bar{A}\bar{B})^{\varphi_i}\rangle}{\langle(\bar{B}\bar{B})^{\varphi_i}\rangle}\right|^{\lambda}\\ &\quad\times\sum_i|\langle(\bar{A}\bar{B})^{\varphi_i}|\rangle|\left|\frac{\langle|(\bar{A}\bar{B})^{\varphi_i}\rangle}{\langle(\bar{A}\bar{A})^{\varphi_i}\rangle}\right|^{\lambda}. \quad (9)\end{aligned}$$

We remark that our generalized uncertainty relation is stronger than Schrödinger's uncertainty relation, as the generalized Cauchy inequality shows that $|\langle\alpha|\beta\rangle|$ is smaller than the right-hand side of our uncertainty relation.

The uncertainty relation (4) can be tightened by optimizing over the sets of complete orthonormal bases. Then we can improve the uncertainty relation by the Callebaut inequality as in the proof of Theorem 1.

Theorem 2 For observables A and B, one has the following uncertainty relation:

$$\begin{aligned}(\Delta A)^2(\Delta B)^2 &\geq \max_{\{|\varphi_i\rangle\}}\Big(\sum_{i=1}^{n}|\alpha_i|^{1+\lambda}|\beta_i|^{1-\lambda}\\ &\quad\times\sum_{i=1}^{n}|\alpha_i|^{1-\lambda}|\beta_i|^{1+\lambda}\Big) := \mathcal{L}_1, \quad (10)\end{aligned}$$

where λ is any fixed number $\in[0,1]$, $\alpha_i = \langle\varphi_i|\bar{A}|\Psi\rangle$, and $\beta_i = \langle\varphi_i|\bar{B}|\Psi\rangle$.

Theorem 3 For observables A and B, we can obtain the following uncertainty relation:

$$(\Delta A)^2(\Delta B)^2 \geq \max_{\{|\varphi_i\rangle\}}\Big(\sum_{i=1}^{n}(|\alpha_i|^2+|\beta_i|^2)$$

$$\times \sum_{i=1}^{n} \frac{|\alpha_i|^2|\beta_i|^2}{|\alpha_i|^2+|\beta_i|^2}\Big) := \mathcal{L}_2, \qquad (11)$$

where $\alpha_i = \langle\varphi_i|\bar{A}|\Psi\rangle$ and $\beta_i = \langle\varphi_i|\bar{B}|\Psi\rangle$.

Proof Consider another generalized Cauchy–Schwarz inequality, the Milne inequality[25]

$$\Big(\sum_{i=1}^{n} a_i b_i\Big)^2 \le \sum_{i=1}^{n}(a_i^2+b_i^2)\sum_{i=1}^{n}\frac{a_i^2 b_i^2}{a_i^2+b_i^2} \le \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2, \quad (12)$$

where $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ are two sequences of real numbers. Therefore

$$\begin{aligned}(\Delta A)^2(\Delta B)^2 &= \langle\alpha|\alpha\rangle\langle\beta|\beta\rangle = \sum_{i}^{n}|\alpha_i|^2\sum_{i}^{n}|\beta_i|^2 \\ &\ge \sum_{i=1}^{n}(|\alpha_i|^2+|\beta_i|^2)\sum_{i=1}^{n}\frac{|\alpha_i|^2|\beta_i|^2}{|\alpha_i|^2+|\beta_i|^2}.\end{aligned}$$

Remark 1 It is similar to generalize the uncertainty relation (11) from the pure states to the mixed states.

Remark 2 For observables A and B, we can also obtain the following uncertainty relation:

$$(\Delta A)^2(\Delta B)^2 \ge \max\{\mathcal{L}_1, \mathcal{L}_2\}. \qquad (13)$$

Both the Callebaut and Milne inequalities (5) and (12) are stronger than the usual Cauchy–Schwarz inequality, thus our uncertainty relations are tighter than those derived from the Cauchy–Schwaz inequality, for example, the Mondal–Bagchi–Pati's uncertainty inequality in Ref. [26] about observables

$$(\Delta A)^2(\Delta B)^2 \ge \frac{1}{4}\Big(\sum_{n}|\langle[\bar{A},\bar{B}_n^{\varphi}]\rangle_{\Psi}+\langle\{\bar{A},\bar{B}_n^{\varphi}\}\rangle_{\Psi}|\Big)^2, \quad (14)$$

where $\bar{B}_n^{\varphi} = |\varphi_n\rangle\langle\varphi_n|\bar{B}$.

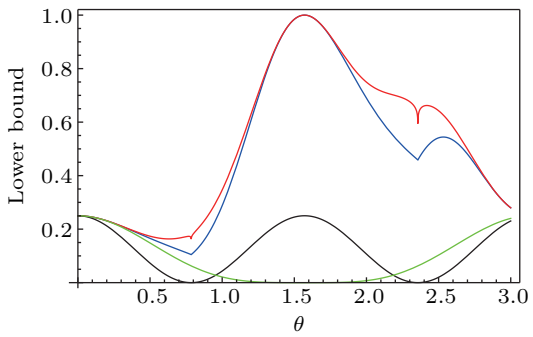


Fig. 1. (color online) The uncertainty relation for observables L_x and L_y at state $|\Psi\rangle$: the blue curve is the lower bound in inequality (4) with the weight $\lambda = 1/10$, the red one is with the weight $\lambda = 9/10$, the black one stands for Mondal–Bagchi–Patis' lower bound in inequality (14), and the green line denotes Schrödinger' lower bound in inequality (3). As observed from the plot, the bound given by inequality (4) is one of the tightest bounds in the literature.

Example 1 We plot in Fig. 1 the lower bound of the product of variances of two incompatible observables $A = L_x$ and $B = L_y$, the two components of the angular momentum of a spin one particle with a state $|\Psi\rangle = \cos\theta|1\rangle - \sin\theta|0\rangle$, where the states $|1\rangle$ and $|0\rangle$ are the eigenvectors of L_z corresponding to eigenvalues 1 and 0, respectively. Take the angular momentum operators with $\hbar = 1$,

$$L_x = \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}, \quad L_y = \frac{1}{\sqrt{2}}\begin{pmatrix}0&-\mathrm{i}&0\\\mathrm{i}&0&-\mathrm{i}\\0&\mathrm{i}&0\end{pmatrix},$$
$$L_z = \begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}. \qquad (15)$$

Example 2 Let us consider a mixed state in the 2×2 quantum system, the isotropic state[27] $\rho_{\rm iso} = (\frac{1-p}{4})I\otimes I + p|\psi_+\rangle\langle\psi_+|$, where $0 \le p < 1$, and $|\psi_+\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d}|ii\rangle$ is the maximal entangled state. We choose observables $A_x = \sigma_x\otimes\sigma_x$ and $B_z = \sigma_z\otimes\sigma_z$, where σ_x and σ_z are the Pauli matrices $\sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\sigma_z = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$. The lower bound of the uncertainty relation (9) is shown in Fig. 2.

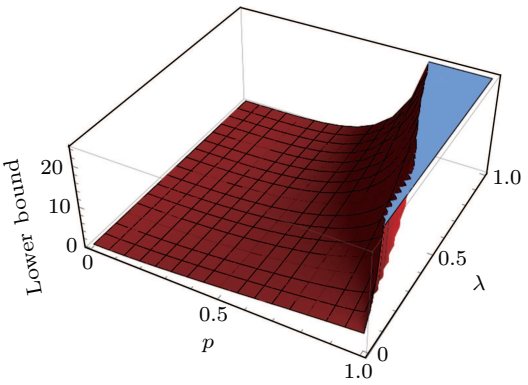


Fig. 2. (color online) The lower bound of the uncertainty relation (9) for observables A_x, B_y at mixed state $\rho_{\rm iso}$, it is clear that the lower bound is bigger as the weight λ and parameter p increase.

3. Conclusion

The uncertainty relations play a central role in the current research in quantum theory and quantum information.[28–30] We have derived a family of new product forms of variance-based uncertainty relations, which are expected to help further investigate the uncertainty relation.

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