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Multi-soliton solutions for the coupled modified nonlinear Schrödinger equations via Riemann–Hilbert approach*

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The coupled modified nonlinear Schrödinger equations are under investigation in this work. Starting from analyzing the spectral problem of the Lax pair, a Riemann–Hilbert problem for the coupled modified nonlinear Schrödinger equations is formulated. And then, through solving the obtained Riemann–Hilbert problem under the conditions of irregularity and reflectionless case, N-soliton solutions for the equations are presented. Furthermore, the localized structures and dynamic behaviors of the one-soliton solution are shown graphically.

Keywords: coupled modified nonlinear Schrödinger equations, Riemann–Hilbert approach, multi-soliton solutions

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1. Introduction

As is well known, the coupled nonlinear Schrödinger type equations[1–3] arise as models to describe different nonlinear waves in an ocean or nonlinear optical fibers, one of which is the coupled modified nonlinear Schrödinger (CMNS) equations

$$\mathrm{i}u_t + u_{xx} + \delta\left[\left(|u|^2+|v|^2\right)u\right] + \mathrm{i}\gamma\left[\left(|u|^2+|v|^2\right)u\right]_x = 0, \quad (1a)$$

$$\mathrm{i}v_t + v_{xx} + \delta\left[\left(|u|^2+|v|^2\right)u\right] + \mathrm{i}\gamma\left[\left(|u|^2+|v|^2\right)u\right]_x = 0. \quad (1b)$$

This system describes the pulse propagation in picosecond or femtosecond regime of the birefringent optical fibers.[4–6] Here u and v represent the slowly varying envelopes for two polarizations in the electric fields, the real parameters δ and γ denote the measure of cubic nonlinear strength and derivative cubic nonlinearity, respectively. In particular, setting $\gamma = 0$, the CMNS system (1) is reduced to the Manakov equations. Taking $\gamma = 0$ as well as $\delta = 2$, the system (1) is called the celebrated Manakov equations.[7] In the case of $\delta = 0$, the CMNS system (1) is changed into the coupled derivative nonlinear Schrödinger equations.[8] There have been plenty of researches on the CMNS system (1). For example, Zhang *et al*.[4] applied the Hirota's bilinear method to obtain the bright one- and two-soliton solutions of the CMNS system (1). The complete or partial energy switching in this coupled system was shown as well by the collision dynamics of the bright vector solitons. In Ref. [5], the integrability was proved by finding a new type of Lax pair and infinitely many conservation laws. In addition, the dark and anti-dark vector soliton solutions were obtained by the Hirota's bilinear method.

Due to the profound theoretical and practical significance, the study of multi-soliton solutions of nonlinear partial differential equations has always been a hot topic. To date, a variety of approaches have been well established to investigate the multi-soliton solutions of nonlinear partial differential equations, such as the inverse scattering method,[9] the Hirota's bilinear method,[10] the Darboux transformation method,[11–13] the Riemann–Hilbert approach,[14] the KP hierarchy reduction method,[15–17] the generalized unified method,[18,19] and so forth. The Riemann–Hilbert approach as a very powerful approach in dealing with nonlinear partial differential equations with initial-boundary value conditions has attracted more and more attention in recent years. By applying the Riemann–Hilbert method, a number of nonlinear partial differential equations have been studied systematically, including the coupled derivative Schrödinger equation,[20] the derivative nonlinear Schrödinger-type equation,[21] the general coupled nonlinear Schrödinger equations,[22] the generalized Camassa–Holm equation[23] with cubic and quadratic nonlinearity, the generalized Sasa–Satsuma equation,[24] the coupled Sasa–Satsuma equation,[25] the two-component Gerdjikov–Ivanov equation,[26] the coupled modified Korteweg–de Vries equation,[27] the coupled higher-order nonlinear Schrödinger equations, and so on.

The aim of the current study is to seek multi-soliton solutions for the coupled modified nonlinear Schrödinger equations (1). The rest of the paper is arranged as follows. In Section 2, we use the spectral problem of the Lax pair related to the CMNS system (1) to formulate a Riemann–Hilbert problem. In Section 3, we discuss the construction of multi-soliton solutions by using the formalism of the Riemann–Hilbert problem. Finally, some discussions and conclusions are given.

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2. The Riemann–Hilbert problem

This section is devoted to formulating a Riemann–Hilbert problem for the CMNS system (1). The Lax pair of the CMNS system (1) reads

$$\psi_x = \boldsymbol{U}\psi = \left[-\frac{\mathrm{i}}{\gamma}\left(\varsigma^2+\frac{\delta}{2}\right)\boldsymbol{\Lambda}+\varsigma\boldsymbol{U}_1\right]\psi, \tag{2a}$$

$$\psi_t = \boldsymbol{V}\psi = \left[\frac{2\mathrm{i}}{\gamma^2}\left(\varsigma^2+\frac{\delta}{2}\right)^2\boldsymbol{\Lambda}-\frac{2}{\gamma}\varsigma^3\boldsymbol{U}_1+\mathrm{i}\varsigma^2\boldsymbol{U}_2-\varsigma\boldsymbol{U}_3\right]\psi. \tag{2b}$$

Here $\psi=(\psi_1,\psi_2,\psi_3)^{\mathrm{T}}$, $\varsigma\in\mathbb{C}$ is a spectral parameter, $\boldsymbol{\Lambda}=\mathrm{diag}(-1,1,1)$,

$$\boldsymbol{U}_1=\begin{pmatrix}0 & u & v\\ u^* & 0 & 0\\ v^* & 0 & 0\end{pmatrix},\quad \boldsymbol{U}_2=\begin{pmatrix}-|u|^2-|v|^2 & 0 & 0\\ 0 & |u|^2 & u^*v\\ 0 & uv^* & |v|^2\end{pmatrix},$$

$$\boldsymbol{U}_3=\begin{pmatrix}0 & A_1 & A_2\\ A_3 & 0 & 0\\ A_4 & 0 & 0\end{pmatrix},$$

where the asterisk stands for the complex conjugation, and

$$\begin{aligned}
A_1&=\frac{\delta}{\gamma}u+\gamma u\left(|u|^2+|v|^2\right)-\mathrm{i}u_x,\\
A_2&=\frac{\delta}{\gamma}v+\gamma v\left(|u|^2+|v|^2\right)-\mathrm{i}v_x,\\
A_3&=\frac{\delta}{\gamma}u^*+\gamma u^*\left(|u|^2+|v|^2\right)+\mathrm{i}u_x^*,\\
A_4&=\frac{\delta}{\gamma}v^*+\gamma v^*\left(|u|^2+|v|^2\right)+\mathrm{i}v_x^*.
\end{aligned}$$

For the sake of simplicity, we only discuss the particular case of $\delta=2$ and $\gamma=1$. Therefore, the Lax pair (2) can be written as

$$\psi_x+\mathrm{i}(\varsigma^2+1)\boldsymbol{\Lambda}\psi=\boldsymbol{V}_1\psi, \tag{3a}$$

$$\psi_t-2\mathrm{i}(\varsigma^2+1)^2\boldsymbol{\Lambda}\psi=\boldsymbol{V}_2\psi, \tag{3b}$$

where

$$\boldsymbol{V}_1=\varsigma\boldsymbol{U}_1,\quad \boldsymbol{V}_2=-2\varsigma^3\boldsymbol{U}_1+\mathrm{i}\varsigma^2\boldsymbol{U}_2-\varsigma\boldsymbol{U}_3.$$

By introducing a new matrix spectral function $\boldsymbol{\mu}=\boldsymbol{\mu}(x,t;\varsigma)$,

$$\psi=\boldsymbol{\mu}\mathrm{e}^{-\mathrm{i}(\varsigma^2+1)\boldsymbol{\Lambda}x+2\mathrm{i}(\varsigma^2+1)^2\boldsymbol{\Lambda}t},$$

the Lax pair (3) is then converted into

$$\boldsymbol{\mu}_x+\mathrm{i}(\varsigma^2+1)[\boldsymbol{\Lambda},\boldsymbol{\mu}]=\boldsymbol{V}_1\boldsymbol{\mu}, \tag{4a}$$

$$\boldsymbol{\mu}_t-2\mathrm{i}(\varsigma^2+1)^2[\boldsymbol{\Lambda},\boldsymbol{\mu}]=\boldsymbol{V}_2\boldsymbol{\mu}, \tag{4b}$$

where $[\boldsymbol{\Lambda},\boldsymbol{\mu}]=\boldsymbol{\Lambda}\boldsymbol{\mu}-\boldsymbol{\mu}\boldsymbol{\Lambda}$ is the commutator.

Now, considering the direct scattering process, we construct two matrix Jost solutions for the spectral problem (4a)

$$\boldsymbol{\mu}_-=([\boldsymbol{\mu}_-]_1,[\boldsymbol{\mu}_-]_2,[\boldsymbol{\mu}_-]_3),$$

$$\boldsymbol{\mu}_+=([\boldsymbol{\mu}_+]_1,[\boldsymbol{\mu}_+]_2,[\boldsymbol{\mu}_+]_3),$$

meeting the asymptotic conditions

$$\boldsymbol{\mu}_-\to\boldsymbol{I},\quad x\to-\infty, \tag{5a}$$

$$\boldsymbol{\mu}_+\to\boldsymbol{I},\quad x\to+\infty, \tag{5b}$$

where each $[\boldsymbol{\mu}_\pm]_l$ $(l=1,2,3)$ denotes the l-th column of the matrices $\boldsymbol{\mu}_\pm$ respectively, $\boldsymbol{I}$ is the 3×3 identity matrix, and $\boldsymbol{\mu}_\pm$ are uniquely determined by the Volterra integral equations

$$\begin{aligned}
\boldsymbol{\mu}_-&=\boldsymbol{I}+\int_{-\infty}^{x}\mathrm{e}^{-\mathrm{i}(\varsigma^2+1)\boldsymbol{\Lambda}(x-\xi)}\boldsymbol{V}_1(\xi)\boldsymbol{\mu}_-(\xi,\varsigma)\\
&\quad\times\mathrm{e}^{\mathrm{i}(\varsigma^2+1)\boldsymbol{\Lambda}(x-\xi)}\mathrm{d}\xi,
\end{aligned} \tag{6a}$$

$$\begin{aligned}
\boldsymbol{\mu}_+&=\boldsymbol{I}-\int_{x}^{+\infty}\mathrm{e}^{-\mathrm{i}(\varsigma^2+1)\boldsymbol{\Lambda}(x-\xi)}\boldsymbol{V}_1(\xi)\boldsymbol{\mu}_+(\xi,\varsigma)\\
&\quad\times\mathrm{e}^{\mathrm{i}(\varsigma^2+1)\boldsymbol{\Lambda}(x-\xi)}\mathrm{d}\xi.
\end{aligned} \tag{6b}$$

Through the direct analysis of Eq. (6), it can be seen that $[\boldsymbol{\mu}_+]_1$, $[\boldsymbol{\mu}_-]_2$, and $[\boldsymbol{\mu}_-]_3$ are analytic for $\varsigma\in\mathbb{D}^-$ and continuous for $\varsigma\in\mathbb{D}^-\cup\mathbb{R}\cup\mathrm{i}\mathbb{R}$, while $[\boldsymbol{\mu}_-]_1$, $[\boldsymbol{\mu}_+]_2$, and $[\boldsymbol{\mu}_+]_3$ are analytic for $\varsigma\in\mathbb{D}^+$ and continuous for $\varsigma\in\mathbb{D}^+\cup\mathbb{R}\cup\mathrm{i}\mathbb{R}$, where

$$\mathbb{D}^-=\left\{\varsigma|\arg\varsigma\in\left(0,\frac{\pi}{2}\right)\cup\left(\pi,\frac{3\pi}{2}\right)\right\},$$

$$\mathbb{D}^+=\left\{\varsigma|\arg\varsigma\in\left(\frac{\pi}{2},\pi\right)\cup\left(\frac{3\pi}{2},2\pi\right)\right\}.$$

Due to the Abel's identity and $\mathrm{tr}\boldsymbol{U}_1=0$, the determinants of $\boldsymbol{\mu}_\pm$ are constants for all x. Based on the asymptotic conditions (5), we obtain

$$\det\boldsymbol{\mu}_\pm=1,\quad \varsigma\in\mathbb{R}\cup\mathrm{i}\mathbb{R}.$$

As to the original spectral problem (3), there exist two fundamental matrix solutions $\boldsymbol{\mu}_-\boldsymbol{E}$ and $\boldsymbol{\mu}_+\boldsymbol{E}$, which are linearly associated with a 3×3 scattering matrix $\boldsymbol{S}(\varsigma)=(s_{kj})_{3\times3}$,

$$\boldsymbol{\mu}_-\boldsymbol{E}=\boldsymbol{\mu}_+\boldsymbol{E}\cdot\boldsymbol{S}(\varsigma),\quad \varsigma\in\mathbb{R}\cup\mathrm{i}\mathbb{R}, \tag{7}$$

where $\boldsymbol{E}=\mathrm{e}^{-\mathrm{i}(\varsigma^2+1)\boldsymbol{\Lambda}x}$. This gives rise to

$$\det\boldsymbol{S}(\varsigma)=1.$$

Furthermore, we find from the property of $\boldsymbol{\mu}_-$ that s_{11} accepts analytic extensions to $\mathbb{D}^+$ and s_{kj} $(k,j=2,3)$ extend to $\mathbb{D}^-$ analytically.

A Riemann–Hilbert problem desired for the CMNS system (1) involves two matrix functions: one is analytic in $\mathbb{D}^+$ and the other is analytic in $\mathbb{D}^-$. Define the first matrix function

$$\boldsymbol{H}_1(x,\varsigma)=([\boldsymbol{\mu}_-]_1,[\boldsymbol{\mu}_+]_2,[\boldsymbol{\mu}_+]_3)(x,\varsigma), \tag{8}$$

which is an analytic function of ς in $\mathbb{D}^+$. And then, we can study the very large-ς asymptotic behavior of $\boldsymbol{H}_1$, which has the asymptotic expansion

$$\boldsymbol{H}_1=\boldsymbol{H}_1^{(0)}+\frac{\boldsymbol{H}_1^{(1)}}{\varsigma}+\frac{\boldsymbol{H}_1^{(2)}}{\varsigma^2}+O\left(\frac{1}{\varsigma^3}\right),\quad \varsigma\to\infty. \tag{9}$$

Inserting Eq. (9) into Eq. (4a) and comparing the coefficients of ς directly bring about

$$\begin{aligned}&O(1): \boldsymbol{H}_{1x}^{(0)}+\mathrm{i}[\boldsymbol{\Lambda},\boldsymbol{H}_1^{(2)}]+\mathrm{i}[\boldsymbol{\Lambda},\boldsymbol{H}_1^{(0)}]=\boldsymbol{U}_1\boldsymbol{H}_1^{(1)},\\&O(\varsigma): \mathrm{i}[\boldsymbol{\Lambda},\boldsymbol{H}_1^{(1)}]=\boldsymbol{U}_1\boldsymbol{H}_1^{(0)},\\&O(\varsigma^2): \mathrm{i}[\boldsymbol{\Lambda},\boldsymbol{H}_1^{(0)}]=0,\end{aligned}$$

from which we have $\boldsymbol{H}_1^{(0)}=\boldsymbol{I}$, namely,

$$\boldsymbol{H}_1\to\boldsymbol{I},\quad \varsigma\in\mathbb{D}^+\to\infty.$$

In order to set up a Riemann–Hilbert problem for the CMNS system (1), it is essential for us to construct a matrix function $\boldsymbol{H}_2$ that is analytic for ς in $\mathbb{D}^-$. Actually, we only need to consider the inverse matrices of $\boldsymbol{\mu}_\pm$

$$\boldsymbol{\mu}_-^{-1}=\begin{pmatrix}[\boldsymbol{\mu}_-^{-1}]^1\\[\boldsymbol{\mu}_-^{-1}]^2\\[\boldsymbol{\mu}_-^{-1}]^3\end{pmatrix},\quad \boldsymbol{\mu}_+^{-1}=\begin{pmatrix}[\boldsymbol{\mu}_+^{-1}]^1\\[\boldsymbol{\mu}_+^{-1}]^2\\[\boldsymbol{\mu}_+^{-1}]^3\end{pmatrix},$$

in which each $[\boldsymbol{\mu}_\pm^{-1}]^l$ $(l=1,2,3)$ denotes the l-th row of the matrices $\boldsymbol{\mu}_\pm^{-1}$, respectively. Moreover, $[\boldsymbol{\mu}_\pm^{-1}]^l$ meet the equation

$$\boldsymbol{K}_x=-\mathrm{i}(\varsigma^2+1)[\boldsymbol{\Lambda},\boldsymbol{K}]-\boldsymbol{K}\boldsymbol{V}_1,$$

which is called the adjoint scattering equation of Eq. (4a). From Eq. (7), it is easy to see that

$$\boldsymbol{E}^{-1}\boldsymbol{\mu}_-^{-1}=\boldsymbol{R}(\varsigma)\cdot\boldsymbol{E}^{-1}\boldsymbol{\mu}_+^{-1},\tag{10}$$

here $\boldsymbol{R}(\varsigma)=(r_{kj})_{3\times3}=\boldsymbol{S}^{-1}(\varsigma)$. The matrix function $\boldsymbol{H}_2$ which is analytic in $\mathbb{D}^-$ is introduced in terms of

$$\boldsymbol{H}_2(x,\varsigma)=\begin{pmatrix}[\boldsymbol{\mu}_-^{-1}]^1\\[\boldsymbol{\mu}_+^{-1}]^2\\[\boldsymbol{\mu}_+^{-1}]^3\end{pmatrix}(x,\varsigma).\tag{11}$$

In a similar process as $\boldsymbol{H}_1$, we can obtain the very large-ς asymptotic behavior of $\boldsymbol{H}_2$,

$$\boldsymbol{H}_2\to\boldsymbol{I},\quad \varsigma\in\mathbb{D}^-\to\infty.$$

Substitution of $\boldsymbol{\mu}_-=([\boldsymbol{\mu}_-]_1,[\boldsymbol{\mu}_-]_2,[\boldsymbol{\mu}_-]_3)$ along with $\boldsymbol{\mu}_+=([\boldsymbol{\mu}_+]_1,[\boldsymbol{\mu}_+]_2,[\boldsymbol{\mu}_+]_3)$ into Eq. (7) gives

$$\begin{aligned}&([\boldsymbol{\mu}_-]_1,[\boldsymbol{\mu}_-]_2,[\boldsymbol{\mu}_-]_3)=([\boldsymbol{\mu}_+]_1,[\boldsymbol{\mu}_+]_2,[\boldsymbol{\mu}_+]_3)\\&\quad\times\begin{pmatrix}s_{11}&s_{12}\mathrm{e}^{2\mathrm{i}(\varsigma^2+1)x}&s_{13}\mathrm{e}^{2\mathrm{i}(\varsigma^2+1)x}\\s_{21}\mathrm{e}^{-2\mathrm{i}(\varsigma^2+1)x}&s_{22}&s_{23}\\s_{31}\mathrm{e}^{-2\mathrm{i}(\varsigma^2+1)x}&s_{32}&s_{33}\end{pmatrix},\end{aligned}$$

from which the expression of $[\boldsymbol{\mu}_-]_1$ reads

$$[\boldsymbol{\mu}_-]_1=s_{11}[\boldsymbol{\mu}_+]_1+s_{21}\mathrm{e}^{-2\mathrm{i}(\varsigma^2+1)x}[\boldsymbol{\mu}_+]_2+s_{31}\mathrm{e}^{-2\mathrm{i}(\varsigma^2+1)x}[\boldsymbol{\mu}_+]_3.$$

Hence, $\boldsymbol{H}_1$ is of the form

$$\boldsymbol{H}_1=([\boldsymbol{\mu}_-]_1,[\boldsymbol{\mu}_+]_2,[\boldsymbol{\mu}_+]_3)=\boldsymbol{\mu}_+\begin{pmatrix}s_{11}&0&0\\s_{21}\mathrm{e}^{-2\mathrm{i}(\varsigma^2+1)x}&1&0\\s_{31}\mathrm{e}^{-2\mathrm{i}(\varsigma^2+1)x}&0&1\end{pmatrix}.$$

On the other hand, through carrying $\boldsymbol{\mu}_-^{-1}=\begin{pmatrix}[\boldsymbol{\mu}_-^{-1}]^1\\[\boldsymbol{\mu}_-^{-1}]^2\\[\boldsymbol{\mu}_-^{-1}]^3\end{pmatrix}$ as well as $\boldsymbol{\mu}_+^{-1}=\begin{pmatrix}[\boldsymbol{\mu}_+^{-1}]^1\\[\boldsymbol{\mu}_+^{-1}]^2\\[\boldsymbol{\mu}_+^{-1}]^3\end{pmatrix}$ into Eq. (10), we derive

$$\begin{aligned}&\begin{pmatrix}[\boldsymbol{\mu}_-^{-1}]^1\\[\boldsymbol{\mu}_-^{-1}]^2\\[\boldsymbol{\mu}_-^{-1}]^3\end{pmatrix}\\&=\begin{pmatrix}r_{11}&r_{12}\mathrm{e}^{2\mathrm{i}(\varsigma^2+1)x}&r_{13}\mathrm{e}^{2\mathrm{i}(\varsigma^2+1)x}\\r_{21}\mathrm{e}^{-2\mathrm{i}(\varsigma^2+1)x}&r_{22}&r_{23}\\r_{31}\mathrm{e}^{-2\mathrm{i}(\varsigma^2+1)x}&r_{32}&r_{33}\end{pmatrix}\begin{pmatrix}[\boldsymbol{\mu}_+^{-1}]^1\\[\boldsymbol{\mu}_+^{-1}]^2\\[\boldsymbol{\mu}_+^{-1}]^3\end{pmatrix},\end{aligned}$$

from which we can express $[\boldsymbol{\mu}_-^{-1}]^1$ as

$$[\boldsymbol{\mu}_-^{-1}]^1=r_{11}[\boldsymbol{\mu}_+^{-1}]^1+r_{12}\mathrm{e}^{2\mathrm{i}(\varsigma^2+1)x}[\boldsymbol{\mu}_+^{-1}]^2+r_{13}\mathrm{e}^{2\mathrm{i}(\varsigma^2+1)x}[\boldsymbol{\mu}_+^{-1}]^3.$$

As a result, $\boldsymbol{H}_2$ takes the form

$$\boldsymbol{H}_2=\begin{pmatrix}[\boldsymbol{\mu}_-^{-1}]^1\\[\boldsymbol{\mu}_+^{-1}]^2\\[\boldsymbol{\mu}_+^{-1}]^3\end{pmatrix}=\begin{pmatrix}r_{11}&r_{12}\mathrm{e}^{2\mathrm{i}(\varsigma^2+1)x}&r_{13}\mathrm{e}^{2\mathrm{i}(\varsigma^2+1)x}\\0&1&0\\0&0&1\end{pmatrix}\boldsymbol{\mu}_+^{-1}.\tag{12}$$

Having discussed the construction of two matrix functions $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ which are analytic in $\mathbb{D}^+$ and $\mathbb{D}^-$, respectively, we are in a position to present a Riemann–Hilbert problem for the CMNS system (1). Here we denote that the limit of $\boldsymbol{H}_1$ is $\boldsymbol{H}^+$ when $\varsigma\in\mathbb{D}^+$ approaches $\mathbb{R}\cup\mathrm{i}\mathbb{R}$ and the limit of $\boldsymbol{H}_2$ is $\boldsymbol{H}^-$ when $\varsigma\in\mathbb{D}^-$ approaches $\mathbb{R}\cup\mathrm{i}\mathbb{R}$, based on which a Riemann–Hilbert problem can be set up as follows:

$$\begin{aligned}&\boldsymbol{H}^-(x,\varsigma)\boldsymbol{H}^+(x,\varsigma)\\&=\begin{pmatrix}1&r_{12}\mathrm{e}^{2\mathrm{i}(\varsigma^2+1)x}&r_{13}\mathrm{e}^{2\mathrm{i}(\varsigma^2+1)x}\\s_{21}\mathrm{e}^{-2\mathrm{i}(\varsigma^2+1)x}&1&0\\s_{31}\mathrm{e}^{-2\mathrm{i}(\varsigma^2+1)x}&0&1\end{pmatrix},\end{aligned}\tag{13}$$

whose canonical normalization conditions are

$$\begin{aligned}&\boldsymbol{H}_1(x,\varsigma)\to\boldsymbol{I},\quad \varsigma\in\mathbb{D}^+\to\infty,\\&\boldsymbol{H}_2(x,\varsigma)\to\boldsymbol{I},\quad \varsigma\in\mathbb{D}^-\to\infty,\end{aligned}$$

and $r_{11}s_{11}+r_{12}s_{21}+r_{13}s_{31}=1$.

3. Multi-soliton solutions

Having established the Riemann–Hilbert problem for the system (1), we will now move on to solve the Riemann–Hilbert problem (13) that is assumed to be irregular. The irregularity means that both $\det\boldsymbol{H}_1$ and $\det\boldsymbol{H}_2$ possess certain zeros in their analytic domains. By the definitions of $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ as well as the scattering relationship (7), we have

$$\begin{aligned}&\det\boldsymbol{H}_1(\varsigma)=s_{11}(\varsigma),\quad \varsigma\in\mathbb{D}^+,\\&\det\boldsymbol{H}_2(\varsigma)=r_{11}(\varsigma),\quad \varsigma\in\mathbb{D}^-.\end{aligned}$$

From these two equalities we find the facts that the zeros of $\det \boldsymbol{H}_1$ and $\det \boldsymbol{H}_2$ are the same as s_{11} and r_{11}, respectively.

Let us now turn to discuss the characteristics of the zeros. As regards the potential matrix $\boldsymbol{U}_1$ possessing the symmetry condition

$$\boldsymbol{U}_1^{\dagger} = -\boldsymbol{\Lambda}\boldsymbol{U}_1\boldsymbol{\Lambda},$$

where the superscript "†" represents the Hermitian of a matrix, we obtain

$$\boldsymbol{\mu}_{\pm}^{\dagger}(\varsigma^*) = \boldsymbol{\Lambda}\boldsymbol{\mu}_{\pm}^{-1}(\varsigma)\boldsymbol{\Lambda}. \tag{14}$$

According to Eq. (7), we have

$$\boldsymbol{S}^{\dagger}(\varsigma^*) = \boldsymbol{\Lambda}\boldsymbol{R}(\varsigma)\boldsymbol{\Lambda},$$

which gives the following relationships

$$s_{21}^*(\varsigma^*) = -r_{12}(\varsigma), \quad \varsigma \in \mathbb{R}\cup \mathrm{i}\mathbb{R}, \tag{15}$$
$$s_{31}^*(\varsigma^*) = -r_{13}(\varsigma), \quad \varsigma \in \mathbb{R}\cup \mathrm{i}\mathbb{R}, \tag{16}$$
$$s_{11}^*(\varsigma^*) = r_{11}(\varsigma), \quad \varsigma \in \mathbb{D}^-. \tag{17}$$

It is apparent from equality (17) that each zero $\pm\varsigma_k$ of s_{11} generates each zero $\pm\varsigma_k^*$ of r_{11} correspondingly. From the relationships (14) and (17), we have

$$\boldsymbol{H}_1^{\dagger}(\varsigma^*) = \boldsymbol{\Lambda}\boldsymbol{H}_2(\varsigma)\boldsymbol{\Lambda}, \quad \varsigma \in \mathbb{D}^-. \tag{18}$$

In addition, there also exists the symmetry relationship for the potential matrix $\boldsymbol{U}_1$

$$\boldsymbol{\Lambda}\boldsymbol{U}_1\boldsymbol{\Lambda} = -\boldsymbol{U}_1, \tag{19}$$

where $\boldsymbol{\Lambda} = \mathrm{diag}(-1,1,1)$. By use of Eq. (19), we have

$$\boldsymbol{\mu}_{\pm}(-\varsigma) = \boldsymbol{\Lambda}\boldsymbol{\mu}_{\pm}(\varsigma)\boldsymbol{\Lambda}, \tag{20}$$

$$\boldsymbol{S}(-\varsigma) = \boldsymbol{\Lambda}\boldsymbol{S}(\varsigma)\boldsymbol{\Lambda},$$

which reveals the relationships

$$s_{21}(-\varsigma) = -s_{21}(\varsigma), \quad \varsigma \in \mathbb{R}\cup \mathrm{i}\mathbb{R}, \tag{21}$$
$$s_{31}(-\varsigma) = -s_{31}(\varsigma), \quad \varsigma \in \mathbb{R}\cup \mathrm{i}\mathbb{R}, \tag{22}$$
$$s_{11}(-\varsigma) = s_{11}(\varsigma), \quad \varsigma \in \mathbb{D}^+. \tag{23}$$

The equality (23) shows that each zero ς_k of s_{11} is accompanied by another zero $-\varsigma_k$, i.e., $s_{11}(\pm\varsigma_k) = 0$. Here we point out that equations (20) and (23) yield the property

$$\boldsymbol{\Lambda}\boldsymbol{H}_1(-\varsigma)\boldsymbol{\Lambda} = \boldsymbol{H}_1(\varsigma), \quad \varsigma \in \mathbb{D}^+. \tag{24}$$

At this point, we posit that $\det \boldsymbol{H}_1$ has $2N$ simple zeros $\{\varsigma_j\}$ $(1 \le j \le 2N)$ in $\mathbb{D}^+$, where

$$\varsigma_{N+l} = -\varsigma_l, \quad 1 \le l \le N,$$

and $\det \boldsymbol{H}_2$ has $2N$ simple zeros $\{\hat{\varsigma}_j\}$ $(1 \le j \le 2N)$ in $\mathbb{D}^-$, where

$$\hat{\varsigma}_l = \varsigma_l^*, \quad 1 \le l \le 2N.$$

In our discussion, the continuous scattering data s_{21}, s_{31} and the discrete scattering data ς_j, $\hat{\varsigma}_j$, $\boldsymbol{\nu}_j$, $\hat{\boldsymbol{\nu}}_j$ are very essential. To determine $\boldsymbol{\nu}_j$ and $\hat{\boldsymbol{\nu}}_j$, the following two equations are needed:

$$\boldsymbol{H}_1(\varsigma_j)\boldsymbol{\nu}_j = 0, \tag{25}$$
$$\hat{\boldsymbol{\nu}}_j\boldsymbol{H}_2(\hat{\varsigma}_j) = 0, \tag{26}$$

where $\boldsymbol{\nu}_j$ and $\hat{\boldsymbol{\nu}}_j$ denote nonzero column vectors and row vectors, respectively. Without loss of generality, taking advantage of Eqs. (18), (24), (25) as well as (26), we find

$$\boldsymbol{\nu}_j = \boldsymbol{\Lambda}\boldsymbol{\nu}_{j-N}, \quad N+1 \le j \le 2N, \tag{27}$$
$$\hat{\boldsymbol{\nu}}_j = \boldsymbol{\nu}_j^{\dagger}\boldsymbol{\Lambda}, \quad 1 \le j \le 2N. \tag{28}$$

Now we shall obtain the spatial evolutions of the vectors $\boldsymbol{\nu}_j$. For this purpose, performing the x-derivative with respect to Eq. (25) directly results in

$$\boldsymbol{\nu}_j = \mathrm{e}^{-\mathrm{i}(\varsigma_j^2+1)\boldsymbol{\Lambda}x}\boldsymbol{\nu}_{j,0}, \quad 1 \le j \le N, \tag{29}$$

in which $\boldsymbol{\nu}_{j,0}$ is independent of x.

Here we examine the Riemann–Hilbert problem (13) with reflectionless case, that is $s_{21} = s_{31} = 0$. We introduce a $2N \times 2N$ matrix $\boldsymbol{M}$ whose entries are

$$m_{kj} = \frac{\hat{\boldsymbol{\nu}}_k\boldsymbol{\nu}_j}{\varsigma_j - \hat{\varsigma}_k}, \quad 1 \le k, j \le 2N.$$

If we assume that the inverse matrix of $\boldsymbol{M}$ exists, then the solution for the Riemann–Hilbert problem (13) can be described as

$$\boldsymbol{H}_1(\varsigma) = \boldsymbol{I} - \sum_{k=1}^{2N}\sum_{j=1}^{2N}\frac{\boldsymbol{\nu}_k\hat{\boldsymbol{\nu}}_j(\boldsymbol{M}^{-1})_{kj}}{\varsigma - \hat{\varsigma}_j}, \tag{30a}$$
$$\boldsymbol{H}_2(\varsigma) = \boldsymbol{I} + \sum_{k=1}^{2N}\sum_{j=1}^{2N}\frac{\boldsymbol{\nu}_k\hat{\boldsymbol{\nu}}_j(\boldsymbol{M}^{-1})_{kj}}{\varsigma - \varsigma_k}. \tag{30b}$$

With the aid of Eq. (30), we are going to restructure the potentials u and v. As a matter of fact, we need to consider the asymptotic expansion of $\boldsymbol{H}_1$,

$$\boldsymbol{H}_1(\varsigma) = \boldsymbol{I} + \frac{\boldsymbol{H}_1^{(1)}}{\varsigma} + \frac{\boldsymbol{H}_1^{(2)}}{\varsigma^2} + O\left(\frac{1}{\varsigma^3}\right), \quad \varsigma \to \infty. \tag{31}$$

Carrying expression (31) into Eq. (4a) yields

$$\boldsymbol{U}_1 = \mathrm{i}[\boldsymbol{\Lambda}, \boldsymbol{H}_1^{(1)}]. \tag{32}$$

Consequently, a direct calculation shows the potential functions

$$u = -2\mathrm{i}(\boldsymbol{H}_1^{(1)})_{12},$$
$$v = -2\mathrm{i}(\boldsymbol{H}_1^{(1)})_{13},$$

where $(\boldsymbol{H}_1^{(1)})_{12}$ and $(\boldsymbol{H}_1^{(1)})_{13}$ are the (1,2)-entry and (1,3)-entry of matrix $\boldsymbol{H}_1^{(1)}$, respectively. Whereas, the $\boldsymbol{H}_1^{(1)}$ can be obtained from Eq. (30) as

$$\boldsymbol{H}_1^{(1)} = -\sum_{k=1}^{2N}\sum_{j=1}^{2N}\boldsymbol{\nu}_k\hat{\boldsymbol{\nu}}_j(\boldsymbol{M}^{-1})_{kj}. \tag{33}$$

So far we merely focus on solving the CMNS system (1) at a fixed time. To seek solutions of the system (1) at any time, it is required to reveal the temporal evolutions of the scattering data. By means of Eqs. (4b) and (7) as well as the sufficiently fast decay of u, v on the boundary, we obtain

$$\boldsymbol{S}_t = 2\mathrm{i}(\varsigma^2+1)^2[\boldsymbol{\Lambda}, \boldsymbol{S}],$$

which leads to the temporal evolutions

$$s_{21,t} = 4\mathrm{i}(\varsigma^2+1)^2 s_{21}, \quad s_{31,t} = 4\mathrm{i}(\varsigma^2+1)^2 s_{31}.$$

Taking account of Eq. (4b), we gain

$$\boldsymbol{\nu}_{j,t} = 2\mathrm{i}(\varsigma_j^2+1)^2\boldsymbol{\Lambda}\boldsymbol{\nu}_j, \quad \frac{\mathrm{d}\varsigma_j}{\mathrm{d}t} = 0.$$

Thus, $\boldsymbol{\nu}_j$ and $\hat{\boldsymbol{\nu}}_j$ are generated based on Eqs. (27)–(29) as

$$\boldsymbol{\nu}_j = \begin{cases} \mathrm{e}^{[-\mathrm{i}(\varsigma_j^2+1)x+2\mathrm{i}(\varsigma_j^2+1)^2t]\boldsymbol{\Lambda}}\boldsymbol{\nu}_{j,0}, & 1\le j\le N, \\ \boldsymbol{\Lambda}\mathrm{e}^{[-\mathrm{i}(\varsigma_{j-N}^2+1)x+2\mathrm{i}(\varsigma_{j-N}^2+1)^2t]\boldsymbol{\Lambda}}\boldsymbol{\nu}_{j-N,0}, & N+1\le j\le 2N; \end{cases}$$

$$\hat{\boldsymbol{\nu}}_j = \begin{cases} \boldsymbol{\nu}_{j,0}^{\dagger}\mathrm{e}^{[-\mathrm{i}(\varsigma_j^2+1)x+2\mathrm{i}(\varsigma_j^2+1)^2t]^*\boldsymbol{\Lambda}}\boldsymbol{\Lambda}, & 1\le j\le N, \\ \boldsymbol{\nu}_{j-N,0}^{\dagger}\mathrm{e}^{[-\mathrm{i}(\varsigma_{j-N}^2+1)x+2\mathrm{i}(\varsigma_{j-N}^2+1)^2t]^*\boldsymbol{\Lambda}}, & N+1\le j\le 2N. \end{cases}$$

Here $\boldsymbol{\nu}_{j,0}$ is a complex column vector that is independent of x and t. In order to obtain the multi-soliton solutions to the CMNS system (1) conveniently, we make the assumption $\boldsymbol{\nu}_{j,0} = (\alpha_j, \beta_j, \gamma_j)^{\mathrm{T}}$, $1\le j\le N$.

After summarizing the above results, the expression of N-soliton solutions for the CMNS system (1) can be immediately obtained as follows:

$$\begin{aligned} u = {} & 2\mathrm{i}\sum_{k=1}^{N}\sum_{j=1}^{N}\alpha_k\beta_j^*\mathrm{e}^{\xi_j^*-\xi_k}(\boldsymbol{M}^{-1})_{kj} \\ & +2\mathrm{i}\sum_{k=1}^{N}\sum_{j=N+1}^{2N}\alpha_k\beta_{j-N}^*\mathrm{e}^{\xi_{j-N}^*-\xi_k}(\boldsymbol{M}^{-1})_{kj} \\ & -2\mathrm{i}\sum_{k=N+1}^{2N}\sum_{j=1}^{N}\alpha_{k-N}\beta_j^*\mathrm{e}^{\xi_j^*-\xi_{k-N}}(\boldsymbol{M}^{-1})_{kj} \\ & -2\mathrm{i}\sum_{k=N+1}^{2N}\sum_{j=N+1}^{2N}\alpha_{k-N}\beta_{j-N}^*\mathrm{e}^{\xi_{j-N}^*-\xi_{k-N}}(\boldsymbol{M}^{-1})_{kj}, \end{aligned} \tag{34a}$$

$$\begin{aligned} v = {} & 2\mathrm{i}\sum_{k=1}^{N}\sum_{j=1}^{N}\alpha_k\gamma_j^*\mathrm{e}^{\xi_j^*-\xi_k}(\boldsymbol{M}^{-1})_{kj} \\ & +2\mathrm{i}\sum_{k=1}^{N}\sum_{j=N+1}^{2N}\alpha_k\gamma_{j-N}^*\mathrm{e}^{\xi_{j-N}^*-\xi_k}(\boldsymbol{M}^{-1})_{kj} \\ & -2\mathrm{i}\sum_{k=N+1}^{2N}\sum_{j=1}^{N}\alpha_{k-N}\gamma_j^*\mathrm{e}^{\xi_j^*-\xi_{k-N}}(\boldsymbol{M}^{-1})_{kj} \\ & -2\mathrm{i}\sum_{k=N+1}^{2N}\sum_{j=N+1}^{2N}\alpha_{k-N}\gamma_{j-N}^*\mathrm{e}^{\xi_{j-N}^*-\xi_{k-N}}(\boldsymbol{M}^{-1})_{kj}, \end{aligned} \tag{34b}$$

where

$$m_{kj} = \begin{cases} \dfrac{(\beta_k^*\beta_j+\gamma_k^*\gamma_j)\mathrm{e}^{\xi_k^*+\xi_j}-\alpha_k^*\alpha_j\mathrm{e}^{-\xi_k^*-\xi_j}}{\varsigma_j-\varsigma_k^*}, & 1\le k,j\le N, \\ \dfrac{(\beta_k^*\beta_{j-N}+\gamma_k^*\gamma_{j-N})\mathrm{e}^{\xi_k^*+\xi_{j-N}}+\alpha_k^*\alpha_{j-N}\mathrm{e}^{-\xi_k^*-\xi_{j-N}}}{-\varsigma_{j-N}-\varsigma_k^*}, & 1\le k\le N, N+1\le j\le 2N, \\ \dfrac{(\beta_{k-N}^*\beta_j+\gamma_{k-N}^*\gamma_j)\mathrm{e}^{\xi_{k-N}^*+\xi_j}+\alpha_{k-N}^*\alpha_j\mathrm{e}^{-\xi_{k-N}^*-\xi_j}}{\varsigma_j+\varsigma_{k-N}^*}, & N+1\le k\le 2N, 1\le j\le N, \\ \dfrac{(\beta_{k-N}^*\beta_{j-N}+\gamma_{k-N}^*\gamma_{j-N})\mathrm{e}^{\xi_{k-N}^*+\xi_{j-N}}-\alpha_{k-N}^*\alpha_{j-N}\mathrm{e}^{-\xi_{k-N}^*-\xi_{j-N}}}{-\varsigma_{j-N}+\varsigma_{k-N}^*}, & N+1\le k,j\le 2N, \end{cases}$$

and $\xi_k = -\mathrm{i}(\varsigma_k^2+1)x+2\mathrm{i}(\varsigma_k^2+1)^2t$, $1\le k\le N$.

At the end of this section, two simplest cases arise by taking $N=1$ and $N=2$ in the N-soliton solution formula (34). Suppose that $N=1$ and $\varsigma_1 = a_1+\mathrm{i}b_1$, $a_1b_1<0$,

$$\boldsymbol{M} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix},$$

the one-soliton solution for the CMNS system (1) follows at once

$$u = 2\mathrm{i}\alpha_1\beta_1^*\mathrm{e}^{\xi_1^*-\xi_1}\frac{m_{22}-m_{12}+m_{21}-m_{11}}{m_{11}m_{22}-m_{12}m_{21}}, \tag{35a}$$

$$v = 2\mathrm{i}\alpha_1\gamma_1^*\mathrm{e}^{\xi_1^*-\xi_1}\frac{m_{22}-m_{12}+m_{21}-m_{11}}{m_{11}m_{22}-m_{12}m_{21}}, \tag{35b}$$

in which $\xi_1 = -\mathrm{i}((a_1+\mathrm{i}b_1)^2+1)x+2\mathrm{i}((a_1+\mathrm{i}b_1)^2+1)^2t$, and

$$m_{11} = \frac{(|\beta_1|^2+|\gamma_1|^2)\mathrm{e}^{\xi_1^*+\xi_1}-|\alpha_1|^2\mathrm{e}^{-\xi_1^*-\xi_1}}{2\mathrm{i}b_1},$$

$$m_{12} = \frac{(|\beta_1|^2+|\gamma_1|^2)\mathrm{e}^{\xi_1^*+\xi_1}+|\alpha_1|^2\mathrm{e}^{-\xi_1^*-\xi_1}}{-2a_1},$$

$$m_{21} = \frac{(|\beta_1|^2+|\gamma_1|^2)\mathrm{e}^{\xi_1^*+\xi_1}+|\alpha_1|^2\mathrm{e}^{-\xi_1^*-\xi_1}}{2a_1},$$

$$m_{22} = \frac{(|\beta_1|^2+|\gamma_1|^2)\mathrm{e}^{\xi_1^*+\xi_1}-|\alpha_1|^2\mathrm{e}^{-\xi_1^*-\xi_1}}{-2\mathrm{i}b_1}.$$

The solution is graphically shown in Figs. 1–3.

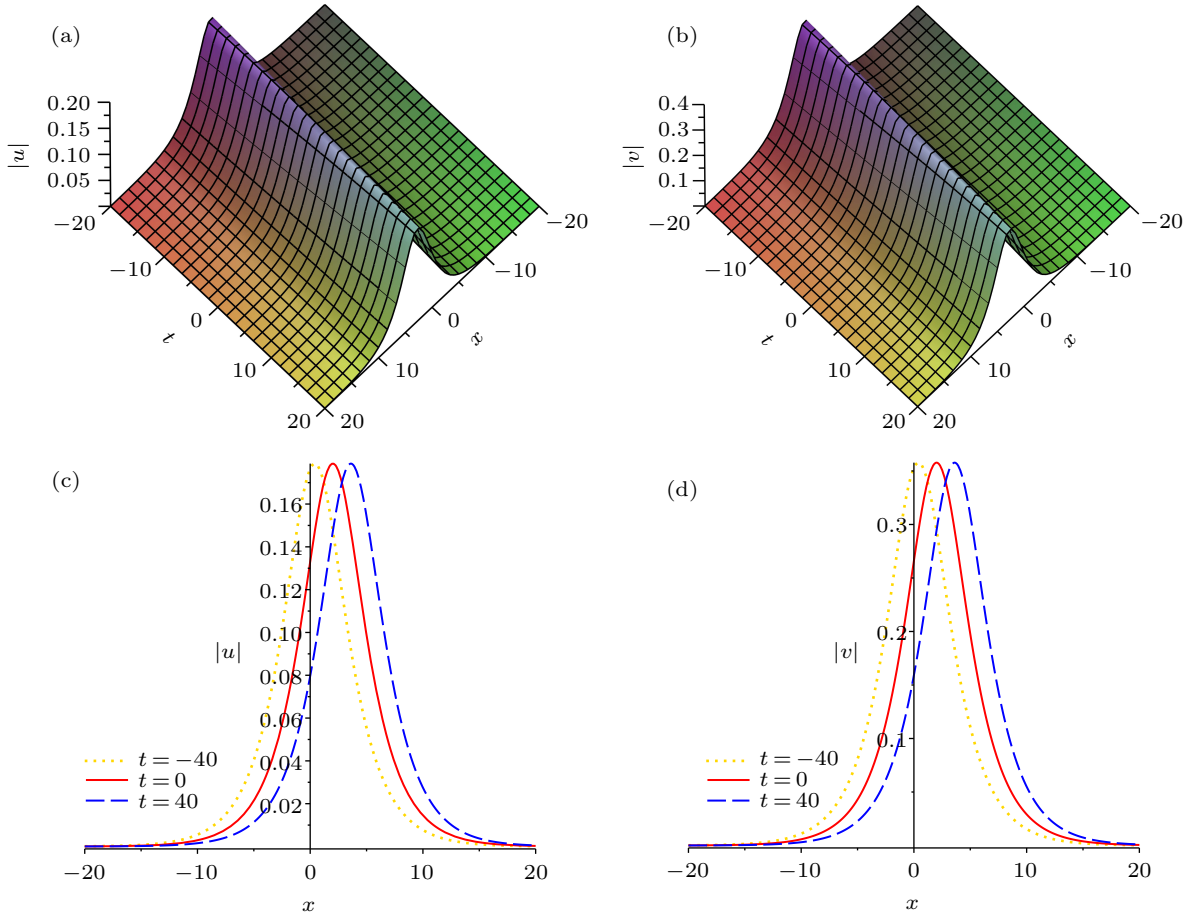


Fig. 1. (color online) One-soliton solution (35) with $\alpha_1 = 1, \beta_1 = 1, \gamma_1 = 2, a_1 = -0.1, b_1 = 1$: (a) perspective view of modulus of u, (b) perspective view of modulus of v. (c) The soliton along the x axis with different time in panel (a). (d) The soliton along the x axis with different time in panel (b).

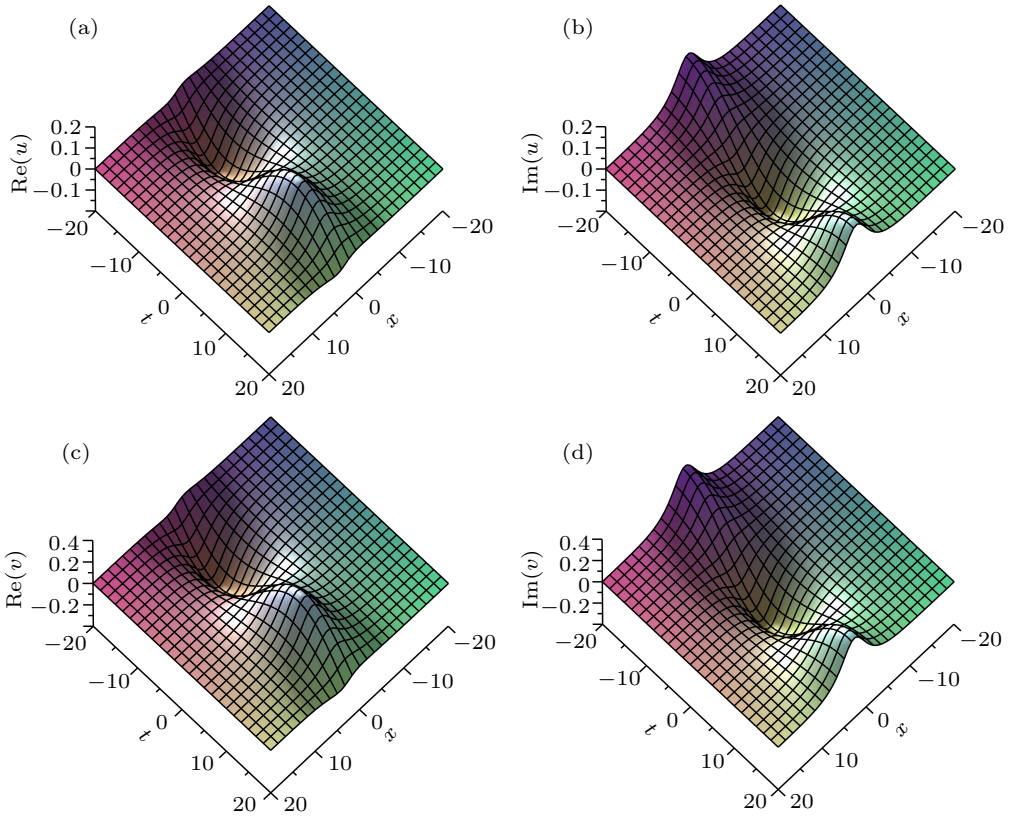


Fig. 2. (color online) One-soliton solution (35) with $\alpha_1 = 1, \beta_1 = 1, \gamma_1 = 2, a_1 = -0.1, b_1 = 1$: (a) perspective view of real part of u, (b) perspective view of imaginary of u, (c) perspective view of real part of v, (d) perspective view of imaginary of v.

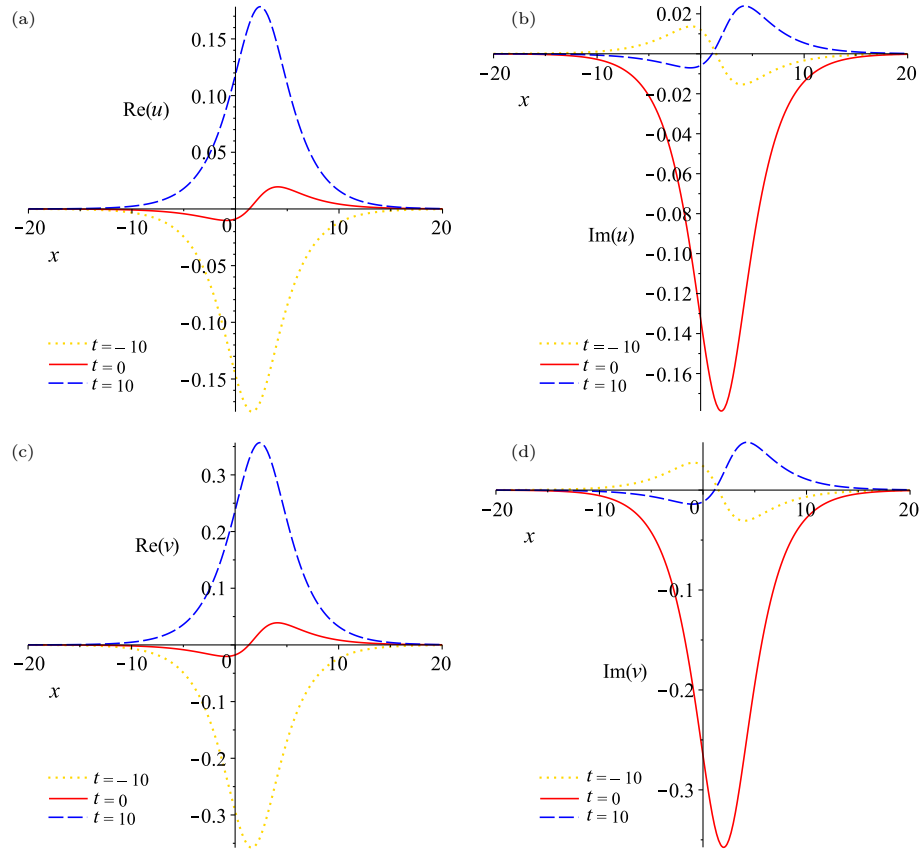


Fig. 3. (color online) One-soliton solution (35) with $\alpha_1 = 1, \beta_1 = 1, \gamma_1 = 2, a_1 = -0.1, b_1 = 1$. (a) The soliton along the x axis with different time in Fig. 2(a); (b) The soliton along the x axis with different time in Fig. 2(b); (c) The soliton along the x axis with different time in Fig. 2(c); (d) The soliton along the x axis with different time in Fig. 2(d).

By taking $N = 2$ and assuming

$$\varsigma_1 = a_1 + \mathrm{i}b_1, \quad a_1 b_1 < 0,$$
$$\varsigma_2 = a_2 + \mathrm{i}b_2, \quad a_2 b_2 < 0,$$

the two-soliton solution for the CMNS system (1) is given by

$$\begin{aligned} u = {} & 2\mathrm{i}\sum_{k=1}^{2}\sum_{j=1}^{2}\alpha_k\beta_j^*\mathrm{e}^{\xi_j^*-\xi_k}(\boldsymbol{M}^{-1})_{kj} \\ & + 2\mathrm{i}\sum_{k=1}^{2}\sum_{j=3}^{4}\alpha_k\beta_{j-2}^*\mathrm{e}^{\xi_{j-2}^*-\xi_k}(\boldsymbol{M}^{-1})_{kj} \\ & - 2\mathrm{i}\sum_{k=3}^{4}\sum_{j=1}^{2}\alpha_{k-2}\beta_j^*\mathrm{e}^{\xi_j^*-\xi_{k-2}}(\boldsymbol{M}^{-1})_{kj} \\ & - 2\mathrm{i}\sum_{k=3}^{4}\sum_{j=3}^{4}\alpha_{k-2}\beta_{j-2}^*\mathrm{e}^{\xi_{j-2}^*-\xi_{k-2}}(\boldsymbol{M}^{-1})_{kj}, \\ v = {} & 2\mathrm{i}\sum_{k=1}^{2}\sum_{j=1}^{2}\alpha_k\gamma_j^*\mathrm{e}^{\xi_j^*-\xi_k}(\boldsymbol{M}^{-1})_{kj} \\ & + 2\mathrm{i}\sum_{k=1}^{2}\sum_{j=3}^{4}\alpha_k\gamma_{j-2}^*\mathrm{e}^{\xi_{j-2}^*-\xi_k}(\boldsymbol{M}^{-1})_{kj} \\ & - 2\mathrm{i}\sum_{k=3}^{4}\sum_{j=1}^{2}\alpha_{k-2}\gamma_j^*\mathrm{e}^{\xi_j^*-\xi_{k-2}}(\boldsymbol{M}^{-1})_{kj} \\ & - 2\mathrm{i}\sum_{k=3}^{4}\sum_{j=3}^{4}\alpha_{k-2}\gamma_{j-2}^*\mathrm{e}^{\xi_{j-2}^*-\xi_{k-2}}(\boldsymbol{M}^{-1})_{kj}, \end{aligned}$$

where $\boldsymbol{M} = (m_{kj})_{4\times4}$ and its entries are

$$\begin{aligned} m_{11} &= \frac{(|\beta_1|^2+|\gamma_1|^2)\mathrm{e}^{\xi_1^*+\xi_1} - |\alpha_1|^2\mathrm{e}^{-\xi_1^*-\xi_1}}{2\mathrm{i}b_1}, \\ m_{12} &= \frac{(\beta_1^*\beta_2+\gamma_1^*\gamma_2)\mathrm{e}^{\xi_1^*+\xi_2} - \alpha_1^*\alpha_2\mathrm{e}^{-\xi_1^*-\xi_2}}{(a_2-a_1)+\mathrm{i}(b_1+b_2)}, \\ m_{13} &= \frac{(|\beta_1|^2+|\gamma_1|^2)\mathrm{e}^{\xi_1^*+\xi_1} + |\alpha_1|^2\mathrm{e}^{-\xi_1^*-\xi_1}}{-2a_1}, \\ m_{14} &= \frac{(\beta_1^*\beta_2+\gamma_1^*\gamma_2)\mathrm{e}^{\xi_1^*+\xi_2} + \alpha_1^*\alpha_2\mathrm{e}^{-\xi_1^*-\xi_2}}{(-a_2-a_1)+\mathrm{i}(b_1-b_2)}, \\ m_{21} &= \frac{(\beta_2^*\beta_1+\gamma_2^*\gamma_1)\mathrm{e}^{\xi_2^*+\xi_1} - \alpha_2^*\alpha_1\mathrm{e}^{-\xi_2^*-\xi_1}}{(a_1-a_2)+\mathrm{i}(b_1+b_2)}, \\ m_{22} &= \frac{(|\beta_2|^2+|\gamma_2|^2)\mathrm{e}^{\xi_2^*+\xi_2} - |\alpha_2|^2\mathrm{e}^{-\xi_2^*-\xi_2}}{2\mathrm{i}b_2}, \\ m_{23} &= \frac{(\beta_2^*\beta_1+\gamma_2^*\gamma_1)\mathrm{e}^{\xi_2^*+\xi_1} + \alpha_2^*\alpha_1\mathrm{e}^{-\xi_2^*-\xi_1}}{(-a_2-a_1)+\mathrm{i}(b_2-b_1)}, \\ m_{24} &= \frac{(|\beta_2|^2+|\gamma_2|^2)\mathrm{e}^{\xi_2^*+\xi_2} + |\alpha_2|^2\mathrm{e}^{-\xi_2^*-\xi_2}}{-2a_2}, \\ m_{31} &= \frac{(|\beta_1|^2+|\gamma_1|^2)\mathrm{e}^{\xi_1^*+\xi_1} + |\alpha_1|^2\mathrm{e}^{-\xi_1^*-\xi_1}}{2a_1}, \\ m_{32} &= \frac{(\beta_1^*\beta_2+\gamma_1^*\gamma_2)\mathrm{e}^{\xi_1^*+\xi_2} + \alpha_1^*\alpha_2\mathrm{e}^{-\xi_1^*-\xi_2}}{(a_1+a_2)+\mathrm{i}(b_2-b_1)}, \\ m_{33} &= \frac{(|\beta_1|^2+|\gamma_1|^2)\mathrm{e}^{\xi_1^*+\xi_1} - |\alpha_1|^2\mathrm{e}^{-\xi_1^*-\xi_1}}{-2\mathrm{i}b_1}, \end{aligned}$$

$$m_{34}=\frac{(\beta_1^*\beta_2+\gamma_1^*\gamma_2)\mathrm{e}^{\xi_1^*+\xi_2}-\alpha_1^*\alpha_2\mathrm{e}^{-\xi_1^*-\xi_2}}{(a_1-a_2)-\mathrm{i}(b_1+b_2)},$$
$$m_{41}=\frac{(\beta_2^*\beta_1+\gamma_2^*\gamma_1)\mathrm{e}^{\xi_2^*+\xi_1}+\alpha_2^*\alpha_1\mathrm{e}^{-\xi_2^*-\xi_1}}{(a_1+a_2)+\mathrm{i}(b_1-b_2)},$$
$$m_{42}=\frac{(|\beta_2|^2+|\gamma_2|^2)\mathrm{e}^{\xi_2^*+\xi_2}+|\alpha_2|^2\mathrm{e}^{-\xi_2^*-\xi_2}}{2a_2},$$
$$m_{43}=\frac{(\beta_2^*\beta_1+\gamma_2^*\gamma_1)\mathrm{e}^{\xi_2^*+\xi_1}-\alpha_2^*\alpha_1\mathrm{e}^{-\xi_2^*-\xi_1}}{(-a_1+a_2)-\mathrm{i}(b_1+b_2)},$$
$$m_{44}=\frac{(|\beta_2|^2+|\gamma_2|^2)\mathrm{e}^{\xi_2^*+\xi_2}-|\alpha_2|^2\mathrm{e}^{-\xi_2^*-\xi_2}}{-2\mathrm{i}b_2},$$

with

$$\xi_1=-\mathrm{i}((a_1+\mathrm{i}b_1)^2+1)x+2\mathrm{i}((a_1+\mathrm{i}b_1)^2+1)^2t,$$
$$\xi_2=-\mathrm{i}((a_2+\mathrm{i}b_2)^2+1)x+2\mathrm{i}((a_2+\mathrm{i}b_2)^2+1)^2t.$$

4. Discussion and conclusion

In this work, we have successfully constructed multi-soliton solutions for the coupled modified nonlinear Schrödinger equations based on the Riemann–Hilbert approach. To sum up, the process of the Riemann–Hilbert approach can be generally outlined as follows: first, we deduce a Riemann–Hilbert problem for the CMNS system (1) via analyzing the related spectral problem. Second, by studying the presented Riemann–Hilbert problem with irregularity and reflectionless case, we gain the solution of the Riemann–Hilbert problem explicitly. After restructuring the potentials and considering the temporal evolutions of the scattering data, we finally derive the N-soliton solutions for the coupled modified nonlinear Schrödinger equations. The localized structures and dynamic behaviors of one-soliton solution are displayed vividly by figures.

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