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General series expression of eddy-current impedance for coil placed above multi-layer plate conductor*

Yin-Zhao Lei(雷银照)†

School of Automation Science and Electrical Engineering, Beihang University, Beijing 100191, *China*

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This paper presents a closed expression of the layered-plate factor used to calculate the coil eddy-current impedance over the multi-layer plate conductor. By using this expression, the general series of eddy-current impedance can be written directly without solving the undetermined constant equations. The series expression is easy to use for theoretical analysis and programming. Experimental results show that calculated values and measured values are in agreement. As an application, when the bottom layer of the layered plate is a non-ferromagnetic thin layer conductor and the product of the thickness and conductivity of the layer remains unchanged, using the layered-plate factor expression proposed in this paper, it can be theoretically predicted that the eddy-current impedance curves corresponding to different thin layer thickness values will coincide.

Keywords: multi-layer plate conductor, layered-plate factor, series expression, eddy-current impedance

PACS: 03.50.De, 41.20.–q, 81.70.Ex **DOI:** 10.1088/1674-1056/27/6/060308

1. Introduction

In industrial eddy-current non-destructive testing, the parameters and surface status of the layered conductor material can be determined by measuring the eddy-current impedance of the coil. The eddy-current analytical methods for an air-core current-carrying cylindrical coil vertically placed above layered conductor have been investigated earlier. In 1968, Dodd and Deeds[1] derived the integral-type analytical solution of magnetic vector potential and the eddy-current impedance of an air-core cylindrical coil carrying a sinusoidal current placed above a two-layer conductor. This paper has been widely cited since its publication. Based on the Dodd–Deeds model, Luquire *et al.*[2] obtained the integral expression of the impedance for the coil above the multi-layer conductive material by using a mathematical induction method. Cheng *et al.*[3] used the 2×2 matrix recursion method to calculate the eddy-current impedance of the coil above layered conductors, furthermore, this method also could be used to calculate the eddy-current field of arbitrary multi-layer conductors. Theodoulidis *et al.*[4] replaced the infinite region of Dodd–Deeds model by a finite-radius coaxial cylindrical region and took the magnetic vector potential to be zero on the cylindrical surface, and therefore, the series-form analytic solution of the sinusoidal eddy-current field has been obtained. The method in Ref. [4] not only greatly improved the efficiency in calculating eddy-current impedance, but also could be used to obtain approximate analytical solutions of some eddy-current problems.[5–7] Furthermore, Fan *et al.*[8] obtained the generalized reflection coefficients of the multi-layer plate conductor, and then obtained the series expression of coil eddy-current impedance by using the recursive method.

The derivation method of Lipskaya–Vanyan formula[9,10] is used in this paper. The method is that the solution of the second order homogeneous ordinary differential equation is divided by its derivative. Starting from the undetermined constant equations, using the natural logarithm identity and the definition of hyperbolic tangent function repeatedly, then a general series expression for eddy-current impedance for air-core cylindrical coil placed above an arbitrary multi-layer conductor is obtained for the first time. Compared with previous results, this series expression needs no iteration or solving linear equations, which is easy to use for theoretical analysis and programming.

2. Eddy-current model for the air-core cylindrical coil and layered plate conductor

As shown in Fig. 1, the central axis of the air-core cylindrical coil is perpendicular to the conductor plate and the coil is uniformly tightly wound. r_1, r_2, h, W, and l are the coil inner radius, outer radius, height, turns, and lift-off (the distance between the coil bottom plane and the multi-layer plate conductor upper plane) respectively. A cylindrical coordinate system $O\rho\phi z$ is established with three coordinate unit vectors, i.e., $\boldsymbol{e}_\rho$, $\boldsymbol{e}_\phi$, and $\boldsymbol{e}_z$. The origin point O is located at the interface between the multi-layer plate conductor upper plane and the vacuum region; and the z axis coincides with the central axis of the coil. The phasor current in the coil is I, with frequency f and angular frequency $\omega = 2\pi f$, and the reference direction of I has a right-hand screw relationship with $\boldsymbol{e}_z$. N is the number of layers of multi-layer plate conductor. The N layers

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†Corresponding author. E-mail: leiyinzhao@buaa.edu.cn

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are numbered along the $-e_z$ direction respectively as: Layer 1 ($z_1 < z < 0$), layer 2 ($z_2 < z < z_1$), $\cdots$, layer N ($z_N < z < z_{N-1}$), where $z_1 = -h_1$, $z_2 = -h_1 - h_2$, $\cdots$, $z_N = -h_1 - h_2 - \cdots - h_N$. The conductivity, relative permeability, and thickness of the n-th layer are σ_n, μ_{rn}, and h_n, $n = 1, 2, \ldots, N$.

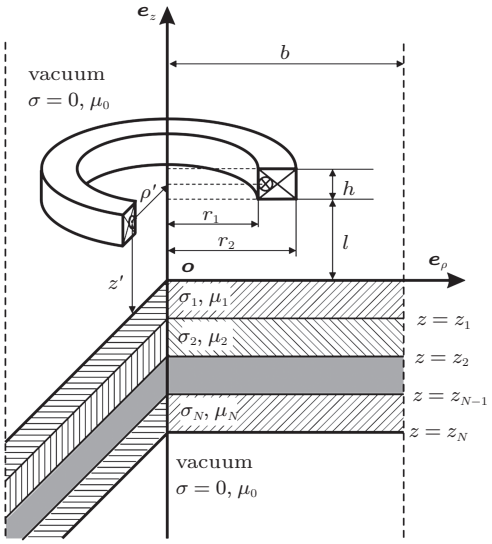


Fig. 1. Model of sinusoidal eddy-current problem composed of N-layer plate conductors and a current-carrying coil.

3. Magnetic vector potential in upper half space generated by the single-turn circular phasor current

3.1. General solutions and undetermined constant equations

In the cross-section of the air-core cylindrical coil shown in Fig. 1, a single-turn circular current I is taken arbitrarily on the plane $S' : z = z'$. Magnetic vector potential A is suitable for solving the axisymmetric electromagnetic field problem, which can be written as $A(\rho, \phi, z) = A_\phi(\rho, z) e_\phi$.[11] In order to facilitate the calculation, the whole field region is truncated by a long straight coaxial cylindrical surface whose radius $\rho = b$ ($b \gg r_2$, $b \gg |z_N|$), and let $A_\phi(b, z) = 0$ on the cylindrical surface.

Based on the assumptions given above, neglecting the displacement current, and then following the analytical methods in Refs. [1] and [4], the general solutions for circumferential components of the magnetic vector potential in each region can be written as follows:

$$A_{\phi 01}(\rho, z) = \sum_{m=1}^{\infty} C_{01m} \mathrm{J}_1(x_m \rho) \exp(-x_m z), \quad z' < z < \infty, \tag{1}$$

$$A_{\phi 02}(\rho, z) = \sum_{m=1}^{\infty} \mathrm{J}_1(x_m \rho) [C_{02m} \exp(-x_m z) + D_{02m} \exp(x_m z)], \quad 0 < z < z', \tag{2}$$

$$A_{\phi n}(\rho, z) = \sum_{m=1}^{\infty} \mathrm{J}_1(x_m \rho) [C_{nm} \exp(-\alpha_{nm} z) + D_{nm} \exp(\alpha_{nm} z)], \quad z_n < z < z_{n-1}, n = 1, 2, \ldots, N, \tag{3}$$

$$A_{\phi 03}(\rho, z) = \sum_{m=1}^{\infty} D_{03m} \mathrm{J}_1(x_m \rho) \exp(x_m z), \quad -\infty < z < z_N, \tag{4}$$

where $x_m = \lambda_m / b$, $\alpha_{nm} = \sqrt{x_m^2 + \mathrm{i}\omega\mu_0\mu_{rn}\sigma_n}$, $\mathrm{i} = \sqrt{-1}$, $\mathrm{Re}\alpha_{nm} > 0$, $\mu_0 = 4\pi \times 10^{-7}$ H/m; $z_0 = 0$; λ_m is the m-th positive zero point of $\mathrm{J}_1(\lambda)$ which denotes the Bessel function of the first kind of order one. C_{01m}, C_{02m}, D_{02m}, C_{nm}, D_{nm}, and D_{03m} are undetermined complex constants which satisfy the following linear equations:

$$C_{01m} = C_{02m} + D_{02m} \exp(2x_m z'), \tag{5}$$

$$C_{01m} - 2g_m \exp(x_m z') = C_{02m} - D_{02m} \exp(2x_m z'), \tag{6}$$

$$C_{02m} + D_{02m} = C_{1m} + D_{1m}, \tag{7}$$

$$C_{02m} - D_{02m} = q_{1m}(C_{1m} - D_{1m}), \tag{8}$$

$$C_{1m} \exp(-\alpha_{1m} z_1) + D_{1m} \exp(\alpha_{1m} z_1) = C_{2m} \exp(-\alpha_{2m} z_1) + D_{2m} \exp(\alpha_{2m} z_1), \tag{9}$$

$$C_{1m} \exp(-\alpha_{1m} z_1) - D_{1m} \exp(\alpha_{1m} z_1) = q_{2m} [C_{2m} \exp(-\alpha_{2m} z_1) - D_{2m} \exp(\alpha_{2m} z_1)], \tag{10}$$

$$\vdots$$

$$C_{Nm} \exp(-2\alpha_{Nm} z_N) + D_{Nm} = D_{03m} \exp[-(\alpha_{Nm} - x_m) z_N], \tag{11}$$

$$C_{Nm} \exp(-2\alpha_{Nm} z_N) - D_{Nm} = -q_{N+1,m} D_{03m} \exp[-(\alpha_{Nm} - x_m) z_N], \tag{12}$$

where

$$g_m = \frac{\mu_0 I \rho' \mathrm{J}_1(x_m \rho')}{b \lambda_m [\mathrm{J}_0(\lambda_m)]^2},$$

$$q_{nm} = \frac{\mu_{r(n-1)} \alpha_{nm}}{\mu_{rn} \alpha_{n-1,m}},$$

$$\mu_{r0} = \mu_{r(N+1)} = 1,$$

$$\alpha_{0m} = \alpha_{N+1,m} = x_m.$$

There are $2(N+2)$ equations and $2(N+2)$ undetermined constants in Eqs. (5)–(12). In theory, whatever the number N of layers is, all undetermined constants can be solved from Eqs. (5)–(12).

3.2. Derivation of the layered-plate factor expression

Let

$$L_{Nm} = -\frac{C_{02m}}{D_{02m}}. \tag{13}$$

Solving Eqs. (5) and (6), we obtain

$$C_{01m} = g_m [\exp(x_m z') - L_{Nm} \exp(-x_m z')], \tag{14}$$

$$C_{02m} = -g_m L_{Nm} \exp(-x_m z'), \tag{15}$$

$$D_{02m} = g_m \exp(-x_m z'). \tag{16}$$

As can be seen, if L_{Nm} has been derived, C_{01m} and C_{02m} can be obtained and then C_{1m} and D_{1m} can be solved from Eqs. (7) and (8). All of the undetermined constants will be obtained by the continuous equivalent transformation with a similar process to the above.

Now, by using the derivation method of Lipskaya–Vanyan formula,[9,10] L_{Nm} can be derived. The ratio of Eq. (8) to Eq. (7) is

$$\frac{L_{Nm}+1}{L_{Nm}-1}=q_{1m}\frac{C_{1m}-D_{1m}}{C_{1m}+D_{1m}}. \tag{17}$$

Using the natural logarithm identity $x=\exp(\ln x)$, equation (17) can be converted into

$$\frac{\exp(\ln L_{Nm})+1}{\exp(\ln L_{Nm})-1}=q_{1m}\frac{\exp\left(\ln\dfrac{C_{1m}}{D_{1m}}\right)-1}{\exp\left(\ln\dfrac{C_{1m}}{D_{1m}}\right)+1}. \tag{18}$$

Further, using the definition of hyperbolic tangent function th$x$ and hyperbolic cotangent function cth$x$, equation (18) can be written as

$$\mathrm{cth}\left(\frac{1}{2}\ln L_{Nm}\right)=q_{1m}\mathrm{th}\left(\frac{1}{2}\ln\frac{C_{1m}}{D_{1m}}\right). \tag{19}$$

Because the inverse function of $y=\mathrm{cth}x$ is $x=\mathrm{arcth}y$, equation (19) can be changed into

$$\frac{1}{2}\ln L_{Nm}=\mathrm{arcth}\left(q_{1m}\mathrm{th}\left(\frac{1}{2}\ln\frac{C_{1m}}{D_{1m}}\right)\right). \tag{20}$$

Using the identity transform

$$\exp(2\mathrm{arcth}(x))=\mathrm{th}\left(\frac{1}{2}\ln(-x)\right),$$

equation (20) can be written as

$$\begin{aligned}L_{Nm}&=\exp(\ln L_{Nm})\\&=\mathrm{th}\left(\frac{1}{2}\ln\left(-q_{1m}\mathrm{th}\left(\frac{1}{2}\ln\frac{C_{1m}}{D_{1m}}\right)\right)\right).\end{aligned} \tag{21}$$

To solve C_{1m}/D_{1m}, we take the ratio of Eq. (10) to Eq. (9) and obtain

$$\begin{aligned}&\frac{C_{1m}\exp(-2\alpha_{1m}z_1)-D_{1m}}{C_{1m}\exp(-2\alpha_{1m}z_1)+D_{1m}}\\&=q_{2m}\frac{C_{2m}\exp(-2\alpha_{2m}z_1)-D_{2m}}{C_{2m}\exp(-2\alpha_{2m}z_1)+D_{2m}}.\end{aligned} \tag{22}$$

Imitating the derivation process of Eq. (20), equation (22) can be written as

$$\begin{aligned}&\frac{1}{2}\ln\frac{C_{1m}}{D_{1m}}\\&=\alpha_{1m}z_1+\mathrm{arth}\left(q_{2m}\mathrm{th}\left(-\alpha_{2m}z_1+\frac{1}{2}\ln\frac{C_{2m}}{D_{2m}}\right)\right).\end{aligned} \tag{23}$$

Similarly,

$$\ln(C_{2m}/D_{2m})/2,\ln(C_{3m}/D_{3m})/2,\ldots,\ln(C_{Nm}/D_{Nm})/2$$

can be obtained successively, and expressed as

$$\frac{1}{2}\ln\frac{C_{Nm}}{D_{Nm}}=\alpha_{Nm}z_N-\mathrm{arth}q_{N+1,m}. \tag{24}$$

Finally, substituting $\ln(C_{Nm}/D_{Nm})/2$ into $\ln(C_{N-1,m}/D_{N-1,m})/2$, $\ln(C_{N-1,m}/D_{N-1,m})/2$ into $\ln(C_{N-2,m}/D_{N-2,m})/2$, …, until $\ln(C_{1m}/D_{1m})/2$ into Eq. (21), we obtain

$$\begin{aligned}L_{Nm}=\mathrm{th}\Bigg(&\frac{1}{2}\ln(q_{1m}\mathrm{th}(\alpha_{1m}h_1+\cdots+\mathrm{arth}(q_{N-1,m}\mathrm{th}(\alpha_{N-1,m}\\&\times h_{N-1}+\mathrm{arth}(q_{Nm}\mathrm{th}(\alpha_{Nm}h_N+\mathrm{arth}q_{N+1,m})))))))\Bigg).\end{aligned} \tag{25}$$

In this paper, L_{Nm} is named layered-plate factor, and it is a complex function determined by thickness, relative permeability, and conductivity of each layer.

3.3. Magnetic vector potential above the multi-layer plate conductor

The region above the multi-layer plate conductor is the measurement region, so the most important step is to solve the magnetic vector potential in this region.

Substituting Eqs. (14)–(16) into Eqs. (1) and (2), the series expression for circumferential components of the magnetic vector potential, under the excitation of single-turn circular sinusoidal current, can be obtained as follows:

$$\begin{aligned}A_{\phi 0}(\rho,z)=&\frac{\mu_0 I}{b}\sum_{m=1}^{\infty}\frac{\rho' \mathrm{J}_1(x_m\rho')}{\lambda_m[\mathrm{J}_0(\lambda_m)]^2}\mathrm{J}_1(x_m\rho)\\&\times\left[\exp\left(-x_m|z-z'|\right)-L_{Nm}\exp\left(-x_m(z+z')\right)\right].\end{aligned} \tag{26}$$

Based on Eq. (26), the series expression for circumferential components of the magnetic vector potential in the upper half space, under the excitation of eddy-current in the multi-layer plate conductor, can be written as

$$\begin{aligned}\Delta A_{\phi 0}(\rho,z)&=A_{\phi 0}(\rho,z)-A_{\phi 0}(\rho,z)\Big|_{L_{Nm}=0}\\&=-\frac{\mu_0 I}{b}\sum_{m=1}^{\infty}\frac{\rho' \mathrm{J}_1(x_m\rho')L_{Nm}}{\lambda_m[\mathrm{J}_0(\lambda_m)]^2}\mathrm{J}_1(x_m\rho)\\&\quad\times\exp\left[-x_m(z+z')\right].\end{aligned} \tag{27}$$

4. Series expression of eddy-current impedance for air-core cylindrical coil

The impedance[11] for the air-core cylindrical coil corresponding to eddy-current is

$$\Delta Z=\mathrm{i}\omega n_{\mathrm{den}}^2\int_V\left(\int_l^{l+h}\mathrm{d}z'\int_{r_1}^{r_2}\frac{\Delta A_{\phi 0}}{I}\mathrm{d}\rho'\right)\mathrm{d}V, \tag{28}$$

where $n_{\mathrm{den}}=W/[h(r_2-r_1)]$ is the density of turns of the air-core cylindrical coil, V is the occupied region of the air-core cylindrical coil. ΔZ is called eddy-current impedance for

short, which is the difference between Z (impedance for air-core cylindrical coil placed above a multi-layer plate conductor) and Z_0 (impedance for the air-core cylindrical coil in the infinite vacuum). Substituting Eq. (27) into Eq. (28), the series expression of eddy-current impedance can be obtained as follows:

$$\Delta Z = -\mathrm{i}2\pi\mu_0 b^5 \omega \sum_{m=1}^{\infty} L_{Nm} C_m^2 \exp(-2x_m l), \tag{29}$$

where

$$C_m = \frac{n_{\mathrm{den}}\left[1-\exp(-x_m h)\right]}{\lambda_m^{7/2} \mathrm{J}_0(\lambda_m)} \int_{x_m r_1}^{x_m r_2} x \mathrm{J}_1(x)\,\mathrm{d}x. \tag{30}$$

In this paper, C_m is named coil factor, determined only by the coil parameters W, r_1, r_2, and h.

5. Experiment verification

In order to verify Eq. (29), a three-layer clinging tightly non-ferromagnetic plate has been made which consists of a copper plate, a paperboard, and an aluminum plate successively from top to bottom. The detailed parameters of the three-layer plate and air-core cylindrical coil are listed in Tables 1 and 2, respectively. Let $b = 1$ m. We pick 32 frequency points between 180 Hz and 6000 Hz and then calculate the sum of the first 300 terms in Eq. (29) of which the MATLAB calculation program is provided in Appendix A. Calculations show that under the above conditions, the calculation values of the eddy-current impedance have at least 4 significant figures. For comparison, we measure Z_{meas} (impedance for the coil above the three-layer plate) and $Z_{0\mathrm{meas}}$ (impedance for the coil in infinity air) respectively with a WK6500B impedance analyzer, and then obtain the eddy-current impedance $\Delta Z_{\mathrm{meas}} = Z_{\mathrm{meas}} - Z_{0\mathrm{meas}}$ corresponding to the 32 frequency points. Figure 2 shows the calculated and measured values of $\Delta R = \mathrm{Re}\Delta Z$ as a function of f, respectively. Figure 3 shows the calculated and measured values of $\Delta X = \mathrm{Im}\Delta Z$ as a function of f, respectively. Both figures 2 and 3 show that the measured values are in agreement with the calculated values. Figure 2 shows that there is a large deviation between the measured values of the real part and calculated values at the frequency of several hundred Hz. The reason for this phenomenon is that the real part of the impedance represents eddy-current power loss, which is small at a low frequency, and it is obviously affected by the change of surrounding temperature. Hence, one of the measures to reduce the large deviation is to place both the coil and the multi-layer plate conductor into the temperature controlled oil sump.[12]

Table 1. Parameters of the three-layer plate.

Symbol	Parameter	Value
h_1	the 1st layer's thickness (copper)	2.00 mm
h_2	the 2nd layer's thickness (paper)	1.04 mm
h_3	the 3rd layer's thickness (aluminum)	3.44 mm
σ_1	the 1st layer's conductivity (copper)	54.55 MS/m
σ_2	the 2nd layer's conductivity (paper)	0
σ_3	the 3rd layer's conductivity (aluminum)	34.45 MS/m

Table 2. Parameters of the air-core cylindrical coil.

Symbol	Parameter	Value
W	turns	43
r_1	inner radius	9.95 mm
r_2	outer radius	10.20 mm
h	height	7.24 mm
l	lift-off	1.20 mm

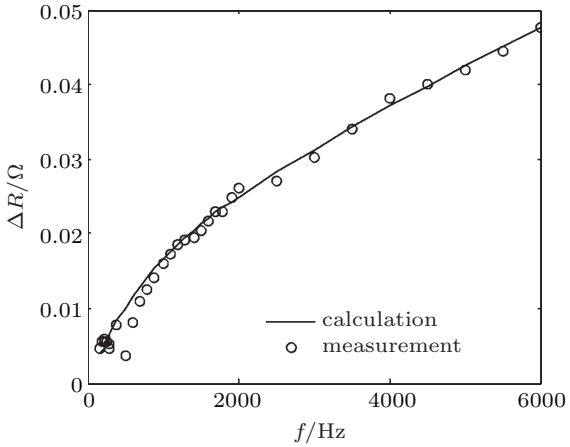


Fig. 2. ΔR changing with f.

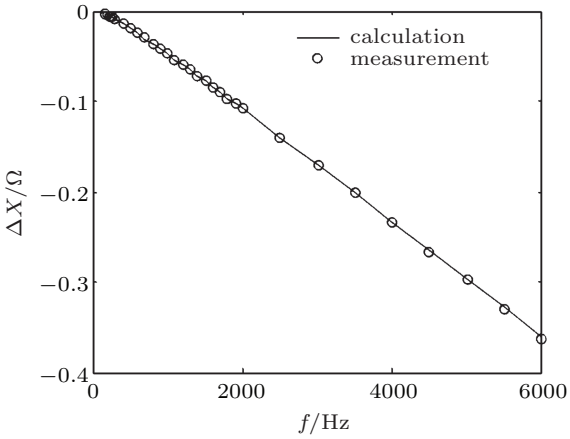


Fig. 3. ΔX changing with f.

6. Discussion

6.1. Writing method of layered-plate factor expression

According to Eq. (25), the layered-plate factor L_{Nm} is composed of N functions shaped like $\mathrm{th}(t + \mathrm{arth}x)$ and the innermost part of the layered-plate factor is written as

$$u_N = q_{Nm}\mathrm{th}(\alpha_{Nm} h_N + \mathrm{arth} q_{N+1,m}),$$

and writing them successively from inner to outer as follows:

$$u_{N-1} = q_{N-1,m}\mathrm{th}(\alpha_{N-1,m} h_{N-1} + \mathrm{arth}\, u_N),$$

$$\vdots,$$
$$u_1 = q_{1m}\mathrm{th}(\alpha_{1m}h_1 + \mathrm{arth}\,u_2).$$

Eventually, we obtain

$$L_{Nm} = \mathrm{th}\left(\frac{1}{2}\ln u_1\right).$$

In this way, the layered-plate factor corresponding to $N = 1,2,3,4$ can be written respectively as

$$L_{1m} = \mathrm{th}\left(\frac{1}{2}\ln(q_{1m}\mathrm{th}(\alpha_{1m}h_1 + \mathrm{arth}\,q_{2m}))\right), \tag{31}$$

$$L_{2m} = \mathrm{th}\Big(\frac{1}{2}\ln(q_{1m}\mathrm{th}(\alpha_{1m}h_1 + \mathrm{arth}\,(q_{2m}\mathrm{th}(\alpha_{2m}h_2 + \mathrm{arth}\,q_{3m}))))\Big), \tag{32}$$

$$L_{3m} = \mathrm{th}\Big(\frac{1}{2}\ln(q_{1m}\mathrm{th}(\alpha_{1m}h_1 + \mathrm{arth}\,(q_{2m}\mathrm{th}(\alpha_{2m}h_2 + \mathrm{arth}\,(q_{3m}\mathrm{th}(\alpha_{3m}h_3 + \mathrm{arth}\,q_{4m})))))\Big), \tag{33}$$

$$L_{4m} = \mathrm{th}\Big(\frac{1}{2}\ln(q_{1m}\mathrm{th}(\alpha_{1m}h_1 + \mathrm{arth}\,(q_{2m}\mathrm{th}(\alpha_{2m}h_2 + \mathrm{arth}\,(q_{3m}\mathrm{th}(\alpha_{3m}h_3 + \mathrm{arth}\,(q_{4m}\mathrm{th}(\alpha_{4m}h_4 + \mathrm{arth}q_{5m})))))))\Big). \tag{34}$$

6.2. Properties of layered plate factor

(i) Even though there are multi-value functions arth$x$ and ln$x$ in L_{Nm}, both

$$\mathrm{th}\,(t + \mathrm{arth}x) = \frac{(1+x)\exp(2t) - (1-x)}{(1+x)\exp(2t) + (1-x)}$$

and

$$\mathrm{th}\left(\frac{1}{2}\ln x\right) = \frac{x-1}{x+1}$$

are single-value functions. Hence, L_{Nm} is single-valued.

(ii) L_{Nm} can also be composed of $\mathrm{cth}(t + \mathrm{arcth}x)$ because

$$\mathrm{cth}\,(t + \mathrm{arcth}x) = \mathrm{th}\,(t + \mathrm{arth}x).$$

(iii) When the coil is in infinite vacuum, $\mu_{r1} = \mu_{r2} = \cdots = \mu_{rN} = 1$, $\sigma_1 = \sigma_2 = \cdots = \sigma_N = 0$, so $L_{Nm} = \mathrm{th}\,(\ln 1/2) = 0$.

6.3. Application example of layered-plate factor expression

Let the N-th layer (bottom layer) of the N-layer plate be non-ferromagnetic conductor, and $c_N = b\sqrt{\omega\mu_0\sigma_N}$. If the N-th layer is a thin-layer conductor and its thickness h_N satisfies

$$\lambda_m h_N \ll 1, \quad c_N h_N \ll 1,$$

then

$$|\alpha_{Nm}h_N| = [(\lambda_m h_N)^4 + (c_N h_N)^4]^{1/4} \ll 1,$$

where $\lambda_m \approx (m + 0.25)\pi$ ($m \le 300$ in general). Using the following approximate expressions:[13]

$$\mathrm{arth}x = x + \frac{1}{3}x^3 + \cdots \approx x, \quad (|x| < 1),$$
$$\mathrm{th}x = x - \frac{1}{3}x^3 + \cdots \approx x, \quad \left(|x| < \frac{\pi}{2}\right),$$

we have

$$q_{Nm}\mathrm{th}(\alpha_{Nm}h_N + \mathrm{arth}q_{N+1,m}) \approx \frac{\lambda_m + \mathrm{i}b^2\omega\mu_0(\sigma_N h_N)}{\alpha_{N-1,m}},$$

where

$$|q_{N+1,m}| = \frac{\lambda_m}{(\lambda_m^4 + c_N^4)^{1/4}} < 1.$$

Hence

$$L_{Nm} \approx \mathrm{th}\Big(\frac{1}{2}\ln(q_{1m}\mathrm{th}(\alpha_{1m}h_1 + \cdots + \mathrm{arth}(q_{N-1,m}\mathrm{th}(\alpha_{N-1,m}h_{N-1} + \mathrm{arth}\frac{\lambda_m + \mathrm{i}b^2\omega\mu_0(\sigma_N h_N)}{\alpha_{N-1,m}})))\Big).$$

As can be seen, L_{Nm} is the function of $\sigma_N h_N$. It means that if $\sigma_N h_N$ remains unchanged, L_{Nm} will not change. In other words,

$$\Delta Z(h_N, \sigma_N) \approx \Delta Z\left(ah_N, \frac{\sigma_N}{a}\right),$$

where $a > 0$.

Taking a three-layer plate for example, the parameters of the first layer and the second layer are the same as those listed in Table 1. The third layer is replaced by the following two kinds of thin-layer plates separately:

Plate 1: $h_3 = 0.01$ mm, $\sigma_3 = 35$ MS/m, $h_3\sigma_3 = 350$ S;

Plate 2: $h_3 = 0.1$ mm, $\sigma_3 = 3.5$ MS/m, $h_3\sigma_3 = 350$ S.

Let the coil parameters be the same as those listed in Table 2. The results show that when the calculated values of eddy-current impedance are accurate to the fourth place after the decimal point, both the impedance values of these two thin-layer plates are $\Delta Z = 0.0473 - \mathrm{i}0.3609\ \Omega$ at $f = 6000$ Hz.

The parameters of the multi-layer plate conductor can be inversed with the least square method,[14–17] i.e., choosing a set of frequency values $f_1, f_2, \cdots, f_n$, and then adjusting parameters to minimize

$$\sum_{i=1}^{n} |\Delta Z_{\mathrm{meas}}(f_i) - \Delta Z_{\mathrm{calcu}}(f_i)|^2,$$

where $\Delta Z_{\mathrm{meas}}(f_i)$ is the measured value and $\Delta Z_{\mathrm{calcu}}(f_i)$ is the calculated value. Numerical calculation shows that when the least square method is used to inverse the parameters of thin-layer metal plate, only the product of the thin-layer metal plate thickness and conductivity can be obtained, i.e., the thickness cannot be obtained directly.[18–20] This phenomenon could be explained by the research results above.

7. Conclusions

In the paper, the closed-form expression and properties of a layered-plate factor are presented. Based on them, it is easy to write a general series expression of eddy-current impedance of the coil. This series expression consists of three independent factors: layered-plate factor, coil factor, and lift-off. The expression has a simple form and is convenient for theoretical analysis. In addition, the method in this paper can also be used to derive the integral solution of eddy-current impedance and analyze the electromagnetic fields of other layered media.

Appendix A: Calculation program of eddy-current impedance series

MATLAB^{TM} calculation program of Eq. (29) is as follows. The conductor plate has three layers and the frequency is 6000 Hz. Parameters of the conductor plate are shown in Table 1, and the coil parameters are shown in Table 2. SI units are used for all quantities.

Input data:

```
f=6000;
W=43; r1=9.95e-3; r2=10.2e-3; h=7.24e-3;
lo=1.2e-3;
h1=2.0e-3; h2=1.04e-3; h3=3.44e-3;
sigma1=54.55e6; sigma2=0; sigma3=34.45e6;
mur1=1; mur2=1; mur3=1;
b=1.0;
M=300;
```

Precomputed quantities:

```
nden=W/h/(r2-r1);
i=complex(0,1);
mu0=4*pi*1e-7;
omega=2*pi*f;
a0=i*omega*mu0;
```

Series computation:

```
syms t
for j=1:1:M
    y0=pi*(j+0.25);
    yj=fzero(@(x)besselj(1,x),[y0-0.096,y0]);
    x(j)=yj/b;
    integral=double(int(t*besselj(1,t),t,x(j)*r1,x(j)*r2));
    J0=besselj(0,yj);
    C(j)=nden*(1-exp(-x(j)*h))*integral/(J0*yj^3.5);
end
c=complex(0,0);
for m=1:1:M
    x2=x(m)*x(m);
    alpha1=sqrt(x2+a0* mur1*sigma1);
    alpha2=sqrt(x2+a0* mur2*sigma2);
    alpha3=sqrt(x2+a0* mur3*sigma3);
    q1=(1.0/mur1)*(alpha1/x(m));
    q2=(mur1/mur2)*(alpha2/alpha1);
    q3=(mur2/mur3)*(alpha3/alpha2);
    q4=(mur3/1.0)*(x(m)/alpha3);
    u3=q3*tanh(alpha3*h3+atanh(q4));
    u2=q2*tanh(alpha2*h2+atanh(u3));
    u1=q1*tanh(alpha1*h1+atanh(u2));
    L3=tanh(0.5*log(u1));
    c=c+L3*C(m)^2*exp(-2*x(m)*lo);
end
```

Result:

```
deltaZ=-2*pi*b^5* a0*c
deltaZ=
0.0477-0.3612i
```

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